PHYS 501
HANDOUT 3 - *Questions on successive applications of vector operators*

3.1 Calculate $\vec{\nabla}\cdot\vec{\nabla}g(r)$. What is the result if $g(r)=r^n$?

3.2 Show that the curl of the gradient of a scalar function is zero.

3.3 Show that the divergence of a curl vanishes, i.e. all curls are solenoidal.

3.4 Show that for an electromagnetic field in vacuum

a) $\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{E}\right)=-\varepsilon_0\mu_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}$, b) $\vec{\nabla}\cdot\vec{\nabla}\mathbf{E}=\varepsilon_0\mu_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}$

given that $\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{V}\right)=\vec{\nabla}\vec{\nabla}\cdot\mathbf{V}-\vec{\nabla}\cdot\vec{\nabla}\mathbf{V}$ and the following Maxwell's equations

$$\vec{\nabla}\cdot\mathbf{B}=0,\quad \vec{\nabla}\cdot\mathbf{E}=0,\quad \vec{\nabla}\times\mathbf{B}=\varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t},\quad \vec{\nabla}\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}.$$

3.5 Prove that $\vec{\nabla}\times\left(\varphi\vec{\nabla}\varphi\right)=0$.

3.6 Prove that $\left(\vec{\nabla}u\right)\times\left(\vec{\nabla}v\right)$ is solenoidal where u and v are differentiable scalar functions.

3.7 If φ is a scalar function satisfying Lapalace's equation $\vec{\nabla}^2\varphi=0$. Show that $\vec{\nabla}\varphi$ is both solenoidal and irrotational.

3.8 From the Navier-Stokes equation for the steady flow of an incompressible viscous fluid we have the term

$$\vec{\nabla}\times\left[\mathbf{v}\times\left(\vec{\nabla}\times\mathbf{v}\right)\right]$$

where **v** is the fluid velocity. Show that this term vanishes for the special case **v** =**i**$v(y, z)$.

3.9 You are given that the curl of **F** equals the curl of **G**. Show that **F** and **G** may differ by (a) a constant and (b) a gradient of a scalar function.

3.10 Verify the relation $\vec{\nabla}\times\left(\vec{\nabla}\times\mathbf{V}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\mathbf{V}\right)-\vec{\nabla}\cdot\vec{\nabla}\mathbf{V}$.

3.11 In the Pauli theory of the electron one encounters the expression

$$\left(\mathbf{p}-e\mathbf{A}\right)\times\left(\mathbf{p}-e\mathbf{A}\right)\psi$$

where ψ is a scalar function. **A** is the magnetic vector potential related to the magnetic induction **B** by $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$. Given that $\mathbf{p} = -i\hbar\vec{\nabla}$, show that this expression reduces to $ie\hbar\mathbf{B}\psi$.

3.12 In a (nonrotating) isolated mass such as a star, the condition for equilibrium is

$$\vec{\nabla}P + \rho\vec{\nabla}\phi = 0 .$$

Here P is the total pressure, ρ the density, and ϕ the gravitational potential. Show that at any given point the normal to the surfaces of a constant pressure and constant gravitational potential are parallel.

3.13 Show that any solution of the equation $\vec{\nabla} \times \left(\vec{\nabla} \times \mathbf{A}\right) - k^2\mathbf{A} = 0$ automatically satisfies the vector Helmholtz equation

$$\vec{\nabla}^2\mathbf{A} + k^2\mathbf{A} = 0$$

and the solenoidal condition

$$\vec{\nabla} \cdot \mathbf{A} = 0 .$$

Hint. Let $\vec{\nabla}\cdot$ operate on the first equation.

3.14 The theory of heat conduction leads to an equation

$$\vec{\nabla}^2\Psi = k\left|\vec{\nabla}\Phi\right|^2$$

where Φ is a potential satisfying Laplace's equation: $\vec{\nabla}^2\Phi = 0$. Show that a solution of this equation is $\Psi = \frac{1}{2}k\Phi^2$.