Discrete time models.

consider a call option its payoff is

$$F = (S_T - K)^+$$

Example: Shehana (seller) $+6$ Lamia (buyer) -6

$$S_T = 26.40$$

$$\begin{cases} S_0 = 26.40 \\ K = 26.42 \end{cases} \qquad T = 1$$

Shehana's portfolio: $6 = \alpha_0 S_0^0 + \Delta_0 S_0 \to (\alpha_0, \Delta_0)$

$(\alpha_0, \Delta_0) \rightsquigarrow$ at maturity: Finance the payoff $(S_T - K)^+ = \max(S_T - K, 0)$

$V_0 = \alpha_0 S_0^0 + \Delta_0 S_0$: the value of the portfolio at time 0.

$V_1 = \alpha_0 S_1^0 + \Delta_0 S_1$, you wish to have $V_1 \geq (S_1 - K)^+$

If $V_1 = \boxed{(S_1 - K)^+ = \alpha_0 S_1^0 + \Delta_0 S_1}$

We need a model:

$$\begin{cases} 0.01 = \alpha_0 \times 1.01 + \Delta_0 26.43 \\ 0 = \alpha_0 \times 1.01 + \Delta_0 26.38 \end{cases}$$

risky asset

$$\begin{array}{ccc} & & 26.43 \\ 26.40 & \nearrow & \\ & \searrow & \\ & & 26.38 \\ S_0 & & S_1 \end{array}$$

riskless asset $\quad S_0^0 = 1 \quad \to \quad S_1^0 = 1.01$

$$\Delta_0 = \frac{0.01}{0.05} = \frac{1}{5} \quad \text{and} \quad \alpha_0 = -\frac{1}{5} \times 26.38 \times \frac{1}{1.01}$$

$$\alpha_0 = -5.22 \quad (\alpha_0, \Delta_0) = \left(-5.22, \frac{1}{5}\right)$$

$$\alpha_0 = -\frac{2638}{5 \times 101} = -\frac{2638}{505}$$

$$\Phi_0 = \left(-\frac{2638}{505}, \frac{1}{5}\right) \longrightarrow V_0 = \alpha_0 S_0^0 + \Delta_0 S_0 = \underline{0.056}$$

0.056 is the necessary amount needed to finance:

$$(S_1 - 26.42)^+.$$

$$\cancel{16 - 0.056}$$

$$V_0 = \text{premium} = \frac{-2638}{505} + \frac{2640}{50} = \frac{142}{2525}$$

$\frac{1}{5}$ shars of the stock costs $\frac{1}{5} \times 26.4 = \frac{264}{50}$

~~The amount 0.056 = $\frac{264}{50}$~~ ~~This amount of money~~

$$\frac{142}{2525} - \frac{264}{50} = -\frac{2638}{505}$$

Shehana will borrow $\frac{2638}{505}$ from the bank at the rate 1% simple compounding.

If it's continuously compounding $e^r = 1.01$

$$\Rightarrow r = \ln(1.01) = 0.995\% \; \boxtimes$$

Shehana + Lamia

over the counter market

OTC

Derivatives markets

$V_0 =$ premium

invested in the real market on stocks and bonds or saving account

$(S_T - K)^+$

$V_0 =$ premium

Market real

(α_0, Δ_0)

$V_T \geq (S_T - K)^+$

We should have for optimality reasons

$$\boxed{V_T = (S_T - K)^+}$$

Assume that we have only two assets.

$$V_0 = \alpha_0 S_0^0 + \Delta_0 S_0$$

$(S_n)_{n \geq 0}$ price of the stock.

(S_n^0) is the price of the riskless asset.

$T = 1$

$\Phi_0 = (\alpha_0, \Delta_0)$ is called the portfolio.

$$V_1 = \alpha_0 S_1^0 + \Delta_0 S_1$$

To hedge the short position of the call. we should have: $V_1 = (S_1 - K)^+$.

$$\boxed{\alpha_0 S_1^0 + \Delta_0 S_1 = (S_1 - K)^+} \quad (*)$$

The replication equation

We consider a simple binomial model:

<u>Risky asset:</u>

$$S_0 \nearrow S_0 u = S_1^u \qquad \boxed{d < u}$$
$$S_0 \searrow S_0 d = S_1^d$$

$$\boxed{S_0 d < K < S_0 u}$$

<u>Riskless asset</u>.

$$\underset{S_0^0}{1} \longrightarrow e^r = S_1^0$$

$$(*) \Leftrightarrow \begin{cases} \alpha_0 e^r + \Delta_0 S_0 u = (S_0 u - K)^+ = S_0 u - K \\ \alpha_0 e^r + \Delta_0 S_0 d = (S_0 d - K)^+ = 0 \end{cases}$$

$$\Delta_0 = \frac{S_0u - K}{S_0u - S_0d} > 0 \quad \text{call.}$$

- The seller of the option has to buy $\frac{S_0u-K}{S_0u-S_0d}$ shares of the stock: the underlying asset.

- $\alpha_0 = -\Delta_0 S_0 d / e^r < 0$.

The investor has to borrow $\frac{\Delta_0 S_0 d}{e^r}$.

$$(\alpha_0, \Delta_0) = \left(-\frac{\Delta_0 S_0 d}{e^r}, \frac{S_0u-K}{S_0(u-d)}\right).$$

- $$V_0 = \frac{-\Delta_0 S_0 d}{e^r} + \Delta_0 S_0 = \Delta_0 S_0 \left(\frac{e^r - d}{e^r}\right)$$

$$= \frac{e^r - d}{e^r} \frac{(S_0u - K)}{u-d}.$$

$$= \frac{e^r - d}{u-d} \cdot \left[\frac{S_0u - K}{e^r}\right] > 0$$

$$\Rightarrow e^r > d$$

The seller will buy Δ_0 shares of the stock; and borrow $V_0 - \Delta_0 S_0$ from the bank.

Exercise: Do the same for the put option.

For the put option $\quad (K - S_1)^+ = \begin{cases} (K - S_0 u)^+ \\ (K - S_0 d)^+ \end{cases}$

Risky: $\qquad\qquad\qquad$ Riskless.

$$S_0 \;\begin{matrix} \nearrow S_0 u \\ \searrow S_0 d \end{matrix} \qquad\qquad 1 \longrightarrow e^r$$

$t=0 \qquad t=1 \qquad\qquad t=0 \qquad t=1$

$$\text{Portfolio } (\alpha_0, \Delta_0) = \left(-\frac{\Delta_0 S_0 u}{e^r} \; ; \; \frac{K - S_0 d}{S_0 u - S_0 d} \right)$$

$$P_0 = V_0 = \alpha_0 + \Delta_0 S_0 = \frac{K - S_0 d}{e^r} \cdot \frac{u - e^r}{u - d} > 0.$$

consequently $u > e^r$. moreover we found for the call that $e^r > d$.

$$C/C: \quad \boxed{d < e^r < u} \iff \ln(d) < r < \ln(u).$$

If this is not the case there will be arbitrage opportunities. We shall come back to this notion later in the course.

<u>Remark:</u> for the put option we have $\Delta_0 < 0$.

This means that the seller of the option will short the stock. So the investor will <u>sell</u> $(-\Delta_0)$ shares: or sell $\dfrac{K - S_0 d}{S_0 u - S_0 d}$ shares of the underlying at S_0.

He will get then: $\dfrac{K - S_0 d}{u - d}$ "units of money".

She will invest the riskless asset the amount $P_0 + \dfrac{K - S_0 d}{u - d} =$

27 $\quad K - S_0 d \quad u - e^r \quad K - S_0 d \quad = \quad K - S_0 d \; \left(\frac{u - e^r}{\;} + 1 \right) \quad K - S d$ ...

the investor will then lend $\frac{K - S_0 d}{u - d} \cdot \frac{u}{e^r}$ to the bank at the rate: r.

Let us go to maturity and see What will happen.

at $t = 1$: The investor will get $\left(\frac{K - S_0 d}{u - d} \cdot u\right)$ from the bank. And will buy $\frac{K - S_0 d}{S_0 u - S_0 d}$ shares of the asset at S_1

This will costs $\frac{K - S_0 d}{S_0 u - S_0 d} \times \begin{cases} S_0 u \\ S_0 d \end{cases} = \frac{K - S_0 d}{u - d} \cdot u$ or $\frac{K - S_0 d}{u - d} \cdot d$

Two cases to discuss:

(1) $S_1 = S_0 u$. The option is not exercised:

then she will buy $\frac{K - S_0 d}{S_0 u - S_0 d}$ shares at $S_0 u$ and give it back to the lender.

(2) $S_1 = S_0 d$. the option is exercised.

She will pay $(K - S_0 d)$ to the buyer and by $\frac{K - S_0 d}{S_0 u - S_0 d}$ shares at $S_0 d$ this will cost for the seller $K - S_0 d + \frac{K - S_0 d}{u - d} \cdot d$

$$= (K - S_0 d)\left(1 + \frac{d}{u - d}\right) = \frac{K - S_0 d}{u - d} \cdot u \; \boxed{}$$

Consider a financial instrument (a derivative) F with the following possible values: $f_1 = F^u$ and $f_2 = F^d$:

risky

$$S_0 \begin{array}{l} \nearrow S_0 u = S_1^u \\ \searrow S_0 d = S_1^d \end{array}$$

$t=0 \qquad t=1$

riskless

$$1 \longrightarrow e^r = S_1^0$$

$t=0 \qquad t=1$

$$f_0 = ? \quad \begin{array}{l} \nearrow f_1 = F^u \\ \searrow f_2 = F^d \end{array}$$

If: $F = (S_1 - K)^+$

then $f_1 = S_0 u - K$.

$f_2 = 0$.

<u>Question</u>: Find the price at time 0 of F and

Find the replicating portfolio.

The seller of the financial instrument F will get a premium f_0 which will be invested in the market to generate at time 1 the payoff F.

in such a way:

$f_0 = \alpha_0 + \Delta_0 S_0$, $\phi = (\alpha_0, \Delta_0)$

$$V_1^{\Phi} = \boxed{F = \alpha_0 S_1^0 + \Delta_0 S_1}$$

$$\begin{cases} \alpha_0 e^r + \Delta_0 S_0 u = F^u \\ \alpha_0 e^r + \Delta_0 S_0 d = F^d \end{cases} \qquad \Delta_0 = \frac{F^u - F^d}{S_0 u - S_0 d}.$$

and $\alpha_0 = \dfrac{F^u - \Delta_0 S_0 u}{e^r} = \dfrac{F^d - \Delta_0 S_0 d}{e^r}$.

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Then the ~~prolem~~ premium F_0 is equal:

$$F_0 = \frac{F^u - \Delta_0 S_0 u}{e^r} + \Delta_0 S_0$$

$$= \frac{F^u}{e^r} + \frac{\Delta_0 S_0 (e^r - u)}{e^r}$$

$$= \frac{F^u}{e^r} + \frac{F^u - F^d}{u-d} \cdot \left(1 - \frac{u}{e^r}\right) \overset{?}{=} q \frac{F^u}{e^r} + (1-q)\frac{F^d}{e^r}$$

$$= \frac{F^u}{e^r} + \frac{e^r - u}{e^r} \cdot \frac{F^u}{u-d} - \frac{e^r - u}{u-d} \cdot \frac{F^d}{e^r}$$

$$= \frac{F^u}{e^r}\left(1 + \frac{e^r - u}{u-d}\right) + \frac{F^d}{e^r} \cdot \frac{u - e^r}{u-d}$$

$$= \frac{e^r - d}{u-d} \cdot \frac{F^u}{e^r} + \frac{u - e^r}{u-d} \cdot \frac{F^d}{e^r}$$

In order to avoid arbitrage opportunities we should: $\boxed{d < e^r < u}$. This implies that

$$0 < q = \frac{e^r - d}{u-d} < 1 \text{ and } 0 < \frac{u - e^r}{u-d} = 1 - q < 1$$

If we set $Q = (q, 1-q)$:

F_0 can be written as the expectation under Q of the discounted payoff $\frac{F}{e^r}$.

That: $F_0 = E_Q\left[\frac{F}{e^r}\right]$ with:

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$$F = \begin{cases} F^u \to \text{with proba } q := \dfrac{e^r - d}{u-d} \\ F^d \to \text{'' '' } 1-q = \dfrac{u-e^r}{u-d} \end{cases}$$

$$E_Q\left[\frac{F}{e^r}\right] = \frac{F^u}{e^r}\left(\frac{e^r-d}{u-d}\right) + \frac{F^d}{e^r}\left(\frac{u-e^r}{u-d}\right) = F_0.$$

<u>Comments:</u> The portfolio (α_0, Δ_0) is called the replicating (or hedging) portfolio.

<u>Example:</u> Find a corresponding example.

S_0, u, d, r, F^u, F^d.

Then find the premium and the replicating portfolio.

$$100 \begin{cases} \nearrow 110 \to u \\ \searrow 90 \to d \end{cases} \qquad 1 \longrightarrow 102 \to r$$

$$F = \begin{cases} 3, \\ 1 \end{cases} \Bigg| \; F \begin{cases} 2 \\ 5 \end{cases} \Bigg| \; F \begin{cases} 1/2 \\ 4 \end{cases}$$

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