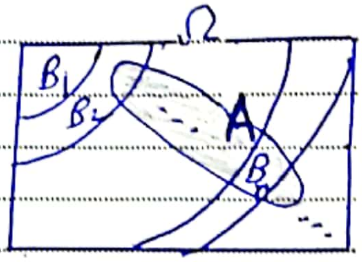
# Tutorial Session (1)

## pb 1.2.1 p.16 Textbook

prove that

$pr(A) = pr(AB) + pr(AB^c)$

proof




$\because pr(A) = \sum_{i=1}^{\infty} pr(A \cap B_i)$

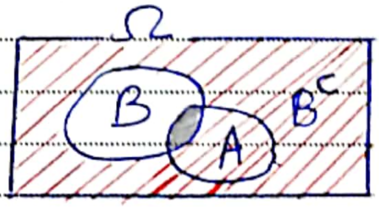
where $\Omega = B_1 \cup B_2 \cup \dots$ and $B_i \cap B_j = \phi$ if $i \neq j$

$\therefore$ For any two subsets A and B, we can write

$pr(A) = pr(A \cap B) + pr(A \cap B^c)$

$\therefore pr(A) = pr(AB) + pr(AB^c)$

where $\Omega = B \cup B^c$ and $B \cap B^c = \phi$

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## pb 1.2.2 p.16

prove that

$pr(A \cup B) = pr(A) + pr(B) - pr(A \cap B)$

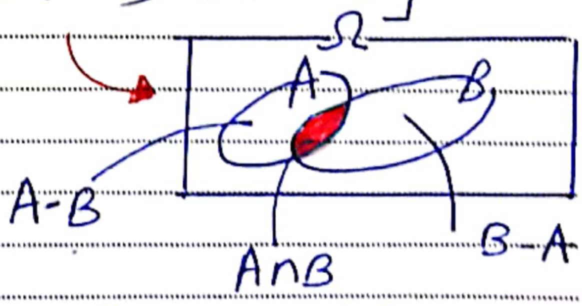
proof

$\because pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} pr(A_i)$ for disjoint sets

&

$\therefore pr(A \cup B) = pr[(A-B) \cup (A \cap B) \cup (B-A)]$

where $A-B$, $A \cap B$ and $B-A$ are disjoint sets




$\therefore pr(A \cup B) = pr(A-B) + pr(A \cap B) + pr(B-A)$

$= pr(A) - \cancel{pr(A \cap B)} + \cancel{pr(A \cap B)} + pr(B) - pr(A \cap B)$

$\therefore pr(A \cup B) = pr(A) + pr(B) - pr(A \cap B)$ ولهم إثباتات أخرى

$= pr(A) + pr(B) - pr(AB)$ #

لا يكتب في هذا الهامش

Another Solution

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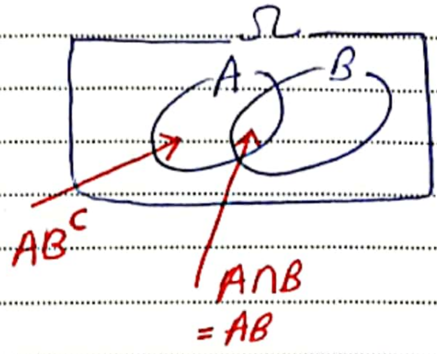
prove that

$pr(A \cup B) = pr(A) + pr(B) - pr(AB)$




proof:

$A \cup B = (A \cup B) \cap (B \cup B^c)$

$\therefore A \cup B = (A \cap B^c) \cup B$ distribution prop.

$\therefore \boxed{A \cup B = (AB^c) \cup B}$ (or you can deduce it directly from the opp. Venn diagram)

$\therefore pr(A \cup B) = pr(AB^c) + pr(B)$ ① where $AB^c$ and $B$ are disjoint events (mutually exclusive events)

$\because A = (AB) \cup (AB^c)$

where $AB$ and $AB^c$ are disjoint events

$\therefore pr(A) = pr(AB) + pr(AB^c)$

$\therefore pr(AB^c) = pr(A) - pr(AB)$ ②

Substitute ② in ① we get

$pr(A \cup B) = pr(A) - pr(AB) + pr(B)$

$\therefore pr(A \cup B) = pr(A) + pr(B) - pr(AB)$ #

3/

## pb 1.2.3 p.16 Textbook

(a) plot the distribution function (C.d.f)

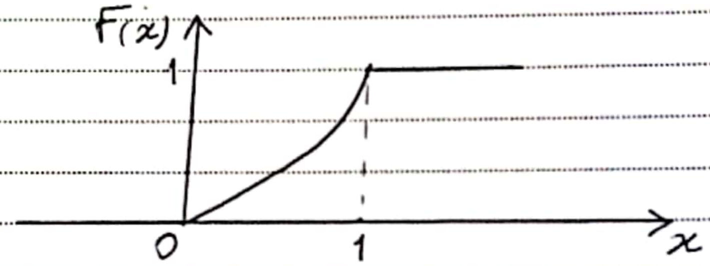
$$F(x)=\begin{cases}0 & \text{for } x\leq 0\\ x^3 & \text{for } 0<x<1\\ 1 & \text{for } x\geq 1\end{cases}$$

(b) Determine the corresponding density function $f(x)$

(c) What is the mean of the distribution?

Ans:

(a)




(b) $\because f(x) = \dfrac{dF}{dx}$

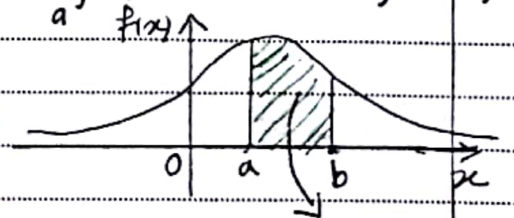
$$\Rightarrow \therefore f(x)=\begin{cases}0 & , x\leq 0\\ 3x^2 & , 0<x<1\\ 0 & , x\geq 1\end{cases} \quad \text{--- is the p.d.f}$$

(c) The mean is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(3x^2)\,dx = 3\int_0^1 x^3\,dx$$

$$= 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}$$

Remember $pr(a\leq X\leq b) = \int_a^b f(x)\,dx = F(b)-F(a)$




$pr(a\leq X\leq b)$

(d)

$$pr\left(\frac{1}{4}\leq X\leq\frac{3}{4}\right)$$

$$=F\left(\frac{3}{4}\right)-F\left(\frac{1}{4}\right)$$

$$=\left(\frac{3}{4}\right)^3-\left(\frac{1}{4}\right)^3=\frac{13}{32} \quad \#$$

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Suppose $X$ is a random variable having the probability density function

$$f(x) = \begin{cases} R x^{R-1} & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

where $R > 0$ is a fixed parameter.

(a) Determine the distribution function $F_X(x)$

(b) Determine the mean $E(X)$

(c) Determine the Variance $Var(X)$

Ans: (a) The cumulative distn fn cdf is

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = R\int_0^x t^{R-1}\,dt$$

$$= \cancel{R}\left[\frac{t^R}{\cancel{R}}\right]_0^x = x^R, \quad 0 \le x \le 1$$

$$\Rightarrow F_X(x) = \begin{cases} 0, & x < 0 \\ x^R, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases} \quad \text{... is the cdf}$$

clearly, $F(0) = 0$, $F(1) = 1$ and $0 \le F(x) \le 1$

(b) the mean, $E(X)$ is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x R x^{R-1}\,dx$$

$$= R\int_0^1 x^R\,dx = R\left[\frac{x^{R+1}}{R+1}\right]_0^1 = \frac{R}{R+1} \quad (1)$$

(c) The variance, $V(X)$ is

$$V(X) = E(X^2) - \mu^2 \quad (2)$$

$$\therefore E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

[illegible]

$$= \int_0^1 x^2 R x^{R-1}\,dx$$

$$= R\int_0^1 x^{R+1}\,dx$$

$$\therefore E(X^2) = R\left[\frac{x^{R+2}}{R+2}\right]_0^1 = \frac{R}{R+2} \quad (3) \Rightarrow$$

$\therefore$ By substituting, (1) and (3) in (2), we get

$$Var(X) = \frac{R}{R+2} - \frac{R^2}{(R+1)^2}$$

$$= R\left[\frac{1}{R+2} - \frac{R}{(R+1)^2}\right]$$

$$= R\left[\frac{(R+1)^2 - R(R+2)}{(R+2)(R+1)^2}\right]$$

$$= R\left[\frac{R^2+2R+1-R^2-2R}{(R+2)(R+1)^2}\right]$$

$$\therefore Var(X) = R/[(R+2)(R+1)^2] \quad \#$$

*pb 1.27 p.18 Textbook

let U and W be jointly distributed random variables. show that U and W are independent if

$$pr\{U>u \text{ and } W>w\} = pr\{U>u\}\, pr\{W>w\}$$
for all $u, w$

Ans:

$\because$ U and W are jointly distributed r.v's such that

$$pr\{U>u, W>w\}$$
$$= [1-F_U(u)][1-F_W(w)] \quad \forall u,w \qquad (1)$$

$\Rightarrow$
$$pr\{U\le u, W\le w\}$$
$$= 1 - pr\{U>u \text{ or } W>w\}$$
$$= 1 - [pr\{U>u\} + pr\{W>w\} - pr\{U>u, W>w\}]$$
$$= 1 - [(1-F_U(u)) + (1-F_W(w)) - (1-F_U(u))(1-F_W(w))]$$
by using (1)
$$= \cancel{1} - [\cancel{1} - F_U(u) + \cancel{1} - F_W(w) - \cancel{1} + F_U(u) + F_W(w) - F_U(u)F_W(w)]$$
$$= F_U(u)\, F_W(w) \qquad (2)$$

$\therefore$ U and W are independent r.v's.

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Note that
A and B are independent events iff $pr(A\cap B) = pr(A)\,pr(B)$

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Random Variables U and V are independent and have the probability mass functions

$P_U(0) = \frac{1}{3}$, $P_V(1) = \frac{1}{2}$

$P_U(1) = \frac{1}{3}$, $P_V(2) = \frac{1}{2}$

$P_U(2) = \frac{1}{3}$

Determine the probability mass function of the sum $W = U + V$.

Ans.

To get the prob. mass fn of the sum $W = U + V$, we have to find the following probabilities

$pr\{W=1\}$, $pr\{W=2\}$, $pr\{W=3\}$ and $pr\{W=4\}$

$pr\{W=1\} = pr\{U=0, V=1\} = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}$

, $pr\{W=2\} = pr\{U=0, V=2\} + pr\{U=1, V=1\}$

$= \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

, $pr\{W=3\} = pr\{U=1, V=2\} + pr\{U=2, V=1\}$

$= \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

and $pr\{W=4\} = pr\{U=2, V=2\} = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}$

$\therefore$ $pr\{W=w\}$ is determined as

| $w$ | 1 | 2 | 3 | 4 | $\Sigma$ |
|---|---|---|---|---|---|
| $pr\{W=w\}$ | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | 1 |