PHYS 501
HANDOUT 4 - *Questions on vector integration*

4.1 The force field acting on a two-dimensional linear oscillator may be described by **F** = -**i**kx-**j**ky. Calculate the work done by moving against this force field when going from (1, 1) to (4, 4) by following straight line paths:

a) $(1,\ 1) \to (4,\ 1) \to (4,\ 4)$, b) $(1,\ 1) \to (1,\ 4) \to (4,\ 4)$, c) along the line $y=x$.

4.2 Find the work done going around a unit circle in the xy-plane:

a) counterclockwise from 0 to π, b) clockwise from 0 to $-\pi$,

doing work *against* a force field given by:

$$\mathbf{F} = -\mathbf{i}\frac{y}{x^2+y^2} + \mathbf{j}\frac{x}{x^2+y^2}.$$

4.3 Calculate the work you do in going from point (1, 1) to point (3, 3). The force you exert is given by

$$\mathbf{F} = \mathbf{i}(x-y) + \mathbf{j}(x+y).$$

Specify clearly the path you choose. Note that this force field is not conservative.

4.4 Evaluate the integral $\oint \mathbf{r}\cdot d\mathbf{r}$.

4.5 Using Gauss's theorem prove that if S is a closed surface then $\int_S \mathbf{n}\, dA = 0$.

4.6 If a volume V is enclosed by a surface S then $\frac{1}{3}\int_S \mathbf{r}\cdot\mathbf{n}\, dA = V$.

4.7 If $\mathbf{B} = \vec{\nabla}\times\mathbf{A}$, show that $\int_S \mathbf{B}\cdot\mathbf{n} ds = 0$ for any closed surface.

4.8 A particular steady-state electric current distribution is localized in space. Choosing a bounding surface far enough out so that the current density **J** is zero everywhere on the surface, show that $\int_V \mathbf{J} d\tau = 0$. Assume that $\vec{\nabla}\cdot\mathbf{J} = 0$.

4.9 Prove the following relation

$$\int \rho\phi\, d\tau = \varepsilon_0 \int E^2\, d\tau$$

Here ϕ is the electrostatic potential, **E** is the electric field and they satisfy the relations $\mathbf{E}=-\vec{\nabla}\phi$, $\vec{\nabla}\cdot\mathbf{E}=\rho/\varepsilon_0$. You may assume that ϕ vanishes at large r at least as fast as r^{-1}.

4.10 Given a vector $\mathbf{t}=-\mathbf{i}y+\mathbf{j}x$. With the help of Stoke's theorem, show that the integral around a continuous closed curve in the xy-plane

$$\frac{1}{2}\oint \mathbf{t}\cdot d\mathbf{l}=\frac{1}{2}\oint\left(xdy-ydx\right)=A$$

the area enclosed by the curve.

4.11 The calculation of the magnetic moment of a current loop leads to the line integral

$$\frac{1}{2}\oint \mathbf{r}\times d\mathbf{r}$$

(a) Integrate around the perimeter of a current loop (in the xy plane) and show that the scalar magnitude of this line integral is twice the area of the enclosed surface.

(b) The perimeter of an ellipse is described by $\mathbf{r}=\mathbf{i}a\cos\theta+\mathbf{j}b\sin\theta$. From part (a) show that the area of the ellipse is πab.

4.12 Evaluate $\oint \mathbf{r}\times d\mathbf{r}$ by using the alternate form of Stoke's theorem given by:

$$\int_S\left(\mathbf{n}dA\times\vec{\nabla}\right)\times\mathbf{P}=\oint d\mathbf{l}\times\mathbf{P}.$$

Take the loop to be entirely in the xy-plane.

4.13 Prove that if S is a closed surface $\int_S \vec{\nabla}\times\mathbf{V}\cdot\mathbf{n}dA=0$.

4.14 Evaluate the integral $\oint \mathbf{r}\cdot d\mathbf{r}$ by Stoke's theorem.

4.15 Prove that $\oint u\vec{\nabla}v\cdot d\mathbf{r}=-\oint v\vec{\nabla}u\cdot d\mathbf{r}$.

4.16 Prove that $\oint u\vec{\nabla}v\cdot d\mathbf{r}=\int_S\left(\vec{\nabla}u\right)\times\left(\vec{\nabla}v\right)\cdot\mathbf{n}dA$.

4.17 Prove that if a force $\mathbf{F}$ derives from a potential ϕ, i.e. $\mathbf{F}=-\nabla\phi$, then its work along a closed loop is zero.

4.18 Show how the Green's theorem is derived from Gauss's theorem.

4.19 Show the alternate forms of Gauss's theorem.