PHYS 501
HANDOUT 5 - *Questions on curvilinear coordinate systems*

5.1 In the spherical polar coordinate system $q_1 = r,\ q_2 = \theta,\ q_3 = \varphi$. The transformation equations are:

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Calculate the spherical polar coordinate scale factors $h_r,\ h_\theta,\ h_\varphi$.

5.2 In Minkowski space (which is used in Relativity) we define $x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = ict$. Show that the metric in Minkowski space is $g_{ij} = \delta_{ij}$ or

$$\left(g_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

5.3 With $\mathbf{e}_1$ a unit vector in the direction of increasing q_1 show that

(a) $\vec{\nabla}\cdot\mathbf{e}_1 = \dfrac{1}{h_1h_2h_3}\dfrac{\partial\left(h_2h_3\right)}{\partial q_1}$, (b) $\vec{\nabla}\times\mathbf{e}_1 = \dfrac{1}{h_1}\left[\mathbf{e}_2\dfrac{\partial h_1}{h_3\partial q_3} - \mathbf{e}_3\dfrac{\partial h_1}{h_2\partial q_2}\right]$. Note that although $\mathbf{e}_1$ is a unit vector, its divergence and curl *do not necessarily vanish.*

5.4 The Navier-Stokes equations of hydrodynamics contain a nonlinear term $\vec{\nabla}\times\left[\mathbf{v}\times\left(\vec{\nabla}\times\mathbf{v}\right)\right]$, where $\mathbf{v}$ is the fluid velocity. Calculate this term in the case of a fluid flowing through a cylindrical pipe in the z-direction where $\mathbf{v} = \mathbf{k}v(\rho)$.

5.5 Resolve the circular cylindrical unit vectors into their Cartesian components and vice versa.

5.6 From the results of the previous problem show that

$$\frac{\partial\hat{\rho}_0}{\partial\varphi} = \hat{\varphi}_0,\ \frac{\partial\hat{\varphi}_0}{\partial\varphi} = -\hat{\rho}_0$$

and that all other first derivatives of the circular cylindrical unit vectors with respect to the circular cylindrical coordinates vanish.

5.7 Compare $\vec{\nabla}\cdot\mathbf{V}$ with the gradient operator $\vec{\nabla}=\hat{\rho}_0\frac{\partial}{\partial\rho}+\hat{\varphi}_0\frac{1}{\rho}\frac{\partial}{\partial\varphi}+\mathbf{k}\frac{\partial}{\partial z}$ dotted into **V**. Note that the differential operator $\vec{\nabla}$ differentiate *both* the unit vectors and the components of **V**. *Hint.* $\hat{\varphi}_0\left(\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)\cdot\hat{\rho}_0 V_\rho$ becomes $\hat{\varphi}_0\cdot\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left(\hat{\rho}_0 V_\rho\right)$ and does not vanish.

5.8 (a) Show that $\mathbf{r}=\hat{\rho}_0\rho+\mathbf{k}z$. (b) Working entirely in circular cylindrical coordinates, show that $\vec{\nabla}\cdot\mathbf{r}=3$ and $\vec{\nabla}\times\mathbf{r}=0$.

5.9 A rigid body is rotating about a fixed axis with a constant angular velocity $\vec{\omega}$. Take $\vec{\omega}$ to lie along the z-axis. Express **r** in circular cylindrical coordinates and using circular cylindrical coordinates. (a) Calculate $\mathbf{v}=\vec{\omega}\times\mathbf{r}$. (b) $\vec{\nabla}\times\mathbf{v}$.

5.10 A particle is moving through space. Find the circular cylindrical components of its velocity and acceleration.

5.11 Solve Laplace's equation $\vec{\nabla}^2\psi=0$, in cylindrical coordinates for $\psi=\psi(\rho)$.

5.12 In right circular cylindrical coordinates a particular vector function is given by $\mathbf{V}(\rho,\varphi)=\hat{\rho}_0 V_\rho(\rho,\varphi)+\hat{\varphi}_0 V_\varphi(\rho,\varphi)$. Show that $\vec{\nabla}\times\mathbf{V}$ has only a z-component.

5.13 For the example shown in question 5.4 we have that $\vec{\nabla}^2\left(\vec{\nabla}\times\mathbf{v}\right)=0$. Show that this relation leads to $\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d^2v}{d\rho^2}\right)=\frac{1}{\rho}\frac{dv}{d\rho}$ and that is satisfied by $v=v_0+a_2\rho^2$.

5.14 A conducting wire along the z-axis carries a current I. The resulting magnetic vector potential is given by $\mathbf{A}=\mathbf{k}\frac{\mu I}{2\pi}\ln\left(\frac{1}{\rho}\right)$. Show the the magnetic induction **B** is given by $\mathbf{B}=\hat{\varphi}_0\frac{\mu I}{2\pi\rho}$.

5.15 A force is described by

$$\mathbf{F}=-\mathbf{i}\frac{y}{x^2+y^2}+\mathbf{j}\frac{x}{x^2+y^2}.$$

(a) Express **F** in circular cylindrical coordinates.

Operating entirely on circular cylindrical coordinates

(b) Calculate the curl of **F**.

(c) Calculate the work of the force in encircling the unit circle once counterclockwise.

(d) How do you reconcile the results of (b) and (c)?

5.16 A transverse E/M wave in a coaxial wave guide has an electric field $\mathbf{E} = \mathbf{E}\left(\rho,\varphi\right)e^{i\left(kz-\omega t\right)}$ and a magnetic induction field of $\mathbf{B} = \mathbf{B}\left(\rho,\varphi\right)e^{i\left(kz-\omega t\right)}$. Since the magnetic field is transverse neither **E** nor **B** has a z component. The two fields satisfy the Laplace equation

$$\vec{\nabla}^2\mathbf{E}\left(\rho,\varphi\right) = 0$$
$$\vec{\nabla}^2\mathbf{B}\left(\rho,\varphi\right) = 0$$

(a) Show that $\mathbf{E} = \hat{\rho}_0 E_0\left(a/\rho\right)e^{i\left(kz-\omega t\right)}$ and $\mathbf{B} = \hat{\varphi}_0 B_0\left(a/\rho\right)e^{i\left(kz-\omega t\right)}$ are solutions. Here a is the radius of the inner conductor and E_0 and B_0 are amplitudes.

(b) Find the ratio B_0/E_0.

5.17 A calculation of the magnetohydrodynamics pinch effect involves the evaluation of $\left(\mathbf{B}\cdot\vec{\nabla}\right)\mathbf{B}$. If the magnetic induction **B** is taken to be $\mathbf{B} = -\hat{\varphi}_0 B_\varphi\left(\rho\right)$, show that $\left(\mathbf{B}\cdot\vec{\nabla}\right)\mathbf{B} = -\hat{\rho}_0 B_\varphi^2/\rho$.

5.18 The linear velocity of particles in a rigid body rotating with angular velocity ω is given by $\mathbf{v} = \hat{\varphi}_0\rho\omega$. Integrate $\oint \mathbf{v}\cdot d\mathbf{l}$ around a circle in the *xy*-plane and verify that

$$\frac{\oint \mathbf{v}\cdot d\mathbf{l}}{area} = \vec{\nabla}\times\mathbf{v}\Big|_z .$$

5.19 Working in spherical coordinates prove the following relations:

$$\vec{\nabla}f(r) = \hat{\mathbf{r}}\frac{df}{dr},\ \ \vec{\nabla}r^n = \hat{\mathbf{r}}nr^{n-1},\ \ \vec{\nabla}\cdot\hat{\mathbf{r}}f(r) = \frac{2f(r)}{r} + \frac{df}{dr},\ \ \vec{\nabla}\cdot\hat{\mathbf{r}}r^n = \left(n+2\right)r^{n-1}$$

$$\vec{\nabla}^2 f(r) = \frac{2}{r}\frac{df}{dr} + \frac{d^2 f}{dr^2},\ \ \vec{\nabla}^2 r^n = n\left(n+1\right)r^{n-2},\ \ \vec{\nabla}\times\hat{\mathbf{r}}f(r) = 0 .$$

5.20 The computation of the magnetic vector potential of a single current loop in the *xy* plane involves the evaluation of $\mathbf{V} = \vec{\nabla}\times\left[\vec{\nabla}\times\hat{\varphi}_0 A_\varphi(r,\theta)\right]$. Evaluate this quantity.

5.21 Resolve the spherical polar unit vectors into their cartesian components and vice versa.

5.22 From the results of 5.21 calculate the partial derivatives of $\hat{\mathbf{r}}$, $\hat{\theta}_0$ and $\hat{\varphi}_0$.

5.23 A rigid body is rotating about a fixed axis with a constant angular velocity $\vec{\omega}$. Take $\vec{\omega}$ to lie along the z-axis. Using spherical coordinates (a) Calculate $\mathbf{v} = \vec{\omega} \times \mathbf{r}$. (b) $\vec{\nabla} \times \mathbf{v}$.

5.24 The direction of one vector is given by the angles θ_1, φ_1. For a second vector the corresponding angles are θ_2, φ_2. Show that the cosine of the included angle γ is given by

$$\cos\gamma = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2).$$

5.25 A certain vector **V** has no radial component. Its curl has no tangential components. What does this imply about the radial dependence of the tangential components of **V**?

5.26 With **A** any vector $\left(\mathbf{A} \cdot \vec{\nabla}\right)\mathbf{r} = \mathbf{A}$. Verify this result in Cartesian coordinates and in spherical polar coordinates.

5.27 Express $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ in spherical polar coordinates. (Hint: Equate $\vec{\nabla}_{xyz}$ and $\vec{\nabla}_{r\theta\varphi}$.

5.28 From the previous question show that

$$-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial \varphi}$$

This is the quantum mechanical operator corresponding to the z-component of angular momentum.

5.29 With the quantum mechanical, angular momentum operator defined as $\mathbf{L} = -i\hbar\left(\mathbf{r} \times \vec{\nabla}\right)$, show that

(a) $L_x + iL_y = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$, (b) $L_x - iL_y = -e^{-i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)$.

5.30 Verify that $\mathbf{L} \times \mathbf{L} = i\hbar\mathbf{L}$ in spherical coordinates. With $\mathbf{L} = -i\hbar\left(\mathbf{r} \times \vec{\nabla}\right)$.

5.31 From the expression for $\vec{\nabla}\psi$ show that

$$\mathbf{L} = -i\hbar\left(\mathbf{r} \times \vec{\nabla}\right) = i\hbar\left(\hat{\theta}_0 \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \hat{\varphi}_0 \frac{\partial}{\partial\theta}\right).$$

5.32 Resolving $\hat{\theta}_0$ and $\hat{\varphi}_0$ into Cartesian components, determine L_x, L_y, L_z in terms of θ and φ.

5.33 With $\mathbf{L} = -i\hbar\left(\mathbf{r} \times \vec{\nabla}\right)$ verify the operator identities

(a) $\vec{\nabla} = \mathbf{r}\dfrac{\partial}{\partial r} - i\dfrac{\mathbf{r} \times \mathbf{L}}{\hbar r^2}$, (b) $r^2\vec{\nabla}^2 - \vec{\nabla}\left(1 + r\dfrac{\partial}{\partial r}\right) = \dfrac{i}{\hbar}\vec{\nabla} \times \mathbf{L}$.

The later identity is useful in relating angular momentum and Legendre's differential equation.

5.34 Show that the following three forms (spherical coordinates) of $\vec{\nabla}^2\psi\left(r\right)$ are equivalent. (a) $\dfrac{1}{r^2}\dfrac{d}{dr}\left[r^2\dfrac{d\psi\left(r\right)}{dr}\right]$, (b) $\dfrac{1}{r}\dfrac{d^2}{dr^2}\left[r\psi(r)\right]$ (c) $\dfrac{d^2\psi\left(r\right)}{dr^2} + \dfrac{2}{r}\dfrac{d\psi\left(r\right)}{dr}$.

5.35 One model of the solar corona assumes that the steady-state equation of heat flow $\vec{\nabla} \cdot \left(k\vec{\nabla}T\right) = 0$ is satisfied. Here, *k*, the thermal conductivity, is proportional to $T^{5/2}$. Assuming that the temperature *T* is proportional to r^n, show that the heat flow equation is satisfied by $T = T_0(r_0 / r)^{2/7}$.

5.36 A certain force field is given by

$$\mathbf{F} = \hat{\mathbf{r}}\frac{2P\cos\theta}{r^3} + \hat{\theta}_0\frac{P}{r^3}\sin\theta, \quad r \geq P/2$$

(in spherical polar coordinates). (a) Calculate $\vec{\nabla} \times \mathbf{F}$, (b) Calculate $\oint \mathbf{F} \cdot d\mathbf{l}$ for a unit circle in the plane $\theta = \pi / 2$.