KING SAUD UNIVERSITY **FULL MARKS: 50**

M - 107 **DEPARTMENT OF MATHEMATICS** **TIME: 90min**

(SEMESTER II, 1433-1434) FIRST MID-TERM

Question: 1 (a) Find polynomial function $p(x) = ax^2 + bx + c$ **such that**

$p(0) = 1, \; p(1) = 4, \; p(2) = 11.$ **[8]**

Solution: Given Polynomial function is

$$p(x) = ax^2 + bx + c \qquad \textbf{Eq.1}$$

at $x = 0$, we obtained $p(0) = 0 + 0 + c = 1$

at $x = 1$, we obtained $p(1) = a + b + c = 1$

at $x = 2$, we obtained $p(2) = 4a + 2b + c = 11$

We obtained system of linear equations

$$c = 1$$
$$a + b + c = 4$$
$$4a + 2b + c = 11$$

Writing the system in the matrix form

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 11 \end{bmatrix}$$

Solving by using Guass Jordon method, Augmented matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 11 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 11 \\ 0 & 0 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & -3 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence ***a = 2, b= 1 c =1*** **Substituting these values in Eq.1.**

The required polynomial function is $p(x) = 2x^2 + x + 1$

(b) Use the reduced row echelon form to solve the system of linear equations

$$x + y + z - 3w = -2$$
$$2x + 3y - 4z = 1$$
$$-3x - 4y - z + 6w = -1$$

[8]

Solution: writing the system of equation in matrix form ,

$$\begin{bmatrix} 1 & 1 & 1 & -3 \\ 2 & 3 & -4 & 0 \\ -3 & -4 & -1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

Solving by using Reduced row echelon form Row echelon form ,Augmented matrix is

$$\approx\begin{bmatrix}1 & 1 & 1 & -3 & -2\\ 2 & 3 & -4 & 0 & 1\\ -3 & -4 & -1 & 6 & -1\end{bmatrix}\approx\begin{bmatrix}1 & 1 & 1 & -3 & -2\\ 0 & 1 & -6 & 6 & 5\\ 0 & -1 & 2 & -3 & -7\end{bmatrix}\approx\begin{bmatrix}1 & 1 & 1 & -3 & -2\\ 0 & 1 & -6 & 6 & 5\\ 0 & 0 & -4 & 3 & -2\end{bmatrix}\approx\begin{bmatrix}1 & 1 & 1 & -3 & -2\\ 0 & 1 & -6 & 6 & 5\\ 0 & 0 & 1 & \frac{-3}{4} & \frac{1}{2}\end{bmatrix}$$

$$\approx\begin{bmatrix}1 & 0 & 7 & -9 & -7\\ 0 & 1 & -6 & 6 & 5\\ 0 & 0 & 1 & \frac{-3}{4} & \frac{1}{2}\end{bmatrix}\approx\begin{bmatrix}1 & 0 & 0 & \frac{-15}{4} & \frac{-21}{2}\\ 0 & 1 & 0 & \frac{3}{2} & 8\\ 0 & 0 & 1 & \frac{-3}{4} & \frac{1}{2}\end{bmatrix}$$

Writing the augmented matrix is the system of equation form

$$x-\tfrac{15}{4}w=\tfrac{-21}{2}$$
$$y-\tfrac{3}{2}w=8$$
$$z-\tfrac{3}{4}w=\tfrac{1}{2}$$

The system will have infinite number of solutions

***x, y, and z* are leading variables and *w* is free variable, let *w* = *t*, where t is any real number**

$$x=\tfrac{-21}{2}+\tfrac{15}{4}w \qquad x=\tfrac{-21}{2}+\tfrac{15}{4}t$$
$$y=8-\tfrac{3}{2}w \qquad \textbf{solutions are} \qquad y=8-\tfrac{3}{2}t$$
$$z=\tfrac{1}{2}+\tfrac{3}{4}w \qquad z=\tfrac{1}{2}+\tfrac{3}{4}t$$
$$w=t,\quad \text{where t} \in \text{R}$$

Question: 2 (a)

If $A=\begin{bmatrix}2 & 1 & -3\\ 1 & 1 & 2\\ 0 & 2 & 1\end{bmatrix}$**, find the inverse of A by using Elementary matrix method. [8]**

Solution: Using Elementary matrix method to find A^{-1}

$$[A:I]=\left[\begin{array}{ccc|ccc}2 & 1 & -3 & 1 & 0 & 0\\ 1 & 1 & 2 & 0 & 1 & 0\\ 0 & 2 & 1 & 0 & 0 & 1\end{array}\right]\approx\left[\begin{array}{ccc|ccc}1 & 1 & 2 & 0 & 1 & 0\\ 2 & 1 & -1 & 1 & 0 & 0\\ 0 & 2 & 1 & 0 & 0 & 1\end{array}\right]R_1\leftrightarrow R_2$$

$-2R_1+R_2,$ $\qquad -R_2$

$$\approx\left[\begin{array}{ccc|ccc}1 & 1 & 2 & 0 & 1 & 0\\ 0 & -1 & -7 & 1 & -2 & 0\\ 0 & 2 & 1 & 0 & 0 & 1\end{array}\right]\approx\left[\begin{array}{ccc|ccc}1 & 1 & 2 & 0 & 1 & 0\\ 0 & 1 & 7 & -1 & 2 & 0\\ 0 & 2 & 1 & 0 & 0 & 1\end{array}\right],$$

$-2R_2+R_3,$ $\qquad -\tfrac{1}{13}R_3$

$$\approx\left[\begin{array}{ccc|ccc}1 & 1 & 2 & 0 & 1 & 0\\ 0 & 1 & 7 & -1 & 2 & 0\\ 0 & 0 & -13 & 2 & -4 & 1\end{array}\right]\approx\left[\begin{array}{ccc|ccc}1 & 1 & 2 & 0 & 1 & 0\\ 0 & 1 & 7 & -1 & 2 & 0\\ 0 & 0 & 1 & -\frac{2}{13} & \frac{4}{13} & \frac{-1}{13}\end{array}\right]$$

$$-7R_3+R_2, -2R_3+R_1 \qquad\qquad -R_2+R_1$$

$$\approx\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{4}{13} & \frac{5}{13} & \frac{2}{13} \\ 0 & 1 & 0 & \frac{1}{13} & \frac{-2}{13} & \frac{7}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{4}{13} & \frac{-1}{13} \end{array}\right] \approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{13} & \frac{4}{13} & \frac{-5}{13} \\ 0 & 1 & 0 & \frac{1}{13} & \frac{-2}{13} & \frac{7}{13} \\ 0 & 0 & 1 & -\frac{2}{13} & \frac{4}{13} & \frac{-1}{13} \end{array}\right] = [I : A^{-1}]$$

$$\text{Hence} \quad A^{-1} = \begin{bmatrix} \frac{3}{13} & \frac{4}{13} & \frac{-5}{13} \\ \frac{1}{13} & \frac{-2}{13} & \frac{7}{13} \\ -\frac{2}{13} & \frac{4}{13} & \frac{-1}{13} \end{bmatrix}$$

(b) If A is a 3x3 matrix and detA=2,
find det(4A), det($4A^{-1}$) and det (A^4) **[6]**

Solution: Using properties of determinant

$$(i) \quad \det(4A) = 4^3 \det A = 64(2) = 128$$

$$(ii) \quad \det(4A^{-1}) = 4^3 \det A^{-1} = \frac{64}{\det A} = \frac{64}{2} = 32$$

$$(iii) \quad \det(A^4) = (\det A)^4 = (2)^4 = 16$$

Question: 3 Solve the system of linear equations by Cramer's Rule

$$x+y+z=7$$
$$-x+y+z=5$$
$$x-y+z=5$$

[8]

Solution: Matrix form of the system of linear equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}$$

$$\det A = \det\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = 4, \det A_1 = \det\begin{bmatrix} 7 & 1 & 1 \\ 5 & 1 & 1 \\ 5 & -1 & 1 \end{bmatrix} = 4, \det A_2 = \det\begin{bmatrix} 1 & 7 & 1 \\ -1 & 5 & 1 \\ 1 & 5 & 1 \end{bmatrix} = 4$$

$$\det A_3 = \det\begin{bmatrix} 1 & 1 & 7 \\ -1 & 1 & 5 \\ 1 & -1 & 5 \end{bmatrix} = 20$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{4} = 1$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{4}{4} = 1$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{20}{4} = 5$$

Question: 4 (a) Let $a =< 2,1,0 >, \quad b =< 2,-1,3 >$

(i) Find the angle between a and b **[6]**

(ii) Find the projection of b onto a

Solution:

$$(i) \quad a.b = 4 - 1 + 0 = 3$$

$$\|a\| = \sqrt{4+1} = \sqrt{5}$$

$$\|b\| = \sqrt{4+1+9} = \sqrt{13}$$

$$\theta = \cos^{-1}\left[\frac{a.b}{\|a\|\|b\|}\right] = \cos^{-1}\left[\frac{3}{\sqrt{5}\sqrt{13}}\right]$$

$$(ii) \quad proj_a^b = \left[\frac{b.a}{\|a\|}\right]\left[\frac{a}{\|a\|}\right] = \left[\frac{3}{\sqrt{5}}\right]\left[\frac{< 2,1,0 >}{\sqrt{5}}\right] = \frac{3}{5} < 2,1,0 >$$

(b) If $u =< 1,1,1 >, v =< 3,-1,1 >$ **, find value of** $\alpha \in$ **R, if**

$$\|\alpha u - v\| = \sqrt{20}$$ **[6]**

Solution: (b)

$$\alpha u = \alpha < 1,1,1 > =< \alpha, \alpha, \alpha >$$

$$-v = - < 3,-1,1 > =< -3,1,-1 >$$

$$\alpha u - v =< \alpha - 3, \alpha + 1, \alpha - 1 >$$

$$\|\alpha u - v\| = \sqrt{(\alpha-3)^2 + (\alpha+1)^2 + (\alpha-1)^2} = \left(\sqrt{20}\right)$$

$$(\alpha^2 - 6\alpha + 9) + (\alpha^2 + 2\alpha + 1) + (\alpha^2 - 2\alpha + 1) = 20$$

$$3\alpha^2 - 6\alpha + 11 = 20$$

$$3\alpha^2 - 6\alpha - 9 = 0$$

$$(\alpha - 3)(\alpha + 1) = 0 \quad \Rightarrow \alpha = 3, \alpha = -1$$