Tut. Session (6)

Pb 3.1.3 p. 81 Textbook

A Markov chain $X_0,\ X_1,\ X_2,\ldots$ has the transition probability matrix

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}$$

If it is known that the process starts in state $X_0 = 1$, determine the probability $pr\{X_0 = 1, X_1 = 0, X_2 = 2\}$

Ans:

$$\because pr\{X_0 = 1, X_1 = 0, X_2 = 2\} = p_1 P_{10} P_{02}\ ,\quad p_1 = pr\{X_0 = 1\} = 1$$

$$= 1(0.3)(0.1)$$

$$\therefore pr\{X_0 = 1, X_1 = 0, X_2 = 2\} = 0.03$$

Pb 3.1.4 p. 82 Textbook

A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.1 & 0.1 & 0.8 \\ 0.2 & 0.2 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{array} \right\| \end{array}$$

Determine the conditional probabilities

$\Pr\{X_1 = 1,\ X_2 = 1 | X_0 = 0\}$ and $\Pr\{X_2 = 1,\ X_3 = 1 | X_1 = 0\}$.

Ans:

First, to find $\Pr\{X_1 = 1,\ X_2 = 1 | X_0 = 0\}$

$$\Pr\{X_1 = 1,\ X_2 = 1 | X_0 = 0\}$$

$$= \Pr\{X_2 = 1 | X_1 = 1, X_0 = 0\}.\Pr\{X_1 = 1 | X_0 = 0\} \text{ Conditional Prob. Property}$$

$$= \Pr\{X_2 = 1 | X_1 = 1\}.\Pr\{X_1 = 1 | X_0 = 0\} = P_{11} P_{01} \text{ Markov definition}$$

$$= 0.2(0.1)$$

$$= 0.02$$

Second, to find $\Pr\{X_2 = 1,\ X_3 = 1 | X_1 = 0\}$

$$\Pr\{X_2 = 1,\ X_3 = 1 | X_1 = 0\}$$

$$= \Pr\{X_3 = 1 | X_2 = 1, X_1 = 0\}.\Pr\{X_2 = 1 | X_1 = 0\} \text{ Conditional Prob. Property}$$

$$= \Pr\{X_3 = 1 | X_2 = 1\}.\Pr\{X_2 = 1 | X_1 = 0\} = P_{11}P_{01} \text{ Markov definition}$$

$$= 0.2(0.1)$$

$$= 0.02$$

Pb 3.1.5 p. 82 Textbook

A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\| \end{array}$$

and initial distribution p_0=0.5 and p_1=0.5. Determine the probabilities

$\text{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\}$ and $\text{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\}$.

Ans:

i) $\text{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \text{pr}\{X_0 = 1\} = 0.5$

$=0.5(0.1)(0.5)$

$=0.025$

ii) $\text{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \text{pr}\{X_1 = 1\} = ?$

$$\because \text{pr}\{X_1 = 1\} = \Pr(X_1 = 1 | X_0 = 0)\Pr(X_0 = 0) + \Pr(X_1 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2)\Pr(X_0 = 2)$$

$$= P_{01}p_0 + P_{11}p_1 + P_{21}p_2$$

$$\therefore \text{pr}\{X_1 = 1\} = 0.2(0.5)+0.1(0.5)+0.2(0)=0.15$$

$$\therefore \text{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = 0.15(0.1)(0.5) = 0.0075$$

Pb 3.1.2 p. 82 Textbook

Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α . Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the nth stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

(i) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.

(ii) Determine the probability that a correct signal is received at stage 2.

Ans:

The transition probability matrix can be written as

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{array} \right\| \end{array}$$

(i) The probability that no error occurs up to stage $n = 2$ is given as follows.

$p_0 = pr(X_0 = 0) = 1$

$$\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} = p_0 P_{00} P_{00}$$
$$= 1 \times (1-\alpha) \times (1-\alpha)$$
$$= (1-\alpha)^2$$

(ii) The probability that a correct signal is received at stage 2 is given as follows.

$$\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} + \Pr\{X_0 = 0, X_1 = 1, X_2 = 0\}$$
$$= p_0 P_{00} P_{00} + p_0 P_{01} P_{10}$$
$$= (1-\alpha)^2 + \alpha^2$$
$$= 1 - 2\alpha + 2\alpha^2$$

Pb 3.1.4 p. 83 Textbook

The random variables $\xi_1, \xi_2, \ldots$ are independent and with the common probability mass function

$k =$	0	1	2	3
$\Pr\{\xi = k\} =$	0.1	0.3	0.2	0.4

Set $X_0 = 0$, and let $X_n = \max\{\xi_1, \ldots, \xi_n\}$ be the largest ξ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.

Ans:

For transition probability matrix of a Markov chain

The elements of first row are given by

$$P_{0,0} = \Pr\{X_1 = 0\} = p_0 = 0.1$$
$$P_{0,1} = \Pr\{X_1 = 1\} = p_1 = 0.3$$
$$P_{0,2} = \Pr\{X_1 = 2\} = p_2 = 0.2$$
$$P_{0,3} = \Pr\{X_1 = 3\} = p_3 = 0.4$$

The elements of second row are given by

$$P_{1,0} = 0 \text{ where } X_n \text{ cannot decrease}$$
$$P_{1,1} = \Pr\{X_n = 1 | X_{n-1} = 1\} = \Pr\{\xi \le 1\} = 0.1 + 0.3 = 0.4$$
$$P_{1,2} = \Pr\{X_n = 2 | X_{n-1} = 1\} = \Pr\{\xi = 2\} = 0.2$$
$$P_{1,3} = \Pr\{X_n = 3 | X_{n-1} = 1\} = \Pr\{\xi = 3\} = 0.4$$

The elements of third row are given by

$$P_{2,0} = P_{2,1} = 0 \text{ where } X_n \text{ cannot decrease}$$
$$P_{2,2} = \Pr\{X_n = 2 | X_{n-1} = 2\} = \Pr\{\xi \le 2\} = 0.1 + 0.3 + 0.2 = 0.6$$
$$P_{2,3} = \Pr\{X_n = 3 | X_{n-1} = 2\} = \Pr\{\xi = 3\} = 0.4$$

The elements of fourth row are given by

$$P_{3,0} = P_{3,1} = P_{3,2} = 0 \text{ where } X_n \text{ cannot decrease}$$
$$P_{3,3} = \Pr\{X_n = 3 | X_{n-1} = 3\} = \Pr\{\xi \le 3\} = 0.1 + 0.3 + 0.2 + 0.4 = 1$$

The transition probability matrix will be of the form

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{Vmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \end{array}$$