جامعة الملك سعود
King Saud University

Solution of Final Examination M - 107 (First Semester 1437-1438)

Question:1. (a) Solve the system of equations by using the Gauss- Jordan method.

[6+6+6]

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 5y + 5z &= 2 \\ x + 4y + z &= 1 \end{aligned}$$

(b) Find inverse of matrix A by method of cofactors

$$A = \begin{bmatrix} -1 & 3 & 1 \\ -3 & 6 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) Let $A = \begin{bmatrix} x & y \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$

Find $x$ and $y$ if AB = C

Solution. (a) $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 5 & 2 \\ 1 & 4 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ -2R_1+R_2 \\ -R_1+R_3 \end{array}$

6

$= \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} -2R_2+R_1 \\ \\ -2R_2+R_3 \end{array}$

$z = t$, $x = 1 - 5t$, $y = t$, $t \in R$.

(b) $A = \begin{bmatrix} -1 & 3 & 1 \\ -3 & 6 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ matrix of cofactors $C = \begin{bmatrix} 6 & 3 & -6 \\ -3 & -2 & 3 \\ -6 & -3 & 3 \end{bmatrix}$

$\det A = -3$, $A^{-1} = \frac{1}{-3} \begin{bmatrix} 6 & -3 & -6 \\ 3 & -2 & -3 \\ -6 & 3 & 3 \end{bmatrix}$ 4

$= \begin{bmatrix} -2 & +1 & 2 \\ -1 & +\frac{2}{3} & 1 \\ 2 & -1 & -1 \end{bmatrix}$ 2

(c) $AB = \begin{bmatrix} x & y \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} x-2y & -x+3y \\ 3 & -4 \end{bmatrix}$ 4

$AB = C \Rightarrow \begin{aligned} x - 2y &= -1 \\ -x + 3y &= 2 \end{aligned} \Rightarrow x = 1, y = 1.$

2

**Question:2. (a)** **Let $L_1$ be the line through points $A(3,1,2)$ and $B(2,0,1)$ and let $L_2$ be the line through points $C(0,1,2)$ and $D(1,2,-1)$. Find the shortest distance between skew lines $L_1$ and $L_2$.**

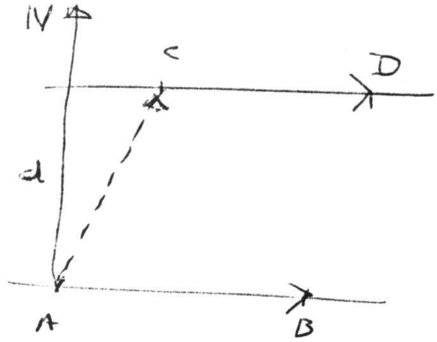
**[6+6+6]** **( b) Find $\lim_{t\to 0} r(t)$, where** $r(t) = e^{-3t}i + \dfrac{t^2}{\sin^2 t}j + \cos 2tk$

**( c) Show that** $\lim_{(x,y)\to(0,0)} \left(\dfrac{x^2y^2}{x^4+2y^4}\right)$ **does not exist.**

Solution

(a) $\vec{AB} = \langle -1,-1,-1\rangle$, $\vec{CD} = \langle 1,1,-3\rangle$

3 $N = \vec{AB}\times\vec{CD} = \langle 4,-4,0\rangle$




$d = \left|\text{Comp}_N \vec{AC}\right| = \dfrac{|\vec{AC}\cdot N|}{\|N\|}$

3 $= \left|\dfrac{-12}{\sqrt{32}}\right| = \dfrac{12}{\sqrt{32}}$ //

(b) $\lim_{t\to 0} r(t) = \left[\lim_{t\to 0} e^{-3t}\right]i + \left[\lim_{t\to 0}\left(\dfrac{t}{\sin t}\right)^2\right]j + \lim_{t\to 0} \cos 2t\, k$

6 $= i + j + k.$ //

(c) Along $y$-axis $x=0$

1 $\lim_{(0,y)\to(0,0)} \left(\dfrac{0}{2y^4}\right) = \lim_{(0,y)\to(0,0)} 0 = 0 \rightarrow 1$

Along $y=x$

2 $\lim_{(x,x)\to(0,0)} \dfrac{x^4}{x^4+2x^4} = \lim_{(x,x)\to(0,0)} \dfrac{1}{3} = \dfrac{1}{3} \rightarrow 2$

2 1 $\neq$ 2

$\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+2y^4}$ does not exist

**Question:3. (a)** **Find** $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ **for** $x^2 \sin(2y-5z) = 1 + y\cos(6xz)$ **and z is differentiable function of x and y.**

**[6+8]**

**(b) Find the total differentials of** $f(x,y) = \left(x^2+y^2\right)^{\frac{1}{3}}$ **and use it to approximate** $\left[(2.1)^2+(1.92)^2\right]^{\frac{1}{3}}$ , **where** ***(x, y)*** **is (2,2).**

Solution.

(a)

$$F(x,y,z) = x^2 \sin(2y-5z) - 1 - y\cos(6xz) = 0$$

$$\frac{\partial F}{\partial x} = 2x\sin(2y-5z) + y\sin(6xz).6z$$

4

$$\frac{\partial F}{\partial y} = 2x^2\cos(2y-5z) - \cos(6xz)$$

$$\frac{\partial F}{\partial z} = -5x^2\cos(2y-5z) + 6xy\sin(6xz)$$

2

$$\frac{\partial z}{\partial x} = -\frac{2x\sin(2y-5z) + 6yz\sin(6xz)}{-5x^2\cos(2y-5z) + 6xy\sin(6xz)}$$

$$\frac{\partial z}{\partial y} = -\frac{2x^2\cos(2y-5z) - \cos(6xz)}{-5x^2\cos(2y-5z) + 6xy\sin(6xz)}$$

(b)

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

4

$$= \frac{1}{3}(x^2+y^2)^{-\frac{2}{3}}.2x\,dx + \frac{1}{3}(x^2+y^2)^{-\frac{2}{3}}2.y\,dy$$

$$= \frac{2}{3}(x^2+y^2)^{-\frac{2}{3}}(x\,dx + y\,dy)$$

2

$x = 2, \quad y = 2, \quad dx = 2.1 - 2.0 = .1$

$dy = 1.92 - 2.00 = -.08$

2

$$df = \frac{2}{3}(2^2+2^2)^{-\frac{2}{3}}(2(0.1) + 2(-.08))$$

$$= 0.0067.$$

**Question:4. (a) Find the equations of the tangent plane and the normal line to the surface**
**[6+8]** $z = x^2 + y^2$ **at the point** $(1,-1,2)$.

**(b) The temperature T at (x,y,z) is given by** $T = 4x^2 - y^2 + 16z^2$.

**( i) Find the rate of change of T at P(4, -2, 1) in the direction of the vector** $< 2,6,-3 >$.
**(ii) In what direction does T increase most rapidly?**
**(iii) What is maximum rate of change?**

Solution. (a) $F(x,y,z) = x^2+y^2-z = 0$

$\nabla F = \langle 2x, 2y, -1 \rangle$

4 $N = \nabla F(1,-1,2) = \langle 2,-2,-1 \rangle$

Point $(1,-1,2)$

1 Equation of tangent plane

$$2(x-1) - 2(y+1) - (z-2) = 0$$

1 Equation of normal line

$$x = 1+2t, \quad y = -1-2t, \quad z = 2-t, \quad t \in \mathbb{R}.$$

(b) $T(x,y,z) = 4x^2 - y^2 + 16z^2$

$\nabla T = \langle 8x, -2y, 32z \rangle$

2 $\nabla T(4,-2,1) = \langle 32, 4, 32 \rangle$

$a = \langle 2, 6, -3 \rangle, \quad \|a\| = \sqrt{49} = 7$

2 $u = \dfrac{a}{\|a\|} = \dfrac{1}{7}\langle 2,6,-3 \rangle$

2 (i) Rate of change $D_u T(4,-2,1) = \langle 32,4,32 \rangle \cdot \frac{1}{7}\langle 2,6,-3 \rangle$

$= \frac{1}{7}(64+24-96) = -\frac{8}{7}$

1 (ii) T increases most rapidly in the direction of

$\nabla T|_P = 32i + 4j + 32k.$

1 (iii) Maximum rate of change is

$\|\nabla T(4,-2,1)\| = \sqrt{2064} \approx 45.43$

**Question:5.** (a) Find local extrema and saddle points, if any, of $f(x,y) = x^3 - y^2 - xy + 1$.
**[8+8]** (b) Use Lagrange multipliers to find greatest and shortest distances from the point $(2, 1, -2)$ to the sphere $x^2 + y^2 + z^2 = 1$.

Solution

(a) $f(x,y) = x^3 - y^2 - xy + 1$

2 $\quad f_x = 3x^2 - y \qquad f_{xx} = 6x$

$f_y = -2y - x \qquad f_{xy} = -1$

$f_{yy} = -2$

critical points

$f_x = 0, \quad f_y = 0 \qquad 3x^2 - y = 0$

$-2y - x = 0 \quad , x = -2y$

2 $\quad 12y^2 - y = 0 \qquad y(12y - 1) = 0 \Rightarrow y = 0, \quad y = \frac{1}{12}$

$x = 0, \quad x = -\frac{1}{6}$

Points are $(0,0)$, $\left(-\frac{1}{6}, \frac{1}{12}\right)$

$D(x,y) = -12x - 1$

1. $D(0,0) = -1 \Rightarrow$ saddle point $(0,0,1)$

4 $\quad$ 2. $D\left(-\frac{1}{6}, \frac{1}{12}\right) = +2 - 1 = 1 > 0 \Rightarrow$ Local extreme

$f_{xx}\left(-\frac{1}{6}, \frac{1}{12}\right) = -2 < 0 \Rightarrow$ Local Maximum $f\left(-\frac{1}{6}, \frac{1}{12}\right) = \frac{433}{432}$

(b) The distance from $(x,y,z)$ to $(2,1,-2)$ is

2 $\quad D = \sqrt{(x-2)^2 + (y-1)^2 + (z+2)^2}$

$f(x,y,z) = (x-2)^2 + (y-1)^2 + (z+2)^2$, $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

$\nabla f = \lambda \nabla g$

2 $\quad \langle 2(x-2), 2(y-1), 2(z+2) \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$2(x-2) = 2\lambda x, \quad 2(y-1) = 2\lambda y, \quad 2(z+2) = 2\lambda z$

$x(1-\lambda) = 2 \qquad x = \frac{2}{1-\lambda}, \quad y = \frac{1}{1-\lambda}, \quad z = -\frac{2}{1-\lambda}$

$y(1-\lambda) = 1$

4 $\quad z(1-\lambda) = -2 \qquad \frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 1, \quad (1-\lambda)^2 = 9$

$x^2 + y^2 + z^2 = 1 \qquad 1 - \lambda = \pm 3$

$\lambda = 4, \quad \lambda = -2$

$\lambda = 4, \quad x = -\frac{2}{3}, \quad y = -\frac{1}{3}, \quad z = \frac{2}{3} \qquad f\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = 16$

$\lambda = -2, \quad x = \frac{2}{3}, \quad y = \frac{1}{3}, \quad z = -\frac{2}{3} \qquad f\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = 4$

$D = \sqrt{4} = 2$ Shortest

$D = \sqrt{16} = 4$ greatest