Tutorials

Problem 2.5.2 p.78 ← To be Solved in Lecture class

* * * First, note that from the properties of conditional expectation

$$E[\underbrace{h(y)}_{\text{determined by } Y} \mid Y=y] = h(y)$$

Similarly, if Z is determined by $X_0, \dots, X_n$, then

$$E[Z \mid X_0, \dots, X_n] = Z$$

Solution for 2.5.2

For $i \geq 1$, U_i indepedent identically distributed with $U_i \sim \text{Uniform}(0,1]$

$X_0 = 1$, $X_n = 2^n \cdot U_1 \cdots U_n$, $n = 1, 2, \dots$

To prove X_n is a martingale, we have to prove:

$\boxed{1}$ $E[|X_n|] < \infty$

and

$\boxed{2}$ $E[X_{n+1} \mid X_0, \dots, X_n] = X_n$

For $\boxed{1}$, as X_n is non-negative random variable,

$$E[|X_n|] = E[X_n] = E[2^n \cdot U_1 \cdots U_n]$$

$$= 2^n \cdot \underbrace{E[U_1] \cdots E[U_n]}$$

As U_i's are independent, $E[U_1 \cdots U_n] = E[U_1] \cdots E[U_n]$

Remember if $U \sim \text{uniform}(a,b)$ then $E(U) = \frac{1}{2}(a+b)$

So, $E[U_i] = \frac{1}{2}(0+1) = \frac{1}{2}$

$$\to = 2^n \cdot \underbrace{\tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \dots \cdot \tfrac{1}{2}}_{n\text{-times}}$$

$$= \frac{2^n}{2^n} = 1$$

$\therefore E[|X_n|] = 1 < \infty$ (finite).

$\boxed{2}$ $E[X_{n+1} \mid X_0, \dots, X_n]$

$$= E\left[2^{n+1} U_1 \cdots U_n \cdot U_{n+1} \mid X_0, \dots, X_n\right]$$

$$= E\left[2^n \cdot U_1 \cdots U_n \cdot 2 \cdot U_{n+1} \mid X_0, \dots, X_n\right]$$

$$= E\left[2^n \cdot U_1 \cdots U_n \cdot 2 \cdot U_{n+1} \mid X_0, \ldots, X_n\right]$$

$$= \underbrace{2^n \cdot U_1 \cdots U_n}_{\text{is out because they are determined by } X_0,\ldots,X_n} \cdot E\left[2 \cdot U_{n+1} \mid X_0, \ldots, X_n\right]$$

$$= 2^n \cdot U_1 \cdots U_n \cdot 2\, E\left[U_{n+1} \mid X_0, \ldots, X_n\right]$$

$$= 2^n \cdot U_1 \cdots U_n \cdot 2\, \underbrace{E\left[U_{n+1}\right]}$$

because U_{n+1} is independent of $U_1, \ldots, U_n$ and so independent of $X_0, \ldots, X_n$.
Remember $E[X \mid Y] = E[X]$ if X is indep. of Y.

$$= 2^n \cdot U_1 \cdots U_n \cdot 2 \cdot \underbrace{\frac{1}{2}}$$

as $E[U_i] = \frac{1}{2}$

$$= 2^n \cdot U_1 \cdots U_n$$

$$= X_n$$

So, we proved the second condition, and hence X_n is a martingale.

Problem 2.5.3

← To be solved in tutorials class.

$S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$, $n \geq 1$
where ξ_i are independent identically distributed with $\xi_i \sim \exp(1)$ (Since $E[\xi_i] = 1$)

* Remember when $X \sim \exp(\lambda)$, then $E[X] = \lambda$.

we should prove that $X_n = 2^n e^{-S_n}$, $n \geq 0$ is a martingale.

$\boxed{1}$ X_n is a non-negative r.v, so

$$E\left[|X_n|\right] = E\left[X_n\right]$$

$$= E\left[2^n e^{-S_n}\right]$$

because if $X \sim \exp(\lambda)$ $E[e^{-X}] = \int_0^\infty e^{-x} \cdot \lambda e^{-\lambda x}\, dx$

because if $X \sim \exp(\lambda)$, $E[e^{-X}] = \int_0^\infty e^{-x} \cdot \lambda e^{-\lambda x} dx$

$$= E\left[2^n e^{-S_n}\right]$$

$$= 2^n E\left[e^{-\xi_1} \cdot e^{-\xi_2} \cdots e^{-\xi_n}\right]$$

$$= 2^n \underbrace{E\left[e^{-\xi_1}\right] \cdots E\left[e^{-\xi_n}\right]}_{\text{since } \xi_i\text{'s are independent}}$$

$$* \; E\left[e^{-\xi_i}\right] = \int_0^\infty e^{-x} e^{-x} dx = \int_0^\infty e^{-2x} dx = -\frac{1}{2} e^{-2x}\Big|_0^\infty = \frac{1}{2}$$

So, $E[|X_n|] = 1 < \infty$.

$\boxed{2}$ Prove that $E[X_{n+1} | X_0, \ldots, X_n] = X_n$

$$E\left[2^{n+1} e^{-S_{n+1}} \mid X_0, \ldots, X_n\right]$$

$$= E\left[2^n \cdot e^{-S_n} \cdot 2 \cdot e^{-\xi_{n+1}} \mid X_0, \ldots, X_n\right]$$

$$= 2^n \cdot e^{-S_n} \quad E\left[2e^{-\xi_{n+1}} \mid X_0, \ldots, X_n\right]$$

because $2^n \cdot e^{-S_n}$ is determined by $X_0, \ldots, X_n$

$$= 2^n \cdot e^{-S_n} \cdot 2 \underbrace{E\left[e^{-\xi_{n+1}}\right]}_{\text{since } \xi_{n+1} \text{ is independent of } X_0, \ldots, X_n}$$

$$= 2^n \cdot e^{-S_n} \cdot \cancel{2} \cdot \frac{1}{\cancel{2}} \quad \leftarrow \text{as } E\left[e^{-\xi_i}\right] = \frac{1}{2}$$

$$= 2^n \cdot e^{-S_n}$$

$$= X_n$$

So, X_n is a martingale.

<u>Problem 2.5.4</u> $\leftarrow$ To be solved in tutorials class. Only <u>Prove X_n is martingale</u>.

ξ_i are independent identically distributed

ε_i are independent identically distributed with $\varepsilon_i \sim$ Bernolli (p) $\quad 0<p<1$

$X_0 = 1 \quad , \quad X_n = p^{-n} \cdot \varepsilon_1 \cdots \varepsilon_n \quad , n = 1, 2, \ldots$

① $E[|X_n|] = E[X_n] = E[p^{-n} \cdot \varepsilon_1 \cdots \varepsilon_n]$

$$= p^{-n} \underbrace{E[\varepsilon_1] \cdots E[\varepsilon_n]}_{\text{as } \varepsilon_i\text{s are independent}}$$

As if $X \sim$ Berolli(p) then $E[X] = p$

$$= p^{-n} \cdot \underbrace{p \cdots p}_{n\text{-times}}$$

$$= p^{-n} \cdot p^n$$

$$= p^0 = 1 < \infty$$

② $E[X_{n+1} \mid X_0, \ldots, X_n]$

$$= E[p^{-(n+1)} \varepsilon_1 \cdots \varepsilon_{n+1} \mid X_0, \ldots, X_n]$$

$$= E[p^{-n} \varepsilon_1 \cdots \varepsilon_n \cdot p^{-1} \cdot \varepsilon_{n+1} \mid X_0, \ldots, X_n]$$

$$= \underbrace{p^{-n} \varepsilon_1 \cdots \varepsilon_n}_{\text{is out because it is determined by } X_0, \ldots, X_n} E[p^{-1} \cdot \varepsilon_{n+1} \mid X_0, \ldots, X_n]$$

$$= p^{-n} \varepsilon_1 \cdots \varepsilon_n \cdot p^{-1} E[\varepsilon_{n+1} \mid X_0, \ldots, X_n]$$

$$= p^{-n} \varepsilon_1 \cdots \varepsilon_n \cdot p^{-1} \cdot \underbrace{E[\varepsilon_{n+1}]}_{\text{as } \varepsilon_{n+1} \text{ is independent of } X_0, \ldots, X_n}$$

$$= p^{-n} \varepsilon_1 \cdots \varepsilon_n \cdot \cancel{p^{-1}} \cdot \cancel{p}$$

$$= p^{-n} \varepsilon_1 \cdots \varepsilon_n$$

$$= X_n$$

So, X_n is a martingale.