King Saud University, Department of Mathematics
Math 204 (1.30H), 25/100, Mid Term Exam 1 S1. 44/45

Question 1. [5] Find and sketch the largest region in $\mathbb{R}^2$, for which the following initial value problem admits a unique solution

$$\begin{cases} (x^2 - x - 2)\frac{dy}{dx} = \sqrt{1 - \ln(2 - y)} \\ y(0) = 0. \end{cases}$$

Question 2. [4,4]. a) Write the following differential equation as a Bernoulli equation and then solve it

$$y\left(\frac{1}{x^2+1} - ye^x\right)dx = \left(\tan^{-1} x\right)dy.$$

b) Obtain the general solution of the following differential equation

$$(y - \sqrt{x^2 + y^2})dx - xdy = 0, \quad x > 0.$$

Question 3. [3,4] a) Solve the differential equtaion

$$\frac{2xy}{x^2+1} - 2x - [2 - \ln(x^2+1)]\frac{dy}{dx} = 0$$

b) Solve the initial value problem

$$\begin{cases} ydx - (y + 3x - 3)dy = 0 \\ y(1) = 1. \end{cases}$$

Question 4 [5] A small metal bar, whose initial temperature was 30^0C, is dropped into a large container of boiling water. How long will it take the bar to reach 70^0C if it is known that its temperature increases 3^0C in one second ?

Question1: Find the largest region of xy-plane on which the IVP

$$(x^2 - x - 2)\frac{dy}{dx} = \sqrt{1 - \ln(2-y)}, \quad y(0) = 0$$

has a unique solution.

Solution: We have

$$f(x,y) = \frac{\sqrt{1-\ln(2-y)}}{x^2 - x - 2} \text{ and } \frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{x^2 - x - 2}\frac{1}{(2-y)\sqrt{1-\ln(2-y)}},$$ 2

which are both continuous on the region

$$\mathbb{R} = \{(x,y) : -1 < x < 2,\ 2 - e < y < 2\}$$ 9 i

moreover the point $(0,0) \in \mathbb{R}$. Hence, $\mathbb{R}$ is the required region.

Q-4. Solve the differential equation

$$y\left(\frac{1}{1+x^2} - ye^x\right) dx = \left(\tan^{-1} x\right) dy.$$

Sol. The differential equation could be arranged as

$$\frac{dy}{dx} - \frac{1}{(1+x^2)\tan^{-1} x} y = -\frac{e^x}{\tan^{-1} x} y^2,$$ 1

which is a Bernoulli's equation that admits substitution $w = y^{-1}$ and transforms the differential equation to the following linear equation

$$\frac{dw}{dx} + \frac{1}{(1+x^2)\tan^{-1} x} w = \frac{e^x}{\tan^{-1} x}.$$ 1

The general solution of this linear differential equation is

$$w \tan^{-1'} x = e^x + c,$$ 1

which gives the general solution of the given differential equation

$$\frac{\tan^{-1} x}{y} = e^x + c.$$ 1

Solution

1) $$\frac{dy}{dx} = \frac{y - \sqrt{x^2+y^2}}{x} = \frac{y}{x} - \sqrt{1+\left(\frac{y}{x}\right)^2} \quad (1)$$ 1

$$\frac{y}{x} = u \Rightarrow y' = u + x u' \xrightarrow{\text{into } (1)}$$

$$u + x u' = u - \sqrt{1+u^2} \Rightarrow \frac{du}{\sqrt{1+u^2}} = -\frac{dx}{x} \Rightarrow$$ 1

$$\sinh^{-1} u = -\ln|x| + \ln|c| \quad \text{or}$$

$$\ln\left(u + \sqrt{1+u^2}\right) = \ln\left|\frac{c}{x}\right| \Rightarrow u + \sqrt{1+u^2} = \frac{c}{x}$$ 1

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = \frac{c_1}{x} \rightarrow y + \sqrt{x^2+y^2} = c_1$$ 1

Scanned with CamScanner

$$\frac{2xy}{x^2+1} - 2x - (2 - \ln(x^2+1))\, y' = 0$$

$$\left(\frac{2xy}{x^2+1} - 2x\right)dx - (2 - \ln(x^2+1))\,dy = 0$$

$$M(x,y) = \frac{2xy}{x^2+1} - 2x \Rightarrow M_y = \frac{2x}{x^2+1}$$

$$N(x,y) = -2 + \ln(x^2+1) \Rightarrow N_x = \frac{2x}{x^2+1}$$

$M_y = N_x \Rightarrow$ Exact ODE

then $\exists\, F(x,y)$ s.t

$$F(x,y) = \int \left(\frac{2xy}{x^2+1} - 2x\right) dx$$

$$= y\ln(x^2+1) - x^2 + h(y) \quad 1$$

$\Longrightarrow$

$$F_y(x,y) = \ln(x^2+1) + h'(y) = \ln(x^2+1) - 2$$

$$\Longrightarrow h'(y) = -2 \Longrightarrow h(y) = -2y + C$$

Hence,

1

$$F(x,y) = y\ln(x^2+1) - x^2 - 2y + C = 0$$

ROCO

Q2) Solve $\quad y\,dx + (3-3x-y)\,dy = 0 \quad (*)$

<u>Ans</u>: $\quad M(x,y) = y \;,\; N(x,y) = 3-3x-y$

$$\frac{\partial M}{\partial y}(x,y) = 1 \;;\; \frac{\partial N}{\partial x}(x,y) = -3 \qquad (1)$$

As $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then $(*)$ is non exact

Integrating factor $\quad g(y) = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -4/y$

$$\mu(y) = e^{\int g(y)\,dy} = y^{-4} \qquad (1)$$

Multiplying $(*)$ by μ:

$$y^{-3}\,dx + (3y^{-4} - 3xy^{-4} - y^{-3})\,dy = 0$$

It is exact then there exists F such

$$\begin{cases} \frac{\partial F}{\partial x}(x,y) = y^{-3} & (1) \\ \frac{\partial F}{\partial y}(x,y) = 3y^{-4} - 3xy^{-4} - y^{-3} & (2) \end{cases}$$

Integrating (1)

$$F(x,y) = \int y^{-3}\,dx = \frac{x}{y^3} + \varphi(y) \qquad (1)$$

Then $\frac{\partial F}{\partial y}(x,y) = -\frac{3x}{y^4} + \varphi'(y) = 3y^{-4} - 3xy^{-4} - y^{-3}$

$$\varphi'(y) = 3y^{-4} - y^{-3} \text{ so } \varphi(y) = -y^{-3} + \frac{1}{2}y^{-2}$$

The General Solution is $F(x,y) = C \iff$

$$\boxed{x - 1 + y/2 = C y^3} \text{ with } C \in \mathbb{R} \qquad (1)$$

$y(1) = 1 \Rightarrow \frac{1}{2} = C \Rightarrow x - 1 + \frac{y}{2} = y^3/2$

Scanned with CamScanner

or $t = \frac{\ldots}{\ln(\frac{19}{20})}$.

Exercise 3 A small metal bar, whose initial temperature was 30° C, is dropped into a large container of boilin water. How long will it take the bar to reach 70° C if it is known that its temperature increases 3° in one second ?

Solution : We have

$$T(t) = T_s + ce^{kt}.$$ 1

at $t = 0, T = 30,\ T_s = 100$ and $T = 33$ at $t = 1$. Then $c = -70$, and

$$33 = 100 - 70e^k \Longrightarrow k = \ln(\frac{67}{70}).$$ 2

So, $T(t) = 100 - 70e^{\ln(\frac{67}{70})t}$. Therefore, for $T = 70$, we have 2

$$t = \frac{\ln(\frac{3}{7})}{\ln(\frac{67}{70})}.$$