

Discrete time models.

consider a call option its payoff is

$$F = (S_T - K)^+$$

Example: shehana (seller)

Lamia (buyer)

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$\bullet \begin{cases} S_0 = 26.40 \\ K = 26.42 \end{cases}$
 $S_T = 26.40$
 $T = 1$

Shehana's portfolio: $6 = \alpha_0 S_0 + \Delta_0 S_0 \rightarrow (\alpha_0, \Delta_0)$

$(\alpha_0, \Delta_0) \rightsquigarrow$ at maturity: Finance the payoff $(S_T - K)^+ = \max(S_T - K, 0)$

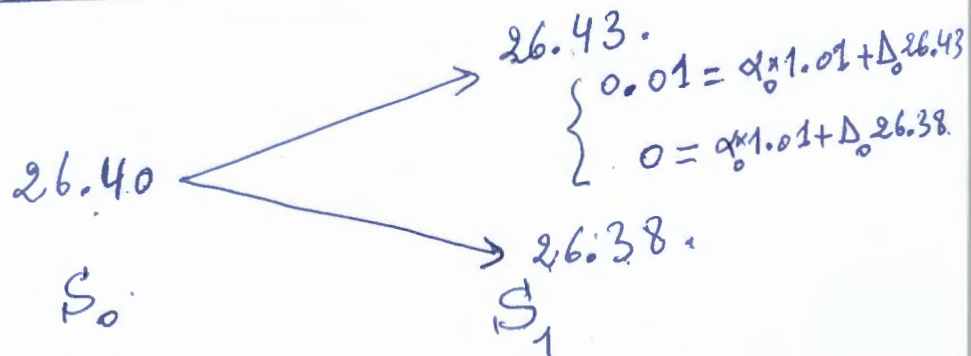
$V_0 = \alpha_0 S_0 + \Delta_0 S_0$: the value of the portfolio at time 0.

$\bullet V_1 = \alpha_0 S_1 + \Delta_0 S_1$, you wish to have $V_1 \geq (S_1 - K)^+$

If $V_1 = (S_1 - K)^+ = \alpha_0 S_1 + \Delta_0 S_1$

We need a model:

risky asset



risky asset

$S_0 = 1$

$S_0 = 1.01$

$$\Delta_0 = \frac{0.01}{0.05} = \frac{1}{5} \quad \text{and} \quad \alpha_0 = -\frac{1}{5} \times 26.38 \times \frac{1}{1.01}$$

$$\alpha_0 = -5.22 \quad (\alpha_0, \Delta_0) = \left(-5.22, \frac{1}{5}\right)$$

$$\alpha_0 = -\frac{2638}{5 \times 101} = -\frac{2638}{505}$$

$$\Phi_0 = \left(-\frac{2638}{505}, \frac{1}{5}\right) \rightarrow V_0 = \alpha_0 S_0 + \Delta_0 S_0 = \underline{0.056}$$

0.056 is the necessary amount needed to finance:

$$(S_1 - 26.42)^+$$

$$V_0 = \text{premium} = \frac{2638}{505} + \frac{2640}{50} = \frac{142}{2525}$$

~~$$V_0 = 0.056$$~~

$$\frac{1}{5} \text{ shares of the stock costs } \frac{1}{5} \times 26.4 = \left(\frac{264}{50}\right)$$

~~the amount 0.056~~ ~~$\frac{264}{50}$~~ This amount of money

$$\frac{142}{2525} - \frac{264}{50} = -\frac{2638}{505}$$

Shehane will borrow $\frac{2638}{505}$ from the bank at the rate (10%) simple compounding.

If it's continuously compounding $e^r = 1.01$.

$$\Rightarrow r = \ln(1.01) = 0.995\% \quad \square$$

Shehana
+ Lamia

Derivatives
markets

over the counter market.

OTC

$$V_0 = \text{premium}$$

is invested
in the real
market
on stocks and bonds
or saving account

$$(S_T - K)^+$$

$$V_0 = \text{premium}$$

Market
(α_0, D_0) | real

$$V_T \geq (S_T - K)^+$$

We should have
for optimality reasons.

$$V_T = (S_T - K)^+$$

Assume that we have only two assets.

$$V_0 = \alpha_0 S_0^r + \Delta_0 S_0$$

(S_n) price of the stock
 $n \geq 0$ stock.

(S_n^r) is the price of the
riskless asset.

T=1

$\Phi_0 = (\alpha_0, D_0)$ is called the portfolio.

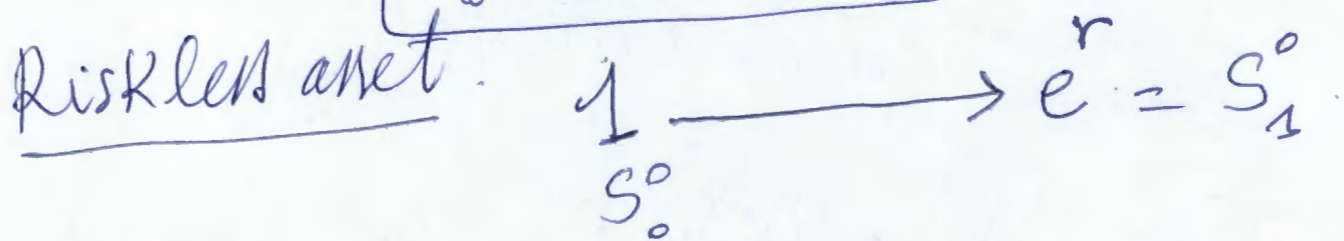
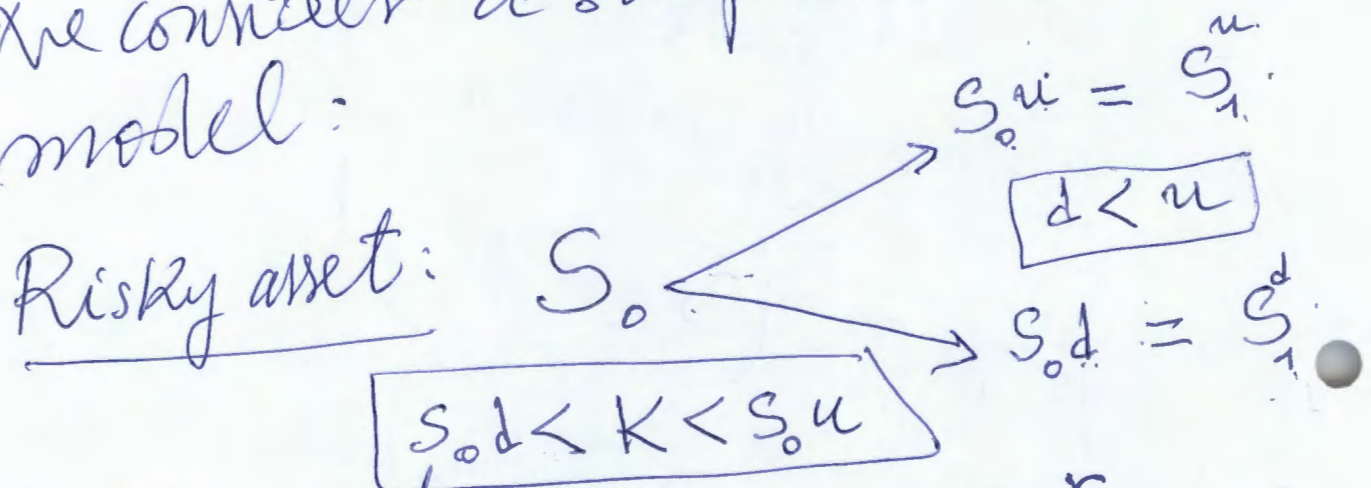
$$V_1 = \alpha_0 S_1^0 + \Delta_0 S_1$$

To hedge the short position of the call, we should have: $V_1 = (S_1 - K)^+$.

$$\alpha_0 S_1^0 + \Delta_0 S_1 = (S_1 - K)^+ \quad (*)$$

The replication equation

We consider a simple binomial model:



$$(*) \Leftrightarrow \begin{cases} \alpha_0 e^r + \Delta_0 S_0^u = (S_0^u - K)^+ = S_0^u - K \\ \alpha_0 e^r + \Delta_0 S_0^d = (S_0^d - K)^+ = 0 \end{cases}$$

$$\Delta_0 = \frac{S_0 u - K}{S_0 u - S_0 d} > 0 \quad \text{call.}$$

- The seller of the option has to buy $\frac{S_0 u - K}{S_0 u - S_0 d}$ shares of the stock, the underlying asset.

$$\alpha_0 = -\Delta_0 S_0 d / e^r < 0$$

The investor has to borrow $\frac{\Delta_0 S_0 d}{e^r}$.

$$(\alpha_0, \Delta_0) = \left(\frac{-\Delta_0 S_0 d}{e^r}, \frac{S_0 u - K}{S_0 (u - d)} \right)$$

$$V_0 = \frac{-\Delta_0 S_0 d}{e^r} + \Delta_0 S_0 = \Delta_0 S_0 \left(\frac{e^r - d}{e^r} \right)$$

$$= \frac{e^r - d}{e^r} \frac{(S_0 u - K)}{u - d}$$

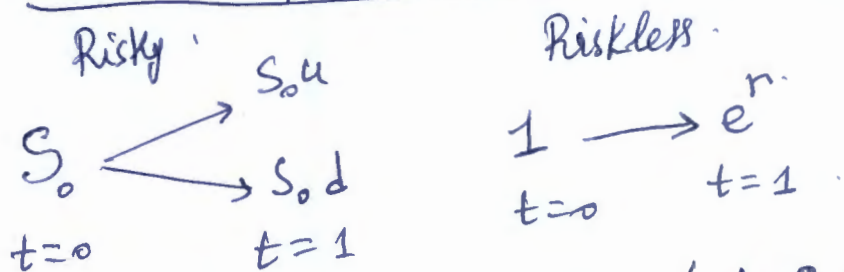
$$= \left(\frac{e^r - d}{u - d} \right) \cdot \left[\frac{S_0 u - K}{e^r} \right] > 0$$

$\Rightarrow e^r > d$

The seller will buy Δ_0 shares of the stock:
and borrow $V_0 - \Delta_0 S_0$ from the bank.

Exercise: Do the same for the put
option.

For the put option $(K - S_1)^+ = \begin{cases} (K - S_0 u)^+ \\ (K - S_0 d)^+ \end{cases}$



Portfolio $(\alpha_0, \Delta_0) = \left(\frac{-\Delta_0 S_0 u}{e^r} ; \frac{K - S_0 d}{S_0 u - S_0 d} \right)$

$P_0 = V_0 = \alpha_0 + \Delta_0 S_0 = \frac{K - S_0 d}{e^r} \cdot \frac{u - e^r}{u - d} > 0$

consequently $u > e^r$ & moreover we found for the call that $e^r > d$.

c/c: $\boxed{d < e^r < u} \Leftrightarrow \ln(d) < r < \ln(u)$

If this is not the case - there will be arbitrage opportunities. We shall come back to this notion later in the course.

Remark: for the put option we have $\Delta_0 < 0$.

This means that the seller of the option will short the stock. So the investor will sell $(-\Delta_0)$ shares:

or sell $\frac{K - S_0 d}{S_0 u - S_0 d}$ shares of the underlying at S_0 .

He will get then: $\frac{K - S_0 d}{u - d}$ "units of money".

She will invest the riskless asset the amount $P_0 + \frac{K - S_0 d}{u - d} =$

$\frac{K - S_0 d}{u - e^r} + \frac{K - S_0 d}{u - d} = \frac{K - S_0 d}{u - d} (1 + \frac{u - e^r}{u - d}) = \frac{K - S_0 d}{u - d} \cdot \frac{u - e^r + u - d}{u - d}$

the investor will then lend $\frac{K - S_0 d}{u - d} \cdot \frac{u}{e^r}$ to the bank at the rate: r .

Let us go to maturity and see what will happen.

at $t=1$: The investor will get $\frac{K - S_0 d}{u - d} \cdot u$ from the bank. And will buy $\frac{K - S_0 d}{S_1 u - S_0 d}$ shares of the asset at S_1

This will cost $\frac{K - S_0 d}{S_1 u - S_0 d} \times \begin{cases} S_1 u \\ S_0 d \end{cases} = \frac{K - S_0 d}{u - d} \cdot u$ or $\frac{K - S_0 d}{u - d} \cdot d$

Two cases to discuss:

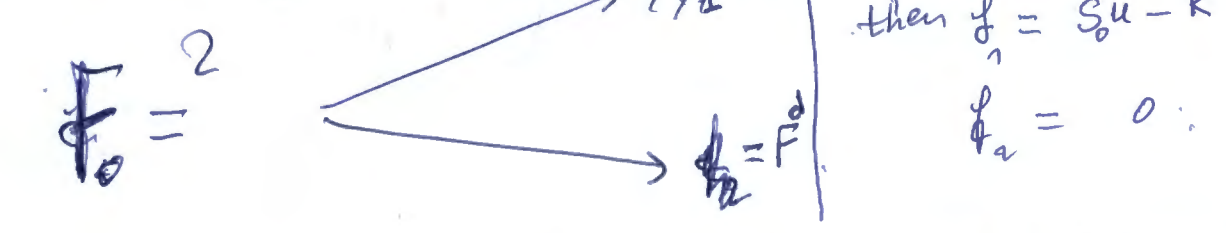
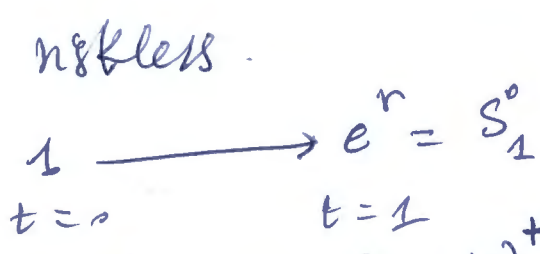
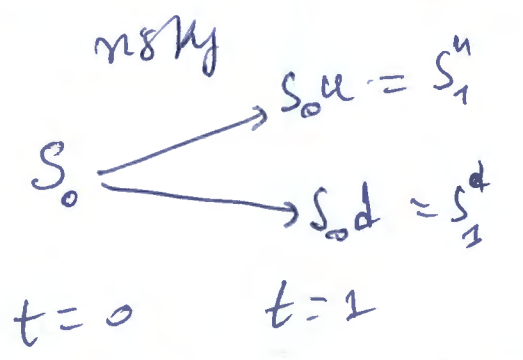
① $S_1 = S_0 u$. The option is not exercised: then she will buy $\frac{K - S_0 d}{S_1 u - S_0 d}$ shares at $S_0 u$ and give it back to the lender.

② $S_1 = S_0 d$. the option is exercised. she will pay $(K - S_0 d)$ to the buyer and buy $\frac{K - S_0 d}{S_1 u - S_0 d}$ shares at $S_0 d$. this will cost for

the seller $K - S_0 d + \frac{K - S_0 d}{u - d} \cdot d$.

$$= (K - S_0 d) \left(1 + \frac{d}{u - d} \right) = \frac{K - S_0 d}{u - d} \cdot u$$

Consider a financial instrument (a derivative) F with the following possible values: F^u and F^d .



Question: Find the price at time 0 of F and Find the replicating portfolio.

The seller of the financial instrument F will get a premium F_0 which will be invested in the market to generate at time 1 the payoff F .

$F_0 = \alpha_0 + \Delta_0 S_0$, $\phi = (\alpha_0, \Delta_0)$ in such a way:

$$V_1^\phi = F = \alpha_0 S_1^0 + \Delta_0 S_1$$

$$\begin{cases} \alpha_0 e^r + \Delta_0 S_0^u = F^u \\ \alpha_0 e^r + \Delta_0 S_0^d = F^d \end{cases} \quad \Delta_0 = \frac{F^u - F^d}{S_0^u - S_0^d}$$

and $\alpha_0 = \frac{F^u - \Delta_0 S_0^u}{e^r} = \frac{F^d - \Delta_0 S_0^d}{e^r}$

Then the ~~premium~~ premium F_0 is equal?

$$F_0 = \frac{F^u - \Delta_0 S_0 u}{e^r} + \Delta_0 S_0$$

$$= \frac{F^u}{e^r} + \frac{\Delta_0 S_0 (e^r - u)}{e^r}$$

$$= \frac{F^u}{e^r} + \frac{F^u - F^d}{u - d} \cdot \left(1 - \frac{u}{e^r}\right) \stackrel{?}{=} q \frac{F^u}{e^r} + (1-q) \frac{F^d}{e^r}$$

$$= \frac{F^u}{e^r} + \frac{e^r - u}{e^r} \cdot \frac{F^u}{u - d} - \frac{e^r - u}{u - d} \cdot \frac{F^d}{e^r}$$

$$= \frac{F^u}{e^r} \left(1 + \frac{e^r - u}{u - d}\right) + \frac{F^d}{e^r} \cdot \frac{u - e^r}{u - d}$$

$$= \frac{e^r - d}{u - d} \cdot \frac{F^u}{e^r} + \frac{u - e^r}{u - d} \cdot \frac{F^d}{e^r}$$

In order to avoid arbitrage opportunities we should: $d < e^r < u$. This implies that

$$0 < q = \frac{e^r - d}{u - d} < 1 \text{ and } 0 < \frac{u - e^r}{u - d} = 1 - q < 1$$

If we set $Q = (q, 1-q)$:

F_0 can be written as the expectation under Q of the discounted payoff $\frac{F}{e^r}$.

That: $F_0 = E_Q \left[\frac{F}{e^r} \right]$ with:

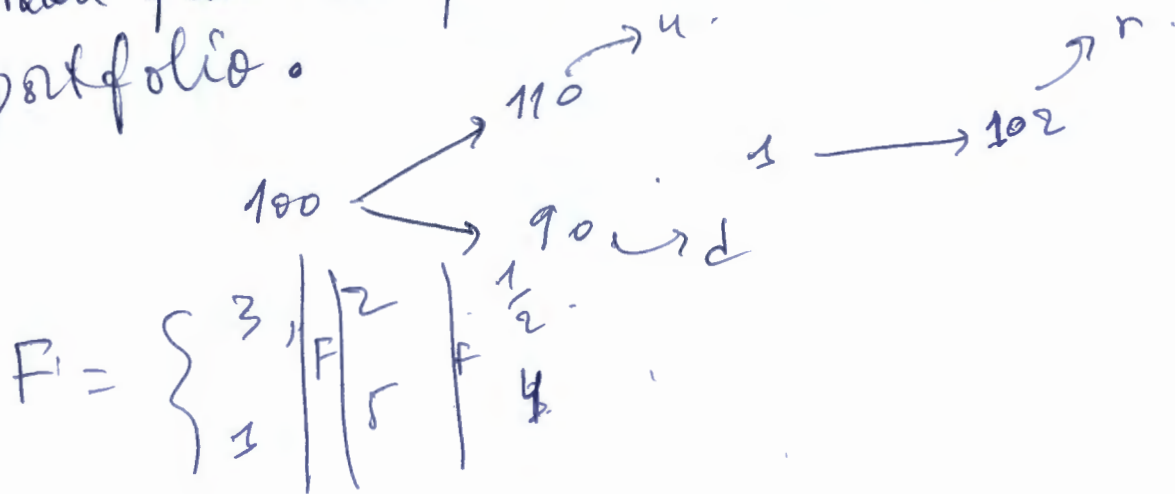
$$F = \left\{ \begin{array}{l} F^u \rightarrow \text{with proba } q = \frac{e^r - d}{u - d} \\ F^d \rightarrow \dots 1 - q = \frac{u - e^r}{u - d} \end{array} \right\}$$

$$E_Q \left[\frac{F}{e^r} \right] = \frac{F^u}{e^r} \left(\frac{e^r - d}{u - d} \right) + \frac{F^d}{e^r} \left(\frac{u - e^r}{u - d} \right) = F_0$$

Comments: The portfolio (d_0, Δ_0) is called the replicating (or hedging) portfolio.

Example: Find a corresponding example.

S_0, u, d, r, F^u, F^d
Then find the premium and the replicating portfolio.



$$F = \left\{ \begin{array}{c|c|c} 3 & F & 2 \\ \hline 1 & F & 5 \\ \hline & & \frac{1}{2} \\ & & 4 \end{array} \right.$$

$$E\left[\frac{D}{C}\right] = \frac{r}{C} \left(\frac{S-d}{r} + \frac{1}{r} \left(\frac{d}{C} \right) \right)$$

Interpret the the yield rate (the Δ for r) as the
 the opportunity (or holding) yield rate

Interpret Δ as a measure of the opportunity cost
 of holding the bond. The higher the yield rate, the
 higher the opportunity cost.

