Tut. Session (8)

(**Pb. 3.4.3 P. 106 Textbook**)

Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

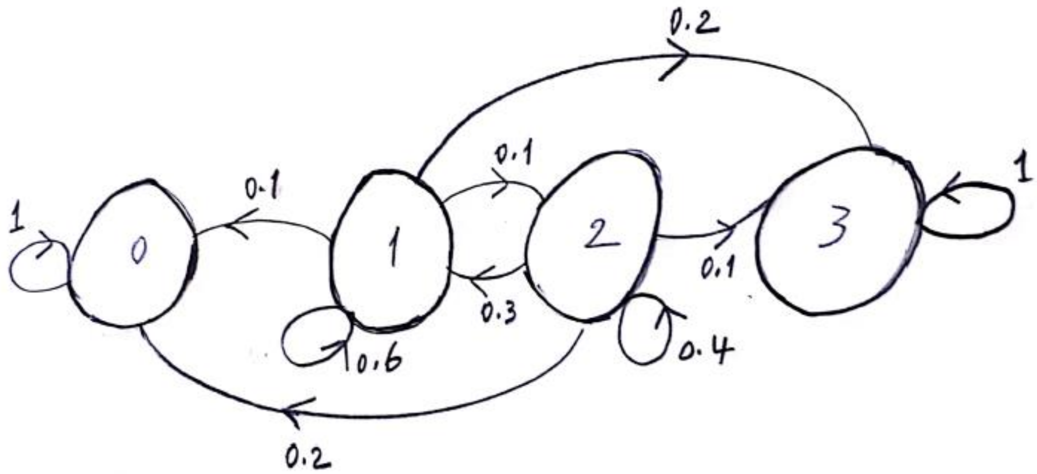
(i) Sketch, the Markov chain diagram, and determine whether it's an absorbing chain or not.

(ii) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(iii) Determine the mean time to absorption.

Ans:

(i) It's an absorbing Markov Chain.



Markov Chain Diagram

$$
\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array}
$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

and v_i =E[$T | X_0 = i$] for i=1,2.

(ii)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.6u_1 + 0.1u_2$$
$$u_2 = 0.2 + 0.3u_1 + 0.4u_2$$

$\Rightarrow$

$$4u_1 - u_2 = 1 \qquad (1)$$
$$3u_1 - 6u_2 = -2 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{8}{21} \text{ and } u_2 = \tfrac{11}{21}$$

Starting in state 1, the probability that the Markov chain ends in state 0 is

$$u_1 = u_{10} = \tfrac{8}{21}$$
$$\approx 0.38$$

(iii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$v_1 = 1 + 0.6v_1 + 0.1v_2$
$v_2 = 1 + 0.3v_1 + 0.4v_2$

$\Rightarrow$

$4v_1 - v_2 = 10$ (1)
$3v_1 - 6v_2 = -10$ (2)

Solving (1) and (2), we get

$v_1 = v_2 = \frac{10}{3}$

$\therefore\ v_1 = v_{10} = \frac{10}{3}$
≈ 3.3

(Pb. 3.4.6 P. 106 Textbook)

Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(i) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

Ans:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,
$and\ \ v_i = E[T | X_0 = i]$ for i=1,2.

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.4u_1 + 0.1u_2$$
$$u_2 = 0.2 + 0.1u_1 + 0.6u_2$$

$\Rightarrow$

$$6u_1 - u_2 = 1 \qquad (1)$$
$$u_1 - 4u_2 = -2 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \frac{6}{23} \text{ and } u_2 = \frac{13}{23}$$

Starting in state 1, the probability that the Markov chain ends in state 0 is

$$u_1 = u_{10} = \frac{6}{23}$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.4v_1 + 0.1v_2$$
$$v_2 = 1 + 0.1v_1 + 0.6v_2$$

$\Rightarrow$

$$6v_1 - v_2 = 10 \qquad (1)$$
$$v_1 - 4v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get $v_2 = v_{20} = \frac{70}{23}$

$$v_1 = \frac{50}{23} \text{ and } v_2 = \frac{70}{23}$$

$\therefore v_1 = v_{10} = \frac{50}{23}$ is the mean time of absorption.