PHYS 501
HANDOUT 2 - *Questions on Vector Operators*

2.1 Show that the gradient operator is a vector.

2.2 Calculate the gradient of the function *r*.

2.3 If $S(x,y,z)=\left(x^2+y^2+z^2\right)^{-3/2}$, find

(a) $\vec{\nabla}S$ at the point (1, 2, 3);

(b) $\left|\vec{\nabla}S\right|$ at the point (1, 2, 3);

(c) the direction cosines of ∇S at the point (1, 2, 3).

2.4 (a) Show that on a surface where a function $\phi(x, y, z)$ maintains a constant value the $\vec{\nabla}\phi$ is perpendicular to $d\mathbf{r}$.

(b) Find a unit vector perpendicular to the surface $x^2+y^2+z^2=3$ at the point (1, 1, 1).

(b) Derive the equation of the plane tangent to the surface (1, 1, 1).

2.5 If a vector function **F** depends on both space coordinates (*x, y, z*) and time *t*, show that

$$d\mathbf{F}=\left(d\mathbf{r}\cdot\vec{\nabla}\right)\mathbf{F}+\frac{\partial\mathbf{F}}{\partial t}dt$$

2.6 Show that $\vec{\nabla}\left(uv\right)=v\vec{\nabla}u+u\vec{\nabla}v$, where *u* and *v* are differentiable scalar functions of *x, y*, and *z*.

2.7 Show that a necessary and sufficient condition that *u*(*x, y, z*) and *v*(*x, y, z*) are related by some function *f*(*u,v*)=0 is that $\left(\nabla u\right)\times\left(\nabla v\right)=0$.

2.8 If *u*=*u*(*x, y*) and *v*=*v*(*x, y*) two differentiable functions, show that the condition $\left(\nabla u\right)\times\left(\nabla v\right)=0$ leads to the two-dimensional Jacobian

$$J\left(\frac{u,v}{x,y}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=0.$$

2.9 Calculate the quantity $\vec{\nabla}\cdot\mathbf{r}$.

2.10 Calculate the quantity $\vec{\nabla}\cdot \mathbf{r}f\left(r\right)$.

2.11 Calculate the quantity $\vec{\nabla}\cdot \mathbf{r}r^{n-1}$.

2.12 For a particle moving in a circular orbit $\mathbf{r}=\mathbf{i}r\cos\omega t+\mathbf{j}r\sin\omega t$,

(a) evaluate $\mathbf{r}\times\dot{\mathbf{r}}$

(b) $\ddot{\mathbf{r}}+\omega^2\mathbf{r}=0$

2.13 Show that $\vec{\nabla}\cdot\left(f\mathbf{V}\right)=\left(\vec{\nabla}f\right)\cdot\mathbf{V}+f\left(\vec{\nabla}\cdot\mathbf{V}\right)$.

2.14 Show by differentiating components, that

(a) $\frac{d}{dt}\left(\mathbf{A}\cdot\mathbf{B}\right)=\frac{d\mathbf{A}}{dt}\cdot\mathbf{B}+\mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$

(b) $\frac{d}{dt}\left(\mathbf{A}\times\mathbf{B}\right)=\frac{d\mathbf{A}}{dt}\times\mathbf{B}+\mathbf{A}\times\frac{d\mathbf{B}}{dt}$

2.15 Prove $\vec{\nabla}\cdot\left(\mathbf{A}\times\mathbf{B}\right)=\mathbf{B}\cdot\vec{\nabla}\times\mathbf{A}-\mathbf{A}\cdot\vec{\nabla}\times\mathbf{B}$.

2.16 The electrostatic field of a point charge is $\mathbf{E}=\frac{q}{4\pi\varepsilon_0}\cdot\frac{\mathbf{r}_0}{r^2}$. Calculate the divergence of **E**. What happens at the origin?

2.17 Show, by rotating the coordinates, that the components of the curl of a vector transform as a vector.

2.18 Show that $\mathbf{u}\times\mathbf{v}$ is solenoidal if **u** and **v** are each irrotational.

2.19 If **A** is irrotational, show that $\mathbf{A}\times\mathbf{r}$ is solenoidal.

2.20 A rigid body is rotating with constant angular velocity **ω**. Show that the linear velocity **v** is solenoidal.

2.21 A vector function $\mathbf{f}(x, y, z)$ is not irrotational but the product of **f** and a scalar function $g(x, y, z)$ is irrotational. Show that $\mathbf{f}\cdot\nabla\times\mathbf{f}=0$.

2.22 If (a) $\mathbf{V}=\mathbf{i}V_x\left(x,y\right)+\mathbf{j}V_y(x,y)$ and (b) $\nabla\times\mathbf{V}\neq 0$, prove that $\nabla\times\mathbf{V}$ is perpendicular to **V**.

2.23 Classically the angular momentum is defined by $\mathbf{L}=\mathbf{r}\times\mathbf{p}$, where **p** is the momentum. To go from classical mechanics to quantum mechanics we replace **p** with the operator $-i\hbar\vec{\nabla}$. Find the Cartesian components of the angular momentum operator.

2.24 Show the relation $\mathbf{L}\times\mathbf{L}=i\hbar\mathbf{L}$ for the quantum mechanical angular momentum operator **L.**

2.25 You are given that: $\vec{\nabla}(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\times\vec{\nabla})\times\mathbf{B}+(\mathbf{B}\times\vec{\nabla})\times\mathbf{A}+(\mathbf{B}\cdot\vec{\nabla})\mathbf{A}+(\mathbf{A}\cdot\vec{\nabla})\mathbf{B}$

Verify the identity $\mathbf{A}\times(\vec{\nabla}\times\mathbf{A})=\frac{1}{2}\vec{\nabla}(A^2)-(\mathbf{A}\cdot\vec{\nabla})\mathbf{A}$.

2.26 If **A** and **B** are constant vectors, show that $\vec{\nabla}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{r})=\mathbf{A}\times\mathbf{B}$.

2.27 A distribution of electric currents creates a constant magnetic moment **m**. The force on **m** in an external magnetic induction **B** is given by $\mathbf{F}=\vec{\nabla}\times(\mathbf{B}\times\mathbf{m})$. Show that $\mathbf{F}=\vec{\nabla}(\mathbf{m}\cdot\mathbf{B})$. *Note.* Assuming no time dependence of the fields, Maxwell's equations yield $\vec{\nabla}\times\mathbf{B}=0$. Also $\vec{\nabla}\cdot\mathbf{B}=0$.

2.28 An electric dipole of moment **p** is located at the origin. The dipole creates an electric potential $V(\mathbf{r})=\frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3}$. Find the electric field $\mathbf{E}=-\vec{\nabla}V$ at **r**.

2.29 The vector potential **A** of a magnetic dipole, with dipole moment **m**, is given by $\mathbf{A}(\mathbf{r})=(\mu_0/4\pi)(\mathbf{m}\times\mathbf{r}/r^3)$. Show that the magnetic induction $\mathbf{B}=\vec{\nabla}\times\mathbf{A}$ is given by

$$\mathbf{B}=\frac{\mu_0}{4\pi}\frac{3\mathbf{r}_0(\mathbf{r}_0\cdot\mathbf{m})-\mathbf{m}}{r^3}$$