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# A new approach for parameter estimation of finite Weibull mixture distributions for reliability modeling

Emad E. Elmahdy [a], Abdallah W. Aboutahoun [b,*]

[a] Department of Mathematics, Teachers College, King Saud University, Riyadh 11491, P.O. 4341, Saudi Arabia
[b] Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt

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ABSTRACT

The aim of this paper is to model lifetime data for systems that have failure modes by using the finite mixture of Weibull distributions. It involves estimating of the unknown parameters which is an important task in statistics, especially in life testing and reliability analysis. The proposed approach depends on different methods that will be used to develop the estimates such as MLE through the EM algorithm. In addition, Bayesian estimations will be investigated and some other extensions such as Graphic, Non-Linear Median Rank Regression and Monte Carlo simulation methods can be used to model the system under consideration. A numerical application will be used through the proposed approach. This paper also presents a comparison of the fitted probability density functions, reliability functions and hazard functions of the 3-parameter Weibull and Weibull mixture distributions using the proposed approach and other conventional methods which characterize the distribution of failure times for the system components. GOF is used to determine the best distribution for modeling lifetime data, the priority will be for the proposed approach which has more accurate parameter estimates.



## 1. Introduction

Mixture distributions arise in practical problems when the measurements of a random variable are taken under two or more different conditions. A mixture model may in fact be physically meaningful, for example, if some items have defects that cause them to fail early, whereas items without defects are susceptible to a more gradual wear out [1]. A mixture models can be used in problems of this type, where the population of sampling units or components of a system consists of a number of subpopulations within each of which a relatively simple model applies [2,3].

Recently the mixed Weibull distribution has been recognized as a suitable model for the lives of electrical and mechanical components (or systems) when the failure of the components (or systems) is caused by more than one failure mode or multi-mode failures in reliability analysis. Due to the lack of an efficient parameter estimation method, the mixed-Weibull has not been used as widely by reliability practitioners as the single-population Weibull distribution [4,5]. In statistics, a mixture model is a probabilistic model for representing the presence of sub-populations within an overall population, without requiring that an observed data-set should identify the sub-population to which an individual observation belongs. Formally a mixture model corresponds to the mixture distribution that represents the probability distribution of observations in the overall population. However, while problems associated with "mixture distributions" relate to deriving the properties of the overall population from those of the sub-populations, "mixture models" are used to make statistical inferences about the properties of the sub-populations given only observations on the pooled population, without sub-population-identity information [6,7].

* Corresponding author.
*E-mail address:* tahoun44@yahoo.com (A.W. Aboutahoun).

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Mixture models play a vital role in many practical applications. For example, direct applications of finite mixture models are in fisheries research, economics, medicine, psychology, palaeoanthropology, botany, agriculture, zoology, life testing and reliability, among others. Indirect applications include outliers, Gaussian sums, cluster analysis, latent structure models, modeling prior densities, empirical Bayes method and nonparametric (kernel) density estimation. In many applications, the available data can be considered as data coming from a mixture population of two or more distributions. This idea enables us to mix statistical distributions to get a new distribution carrying the properties of its components [8].

Different methods are used to determine the parameters and the mixing parameter of the mixture Weibull distribution. A graphical approaches introduced by Jiang et al. [9] have been used extensively to decide on the appropriateness of a mixture of 2 Weibull distribution to model a given failure data set [10,9]. It involves plotting the data on Weibull plotting paper (WPP). The method of moments and generalizations has been applied by Rider to the decomposition of the mixture of two exponential distributions [5]. MLE (maximum likelihood estimate), EM (expectation and maximization) algorithm are developed in determining whether mixed Weibull distribution is suited to modeling a given failure data set [11,12]. The EM algorithm is introduced by Dempster et al. [13] in their fundamental paper, which is often referred to as DLR paper is a powerful algorithm for ML estimation for data containing missing values or being considered as containing missing values [13]. This formulation is particularly suitable for distributions arising as mixtures since the mixing operation can be considered as producing missing data. An important feature of EM algorithm is that it is not merely a numerical technique but it also offers useful statistical insight. EM is a numerical technique which finds the posterior mode [14,15,22]. It is an iterative method starting from some plausible guess for a certain parameter.

The Monte Carlo EM (MCEM) algorithm introduced by Wei and Tanner (1990) is a modification of the EM algorithm where the expectation in the E-step is computed numerically through Monte Carlo simulations [16]. The MCEM algorithm relates to EM as a forerunner by its data augmentation step that replaces maximization by simulation.

In this paper we will describe an EM algorithm for maximum likelihood estimation of finite Weibull mixture distributions. One of the difficulties of general EM is in Finding the expected log-likelihood when the augmented data enter the complete likelihood in a non- linear fashion.

The main achievement is to introduce a proposed approach that it reduces the problem of estimation to one of estimation of the mixing distribution, variants of the algorithm work even when the probability function of the mixed distribution is not known explicitly but we have only an approximation of it. We make an algorithm for the proposed approach which can be applied to the complete and censored samples, it also can find proper estimates for any data sample. We apply the proposed approach for an engineering application for modeling the times to failure of system components by the mixed Weibull distribution.

The outline of the paper is as follows. In Section 2 we describe statistical methods for modeling failure data by 3-parameter Weibull and mixed Weibull distributions and we present a formulation of an algorithm for the proposed approach. Goodness of fit tests which is used to determine the best distribution for modeling lifetime data is presented in Section 3. In Section 4 we present an application in which we model lifetime data by using different methods, we also find the fitted probability density functions, reliability functions and hazard functions of the Weibull mixture distribution using the proposed approach and other conventional methods. Section 5 summarizes the conclusions of this paper and discusses future extensions of this research.

## 2. Statistical methods for modeling failure data by 3-parameter Weibull and mixed Weibull distributions

### 2.1. Weibull model

The probability density function of Weibull distribution is defined mathematically as

$$f(t|\beta,\alpha,\gamma)=\frac{\beta}{\alpha}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1}e^{-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}},\quad t>0, \tag{1}$$

where $\beta>0,\ \alpha>0$ and $-\infty<\gamma<\infty$ are the shape, scale and location parameters of the distribution. The shape parameter is responsible for the skew of the distribution, the scale parameter is sometimes referred to as the characteristic life and the location parameter is used to shift the distribution in one direction or another to define the location of its origin and can be either positive or negative, it is sometimes called minimum life. The corresponding Reliability and hazard functions are defined respectively as

$$R(t|\beta,\alpha,\gamma)=e^{-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}}, \tag{2}$$

$$h(t|\beta,\alpha,\gamma)=\frac{\beta}{\alpha}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1}. \tag{3}$$

If the location parameter $\gamma$ is equal to zero, the three parameter model becomes two parameter model (simple Weibull model), then we have

$$f(t|\beta,\alpha)=\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}\exp\left[-(\frac{t}{\alpha})^{\beta}\right],\quad t>0. \tag{4}$$

This distribution is applied to a wide range of problems. The Weibull distribution is by far the world's most popular statistical model for life data. It is also used in many other applications, such as weather forecasting and fitting data of all kinds. It may be employed for engineering analysis with smaller sample sizes than any other statistical distribution [17].

### 2.2. Maximum likelihood estimation (MLE) method

Let $\{t_j : j = 1, 2, \ldots, n\}$ be a complete data random sample of ordered time-to-failures, the likelihood function $L$ is defined as follows:

$$L(t;\theta) = \prod_{j=1}^{n} f(t_j|\theta). \tag{5}$$

The log-likelihood function $l$ can be expressed as:

$$l(t;\theta) = \sum_{j=1}^{n} \ln\left[f(t_j|\theta)\right], \tag{6}$$

where $f(t)$ is given in (1) and $\theta = (\beta, \alpha, \gamma)$ which can be written as

$$l(t;\beta,\alpha,\gamma) = n\ln\beta - n\beta\ln\alpha + (\beta-1)\sum_{j=1}^{n}\ln(t_j-\gamma) - \alpha^{-\beta}\sum_{j=1}^{n}(t_j-\gamma)^{\beta}.$$

The values of $\beta, \alpha$ and $\gamma$ can be estimated by taking the partial derivatives of $l(t;\beta,\alpha,\gamma)$ with respect to $\alpha, \beta$ and $\gamma$ respectively and equating each one by zero, we get

$$\frac{\partial l}{\partial \alpha} = -\frac{n\beta}{\alpha} + \frac{\beta}{\alpha^{\beta+1}}\sum_{j=1}^{n}(t_j-\gamma)^{\beta} = 0,$$

$$\alpha = \sqrt[\beta]{\frac{1}{n}\sum_{j=1}^{n}(t_j-\gamma)^{\beta}} \tag{7}$$

which gives $\alpha$ in terms of $\beta$ and $\gamma$.

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - n\ln\alpha + \sum_{j=1}^{n}\ln(t_j-\gamma) + \frac{\ln\alpha}{\alpha^{\beta}}\sum_{j=1}^{n}(t_j-\gamma)^{\beta} - \alpha^{-\beta}\sum_{j=1}^{n}(t_j-\gamma)^{\beta}\ln(t_j-\gamma) = 0.$$

By using 7, we can write

$$f_1(\beta,\gamma) = \frac{1}{\beta} + \frac{1}{n}\sum_{j=1}^{n}\ln(t_j-\gamma) - \frac{\sum_{j=1}^{n}(t_j-\gamma)^{\beta}\ln(t_j-\gamma)}{\sum_{j=1}^{n}(t_j-\gamma)^{\beta}} = 0 \tag{8}$$

Similarly,

$$\frac{\partial l}{\partial \gamma} = (\beta-1)\sum_{j=1}^{n}\frac{1}{(\gamma-t_j)} + \beta\alpha^{-\beta}\sum_{j=1}^{n}(t_j-\gamma)^{\beta-1} = 0$$

So, we can write

$$f_2(\beta,\gamma) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{(t_j-\gamma)}\cdot\frac{\sum_{j=1}^{n}(t_j-\gamma)^{\beta}}{\sum_{j=1}^{n}(t_j-\gamma)^{\beta-1}} + \frac{\beta}{1-\beta} = 0 \tag{9}$$

We can estimate the parameters $\beta, \alpha$ and $\gamma$ by using a specified statistical software or by solving the two simultaneous, nonlinear Eqs. (8)and (9) numerically by using, for example Newton–Raphson Method with a good initial guess of $\beta^{(0)}$ and $\gamma^{(0)}$, and update by the following iterative relation

$$\theta^{(k+1)} = \theta^{(k)} - \frac{f\left(\theta^{(k)}\right)}{f'\left(\theta^{(k)}\right)}, \tag{10}$$

until the change in two successive steps is negligible, where

$$\theta = \begin{bmatrix}\beta\\ \gamma\end{bmatrix}, \quad f(\theta) = \begin{bmatrix}f_1(\beta,\gamma)\\ f_2(\beta,\gamma)\end{bmatrix}, \quad f'(\theta) = \begin{bmatrix}\frac{\partial f_1}{\partial\beta} & \frac{\partial f_1}{\partial\gamma}\\ \frac{\partial f_2}{\partial\beta} & \frac{\partial f_2}{\partial\gamma}\end{bmatrix}.$$

Once $\beta$ and $\gamma$ have been estimated by Eq. (10), estimate of $\alpha$ is obtained from Eq. (7). Note that to get a good initial estimates of $\beta$ and $\gamma$, we use the graphical approach which is introduced in [18].

### 2.3. Weibull mixture model

When we have $m$-fold mixture model that involves m sub-populations, then the probability density function $f(t|\boldsymbol{\theta})$ of the mixture distribution is given as follows:

$$f(t|\boldsymbol{\theta}) = \sum_{i=1}^{m} \omega_i f_i(t|\beta_i, \alpha_i), \tag{11}$$

where $\omega_i > 0, \alpha_i > 0, \ \beta_i > 0$ are mixing weight, scale and shape parameters of subpopulatin $i$ respectively, $\sum_{i=1}^{m}\omega_i = 1$ and $\boldsymbol{\theta} = (\omega_1, \omega_2, \ldots, \omega_m, \alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_m)$ is called the parameter vector of an m- mixed Weibull distribution.

$$f(t|\boldsymbol{\theta}) = \sum_{i=1}^{m} \omega_i \left(\frac{\beta_i}{\alpha_i}\right)\left(\frac{t}{\alpha_i}\right)^{\beta_i - 1} \exp\left[-\left(\frac{t}{\alpha_i}\right)^{\beta_i}\right]. \tag{12}$$

The reliability (survivor) function $R(t|\boldsymbol{\theta})$ of the mixture distribution is given as follows:

$$R(t|\boldsymbol{\theta}) = \sum_{i=1}^{m} \omega_i \exp\left[-\left(\frac{t}{\alpha_i}\right)^{\beta_i}\right]. \tag{13}$$

Another function to describe the Reliability of system components is the hazard (failure rate) function $h(t|\boldsymbol{\theta})$ of the mixture Weibull distribution is given as follows:

$$h(t|\boldsymbol{\theta}) = \sum_{i=1}^{m} \omega_i \left(\frac{\beta_i}{\alpha_i}\right)\left(\frac{t}{\alpha_i}\right)^{\beta_i - 1} \tag{14}$$

The appropriate mixed Weibull distribution must be used when a product has two or more failure modes or causes. This occurs in many situations, such as both early failures (infant mortality) and chance failures might be involved in a burn-in test and also in the case of quality control mode with an infant mortality followed by a wear out mode.

#### 2.3.1. The proposed method

Consider a reliability life testing is applied on $n$ units of a product which has two failure modes, a complete data of ordered time-to-failure sample $\{t_j, \ j = 1, 2, \ldots, n\}$ is obtained.

The likelihood function $L$ is defined as follows:

$$L(t; \boldsymbol{\theta}) = \prod_{j=1}^{n} f(t_j|\boldsymbol{\theta}). \tag{15}$$

The log-likelihood function $l$ can be expressed as:

$$l(t; \boldsymbol{\theta}) = \sum_{j=1}^{n} \ln[f(t_j|\boldsymbol{\theta})]. \tag{16}$$

For the EM algorithm, we augment the observed (measured) data with some unobserved (missing data). This means that we embed the observed data in a larger complete data space. Note that, missing data are not necessarily missing in the classical way. This process is called data augmentation [12,13]. By Bayes formula the concept of belonging probability, $P_i(t_j, \boldsymbol{\theta}^{(k)})$, which is the posterior probability that the unit belongs to the $i$th subpopulation $(i = 1, 2, \ldots, m)$, knowing that it failed at time $t_j$ is introduced as:

$$P_i(t_j, \boldsymbol{\theta}^{(k)}) = \frac{\omega_i^{(k)} f_i(t_j|\beta_i^{(k)}, \alpha_i^{(k)})}{f(t_j|\boldsymbol{\theta}^{(k)})}, \tag{17}$$

$$P_i(t_j, \boldsymbol{\theta}^{(k)}) = \frac{\omega_i^{(k)} f_i(t_j|\beta_i^{(k)}, \alpha_i^{(k)})}{\sum_{i=1}^{m} \omega_i^{(k)} f_i(t_j|\beta_i^{(k)}, \alpha_i^{(k)})}. \tag{18}$$

Given a current estimate $\boldsymbol{\theta}^{(k)}$ define the expectation of the log-likelihood function as;

$$Q(\theta, \theta^{(k)}) = \sum_{j=1}^{n} P_i(t_j, \boldsymbol{\theta}^{(k)}) \ln[f(t_j|\boldsymbol{\theta})] \tag{19}$$

which can also be written as:

$$Q(\theta,\theta^{(k)})=\sum_{j=1}^{n}\sum_{i=1}^{m}P_i(t_j,\boldsymbol{\theta}^{(k)})\ln[\omega_i f_i(t_j|\beta_i,\alpha_i)], \tag{20}$$

$$Q(\theta,\theta^{(k)})=\sum_{j=1}^{n}\sum_{i=1}^{m}P_i(t_j,\boldsymbol{\theta}^{(k)})\ln(\omega_i)+\sum_{j=1}^{n}\sum_{i=1}^{m}P_i(t_j,\boldsymbol{\theta}^{(k)})\ln(f_i(t_j|\beta_i,\alpha_i)+\lambda\left(\sum_{i=1}^{m}\omega_i-1\right), \tag{21}$$

where $\lambda$ is the lagrange multiplier with the constraint that $\sum_{i=1}^{m}\omega_i=1$. The evaluation of this expectation is called the E step of the algorithm. In the second step, the M-step (the maximization step), we find that value $\boldsymbol{\theta}^{(k+1)}$ of $\boldsymbol{\theta}$ which maximizes $Q(\theta,\theta^{(k)})$.

To find $\omega_i^{(k+1)}$ of $\omega_i$ which maximize $Q(\theta,\theta^{(k)})$, taking the derivative of Eq. (21) with respect to $\omega_i$ equal to zero, we ge t:

$$\sum_{j=1}^{n}\frac{1}{\omega_i}P_i\left(t_j,\boldsymbol{\theta}^{(k)}\right)+\lambda=0. \tag{22}$$

Summing both sides over $i$ and using the fact that $\sum_{i=1}^{m}P_i(t_j,\boldsymbol{\theta}^{(k)})=1$ we get $\lambda=-n$ , consequently

$$\omega_i^{k+1}=\frac{1}{n}\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)}). \tag{23}$$

To find the value $\alpha_i^{(k+1)}$ of $\alpha_i$ which maximize $Q(\theta,\theta^{(k)})$, taking the derivative of Eq. (21) with respect to $\alpha_i$ equal to zero, we ge t:

$$\frac{\partial Q(\theta,\theta^{(k)})}{\partial\alpha_i}=0 \tag{24}$$

$$\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})\frac{\partial\ln(f_i(t_j|\beta_i,\alpha_i)}{\partial\alpha_i}=0 \tag{25}$$

$$\sum_{j=1}^{n}\frac{P_i\left(t_j,\boldsymbol{\theta}^{(k)}\right)\beta_i^{(k+1)}\left[-1+(\frac{t_j}{\alpha_i})^{\beta_i^{(k+1)}}\right]}{\alpha_i^{(k+1)}}=0 \tag{26}$$

$$\sum_{j=1}^{n}-P_i\left(t_j,\boldsymbol{\theta}^{(k)}\right)+\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})\left(\frac{t_j}{\alpha_i^{(k+1)}}\right)^{\beta_i^{(k+1)}}=0 \tag{27}$$

$$\sum_{j=1}^{n}P_i\left(t_j,\boldsymbol{\theta}^{(k)}\right)=\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})\left(\frac{t_j}{\alpha_i^{(k+1)}}\right)^{\beta_i^{(k+1)}} \tag{28}$$

$$(\alpha_i^{(k+1)})^{\beta_i^{(k+1)}}=\frac{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})(t_j)^{\beta_i^{(k+1)}}}{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})} \tag{29}$$

$$\alpha_i^{(k+1)}=\left[\frac{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})(t_j)^{\beta_i^{(k+1)}}}{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})}\right]^{1l\beta_i^{(k+1)}} \tag{30}$$

Similarly, we can find the value $\beta_i^{(k+1)}$ of $\beta_i$ which maximize $Q(\boldsymbol{\theta},\boldsymbol{\theta}^{(k)})$, taking the derivative of Eq. (21) with respect to $\beta_i$ equal to zero, we ge t:

$$\frac{1}{\beta_i^{(k+1)}}\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})+\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})\ln(t_j)-\frac{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})(t_j)^{\beta_i^{(k+1)}}\ln(t_j)\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})}{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})(t_j)^{\beta_i^{(k+1)}}}=0 \tag{31}$$

$$g(\beta_i^{(k+1)})=\frac{1}{\beta_i^{(k+1)}}+\frac{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})\ln(t_j)}{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})}-\frac{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})(t_j)^{\beta_i^{(k+1)}}\ln(t_j)}{\sum_{j=1}^{n}P_i(t_j,\boldsymbol{\theta}^{(k)})(t_j)^{\beta_i^{(k+1)}}}=0 \tag{32}$$

Taking a good initial guess of $\theta^{(k)}$, consequently knowing $P_i(t_j, \theta^{(k)})$, and solving Eq. (32) using a numerical method such as Newton–Raphson, updating Eqs. (23), (30) and (32) we can find MLE estimates of $\omega_i^{k+1}, \beta_i^{(k+1)}$and $\alpha_i^{(k+1)}$of subpopulation *i*.

The formulation of the proposed approach algorithm is given as a flow chart in Fig. 1.

*2.3.2. The graphic method*

In graphic method we separate the observed times in such a way to failure data into two sub-populations or more, then we model each subpopulation to a single Weibull distribution. Realy precise estimation of parameters in a mixture model is not possible with only moderate amounts of data, but a main concern is to determine whether some model in the given class produces a reasonable fit to the data [1]. Relatively informal methods are often helpful in examining this possibility, particularly if the data appear separable into two or more fairly distinct parts. One can estimate by plotting the sample cdf (cumulative distribution function) on Weibull plotting paper (WPP) and fit it by inspection a smooth curve. There will usually be a knee in the curve where the slope decreases sharply [19]. The cumulative failure at this point (e.g., 20%) should be used for an estimate of $\omega$. If there is no Knee, use the plotting position for the first data point. Probability plots of the two sets of observations are especially useful, providing both parameter estimates and a check on the assumed form of two sub-populations distribution (simple mixture distributions), also we use these graphic parameter estimates as initial estimates in the proposed method. We can also use the facilities of Super SMITH software package to find the graphic parameter estimates of mixture Weibull distribution.

*2.3.3. The non-linear median rank regression method*

We estimate the values of parameters for mixed Weibull distribution using Weibull++ software package which performs statistical analysis depends on non-linear rank regression and median rank methods. This regression analysis determines the values of the parameters that cause the mixed Weibull distribution to best fit the observed data that we provide. This process is also called "curve fitting".

*2.3.4. Simulation*

One can generate a random data sample such that it follows a mixture Weibull distribution of many sub- populations with known parameters using Monte Carlo method which is included in Weibull++ software package. By applying The proposed method on this sample. The estimated parameters results will be accurate. This means that the performance of the proposed method is efficient.

## 3. Goodness of fit tests (GOF)

GOF is used to determine the best distribution for modeling lifetime data, we present Akaike's Information Criteria (AIC) [20], which is of the form

$$AIC = -2\ln\left(L\left(\hat{\theta}\right)\right) + 2k \tag{33}$$

where $\ln\left(L\left(\hat{\theta}\right)\right)$ is the natural logarithm of the maximum likelihood for the proposed model and $k$ is the number of independently adjusted parameters within the model. The result of AIC is directly dependent with sample size of observations. AIC is asymptotically effective and unbiased since the test is based on the maximum likelihood function and if the sample size is sufficiently larger than 30, the test will yield fairly accurate result.

The best model for the data, as calculated by the AIC, is the model with the lowest AIC value. We use $AIC_c$ value [21] in our study which is defined as:

$$AIC_c = AIC + 2k(k+1)/(n-k-1), \tag{34}$$

where $n$ is the sample size.

$AIC_c$ is recommended to use when the ratio $\frac{n}{k}$ is small (say $< 40$) [19,21].

## 4. Application

The data in Table 1 [1] show the numbers of cycles to failure for a group of 60 electrical appliances in a life test, The failure times have been ordered.

*4.1. Statistical inferance for 3-parameter Weibull and mixed Weibull distributions*

The first step for modeling these data sample is to plot the cdf (cumulative distribution function) by using the Median Rank approach [17,19] on Weibull probability paper (WPP) versus ordered time-to-failure $t_j$, if the data points appear curved, we may model these data to 3-parameter Weibull distribution, then a location parameter $\gamma$ will exist which may straighten out these points and if these points make other shapes particularly S shapes we suggest the extend of more than one

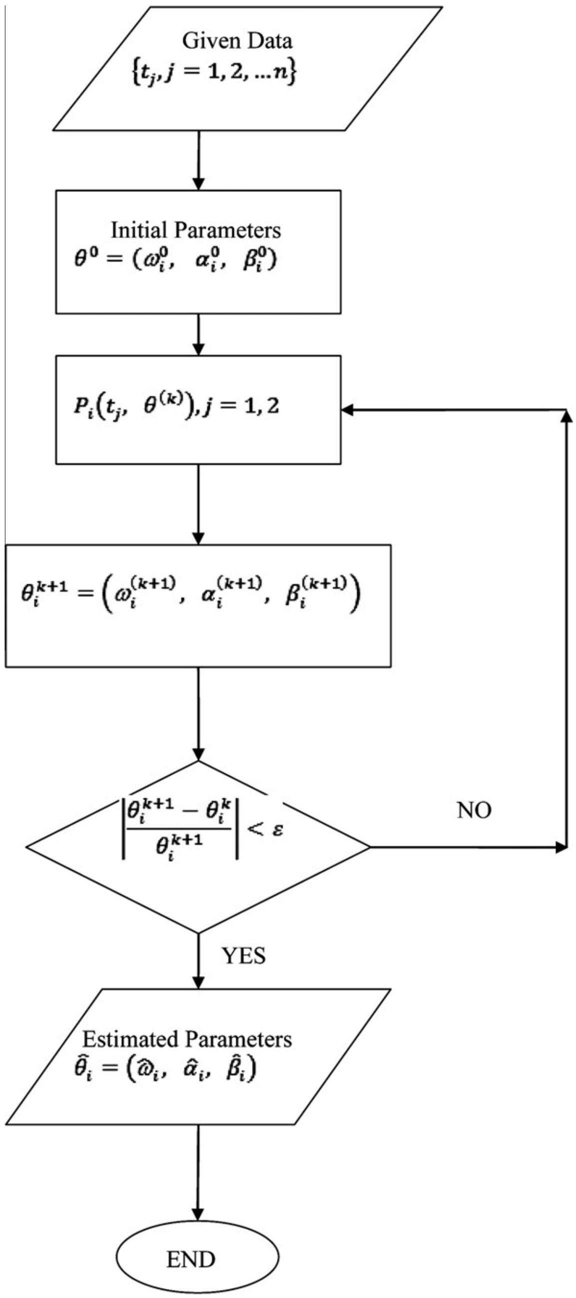


**Fig. 1.** The flow chart of the proposed algorithm.

**Table 1**
Ordered failure times in a life test.

| |
|---|
| 14, 34, 59, 61, 69, 80, 123, 142, 165, 210, 381, 464, 479, 556, 574, 839, 917, 969, 991, |
| 1064, 1088, 1091, 1174, 1270, 1275, 1355, 1397, 1477, 1578, 1649, 1702, 1893, |
| 1932, 2001, 2161, 2292, 2326, 2337, 2628, 2785, 2811, 2886, 2993, 3122, 3248, 3715, |
| 3790, 3857, 3912, 4100, 4106, 4116, 4315, 4510, 4584, 5267, 5299, 5583, 6065, 9701 |

population (batch problem), in this case the multiple population mixed Weibull distribution will be appropriate. Thus we have the following comparison for the two cases:

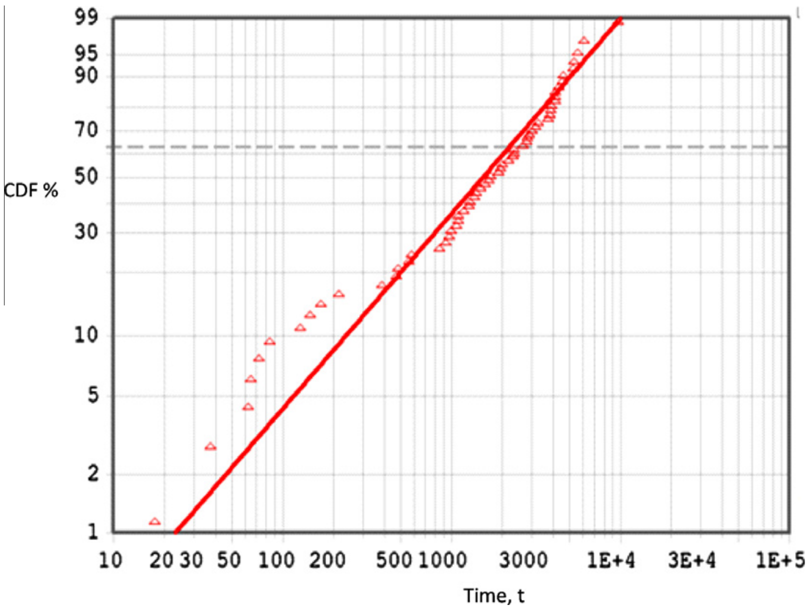


**Fig. 2.** Cumulative distribution function of 3- parameter Weibull distribution for data given in Table 1.

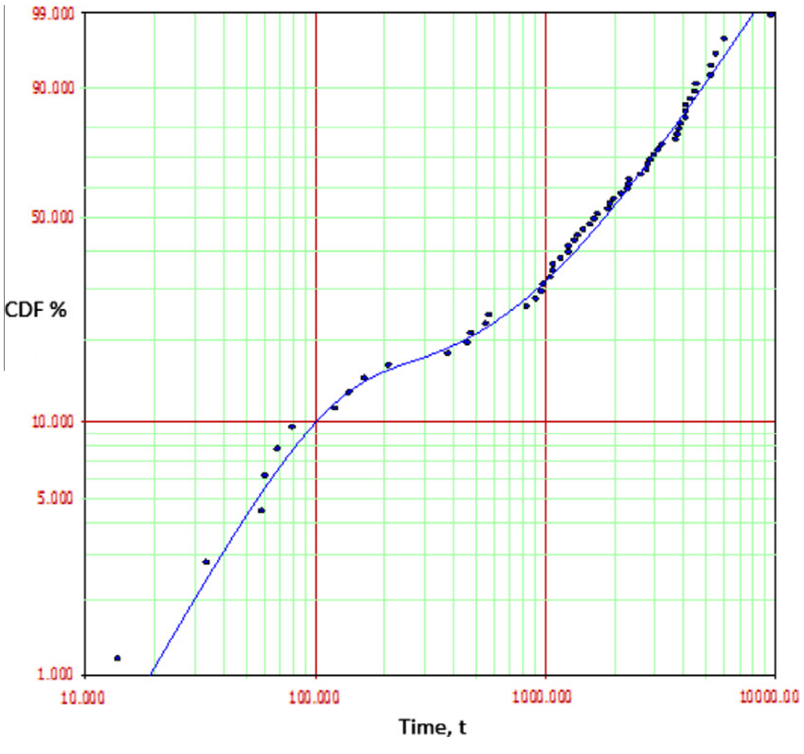


**Fig. 3.** Cumulative distribution function of mixed- Weibull distribution for data given in Table 1.

**Case I**

If we model these failure times data to 3-parameter Weibull distribution, see Fig. 2, we obtain the estimated parameters using MLE method as $\left(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\right) = (2204, 1.009, -3.438)$.

**Case II**

If we model these failure times data to mixed Weibull distribution, the model plot fits the data points well, see Fig. 3. Table 2 shows the estimated parameters obtained by different methods explained before.

**Table 2**
Estimated parameters obtained by different methods for mixture Weibull model.

| Estimation method | Estimated parameters | | | | | |
|---|---|---|---|---|---|---|
| | $\hat{\omega}_1$ | $\hat{\beta}_1$ | $\hat{\alpha}_1$ | $\hat{\omega}_2$ | $\hat{\beta}_2$ | $\hat{\alpha}_2$ |
| Graphic estimation method | 0.2 | 1.704 | 115.3 | 0.8 | 1.733 | 4511 |
| Non-linear regression estimation method | 0.1197 | 1.7278 | 80.9094 | 0.8803 | 1.3036 | 2743.7605 |
| | 0.1636 | 1.5276 | 105.6869 | 0.8364 | 1.3576 | 2787.2 |

**Table 3**
Confidence intervals of estimated parameters of mixed Weibull distribution.

| Parameter method | Proposed estimation | Approximated 95% | |
|---|---|---|---|
| | Confidence intervals | Lower | Upper |
| $\hat{\omega}_1$ | 0.1636 | 0.0755 | 0.3248 |
| $\hat{\beta}_1$ | 1.5276 | 0.8556 | 2.7273 |
| $\hat{\alpha}_1$ | 105.6869 | 62.2314 | 179.4868 |
| $\hat{\omega}_2$ | 0.8364 | 0.1233 | 0.9658 |
| $\hat{\beta}_2$ | 1.3576 | 1.0525 | 1.7513 |
| $\hat{\alpha}_2$ | 2787.2 | 2234.4633 | 3476.6665 |

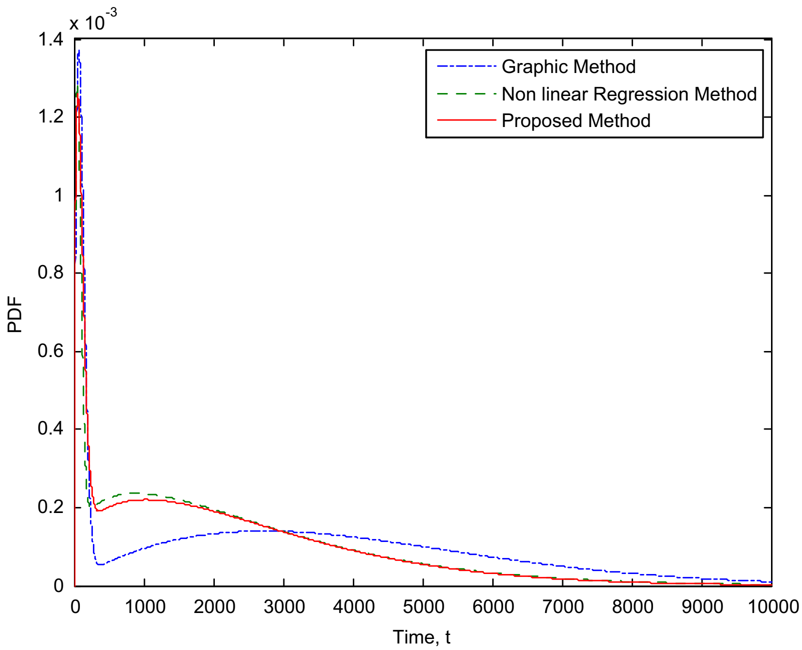


**Fig. 4.** A comparison of fitted probability density functions of failure times obtained by different methods.

Furthermore, we obtain 95% confidence intervals for the estimated parameters obtained by using the proposed approach, see Table 3.

The shape of the data points on the Weibull plot paper (WPP) in Fig. 3 can be considered as a concave upward curve has a cusp or may be a straight line, curving into a second with steeper slope which probably caused by a mixture of failure modes, so one can suggest a mixed Weibull distribution as a good fit than other distributions. We can prove this statistically as follows.

We can use $AIC_c$ expained in equation 34 to get the best model fit the data. We estimate $AIC_c$ for 3-parameter Weibull model and Weibull mixture model respectively to be 1049.8 and 1045.2. This implies that the Weibull mixture model is better than 3-parameter Weibull (extended Weibull model) to model the given sample data.

We model these failure times data to mixed Weibull distribution. Table 2 shows the estimated parameters obtained by different methods explained above.

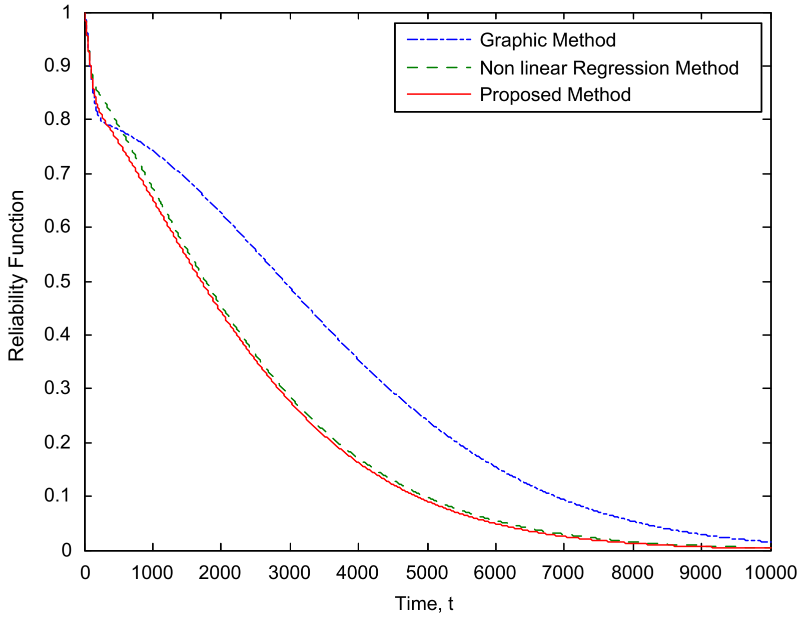


**Fig. 5.** A comparison of fitted reliability functions of failure times obtained by different methods.

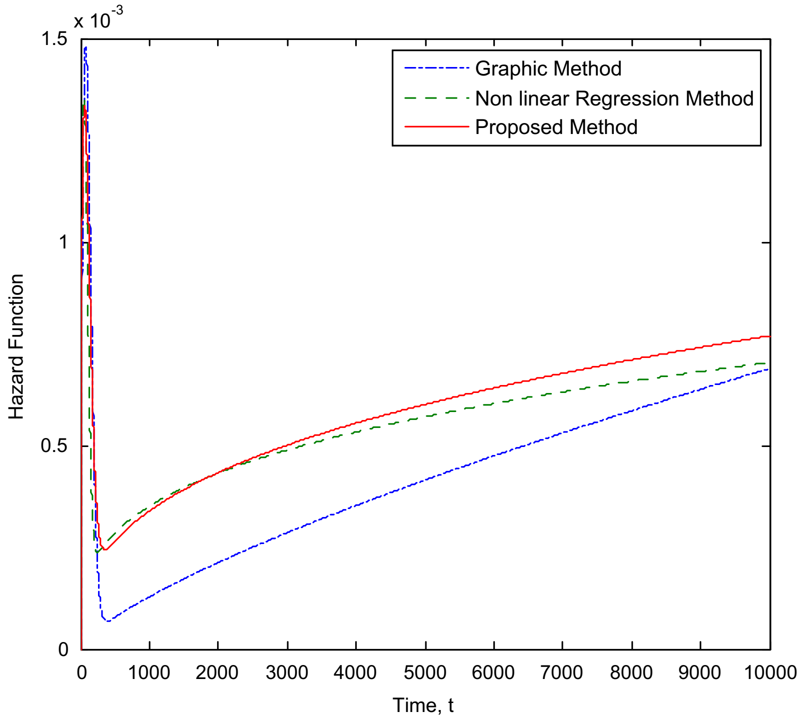


**Fig. 6.** A comparison of fitted hazard functions of failure times obtained by different methods.

Based on the estimated parameters in Table 2 and using Eq. (12), Eq. (13) and Eq. (14), we obtain the probability density functions, reliability (survivor) and hazard functions of the mixture distributions by the three different methods as illustrated in Figs. 4–6. We can deduce that the proposed method is the best fit.

Moreover, we find the hazard function in Fig. 6 is a compound of two periods, the first period with a decreasing hazard rate and the second period where the rate is increasing, this is a strong evidence that failure times under consideration follow a mixed Weibull distribution. In Reliability modeling theory the first period is called the burn-in failure period or the period of infant mortality, the second period is called wear-out failure period or old age period and there is no chance failures that have a constant failure rate in our application. This agrees with our assumption that a mixture model may in fact be physically meaningful, for example, if some items have defects during design or manufacture process that cause them to fail early, whereas items without defects are susceptible to a more gradual wear out.

## 5. Conclusion

This paper presented a powerful approach for modeling the failure data for systems that have different failure modes by using the finite mixture of Weibull distributions. It involves estimating of the unknown parameters which is an important task in statistics, especially in life testing and reliability analysis. The proposed approach depends on different methods that will be used to develop the estimates such as MLE through the EM algorithm. In addition, Bayesian estimations were investigated and some other extensions such as Graphic, Non-Linear Median Rank Regression and Monte Carlo simulation methods were used to model the system under consideration. A numerical application used through the proposed approach. This paper also presents a comparison of the fitted probability density functions, reliability functions and hazard functions of the 3-parameter Weibull and Weibull mixture distributions using the proposed approach and other conventional methods which characterize the distribution of failure times for the system components, GOF is used to determine the best distribution for modeling lifetime data, the priority is for the proposed approach which has more accurate parameter estimates.

In concluding this paper, the proposed approach characterizes the mixed Weibull distribution of the times to failure for the system components and it's an efficient approach especially when the mixture is well mixed for moderate complete sample size. It can be applied to the complete, censored, grouped and ungrouped samples. We can use the proposed method for other finite mixture distribution. The proposed method can be applied on a simple mixture and a competing risk mixture due to infant mortality and chance failure modes or a competing risk mixture due to chance and wear-out failure modes.

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