Pb. 4.1.1 p. 173

A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{array} \right\| \end{array}$$

Determine the limiting distribution.

Answer

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

$$\because \pi_j = \sum_{k=0}^{2} \pi_k P_{kj}, \; j = 0,1,2$$

$$\text{and } \sum_{k=0}^{2} \pi_k = 1$$

$\Rightarrow$

$$\pi_0 = 0.7\pi_0 + 0.5\pi_2$$
$$\therefore \pi_2 = 0.6\pi_0 \qquad (1)$$

$$\pi_1 = 0.2\pi_0 + 0.6\pi_1$$
$$\therefore \pi_1 = 0.5\pi_0 \qquad (2)$$

$$\because \pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$$

$\therefore$ by substituting (1) and (2) in (3)

we get $\pi_0 + 0.5\pi_0 + 0.6\pi_0 = 1$

$$\therefore \pi_0 = \tfrac{10}{21}$$

$$\therefore \pi_1 = 0.5(\tfrac{10}{21}) = \tfrac{5}{21}, \; \pi_2 = 0.6(\tfrac{10}{21}) = \tfrac{6}{21}$$

$\therefore$ The limitting distribution is $\pi = (\pi_0, \pi_1, \pi_2) = (10/21, 5/21, 6/21)$

Pb. 4.1.2 p. 173

A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{ccc} 0 & 1 & 2 \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}$$

Determine the limiting distribution.

Answer

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

$\Rightarrow$

$$\pi_0 = 0.6\pi_0 + 0.3\pi_1 + 0.4\pi_2$$

$$\pi_1 = 0.3\pi_0 + 0.3\pi_1 + 0.1\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.4\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solving the following equations

$$4\pi_0 - 3\pi_1 - 4\pi_2 = 0 \qquad (1)$$
$$3\pi_0 - 7\pi_1 + \pi_2 = 0 \qquad (2)$$
$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$$

By solving equations using Cramer's rule, we get

$$\Delta = \begin{vmatrix} 4 & -3 & -4 \\ 3 & -7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -66,\ \Delta_0 = \begin{vmatrix} 0 & -3 & -4 \\ 0 & -7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -31$$

$$\Delta_1 = \begin{vmatrix} 4 & 0 & -4 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -16,\ \Delta_2 = \begin{vmatrix} 4 & -3 & 0 \\ 3 & -7 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -19$$

$$\therefore\ \pi_0 = \tfrac{\Delta_0}{\Delta} = \tfrac{31}{66},\ \pi_1 = \tfrac{\Delta_1}{\Delta} = \tfrac{16}{66},\ \pi_2 = \tfrac{\Delta_2}{\Delta} = \tfrac{19}{66}$$

$\therefore$ The limitting distribution is $\pi = (\pi_0, \pi_1, \pi_2) = (31/66, 16/66, 19/66)$

Pb. 4.1.8 p. 174

Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are upper class in the long run?

Answer

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

$\Rightarrow$

$$\pi_0 = 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2$$

$$\pi_1 = 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solving the following equations

$$3\pi_0 - 2\pi_1 - \pi_2 = 0 \qquad (1)$$
$$\pi_0 + 2\pi_1 - 5\pi_2 = 0 \qquad (2)$$
$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$$

By solving equations using Cramer’s rule, we get

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ 1 & 2 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 34, \ \Delta_0 = \begin{vmatrix} 0 & -2 & -1 \\ 0 & 2 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 12$$

$$\Delta_1 = \begin{vmatrix} 3 & 0 & -1 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{vmatrix} = 14, \ \Delta_2 = \begin{vmatrix} 3 & -2 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 8$$

$\therefore \ \pi_0 = \frac{\Delta_0}{\Delta} = \frac{6}{17}, \ \pi_1 = \frac{\Delta_1}{\Delta} = \frac{7}{17}, \ \pi_2 = \frac{\Delta_2}{\Delta} = \frac{4}{17}$

$\therefore$ The limitting distribution is $\pi = (\pi_0, \pi_1, \pi_2) = (6/17, 7/17, 4/17)$

$\therefore$ In the long run, approximately 23.53% of families are upper class.