Tutorial Session ④

1//

Random Sums

- **Theorem**

For the random sum $X = \xi_1 + \xi_2 + \cdots + \xi_k + \cdots + \xi_N$, $N > 0$

If $E(\xi_k) = \mu$, $Var(\xi_k) = \sigma^2$

, $E(N) = \nu$, $Var(N) = \tau^2$

then

$E(X) = \mu\nu$ and $Var(X) = \nu\sigma^2 + \mu^2\tau^2$

* pb 2.3.5 p.64 Textbook

let N is # accidents in a week, and ξ_k is # individuals injured for k^{th} accident. $X = \xi_1 + \xi_2 + \cdots + \xi_N$, $N > 0$

$N \sim$ poisson (2)

$E(N) = 2$, $Var(N) = 2$

& $\because$ $E(\xi_k) = 3$, $Var(\xi_k) = 4$

$\therefore$ $E(X) = \mu\nu = 3(2) = 6$,

$Var(X) = \nu\sigma^2 + \mu^2\tau^2 = 2(4) + 9(2) = 26$ #

* pb 2.3.5 p.65 Textbook

$\because$ $E(\xi_k) = \mu$, $Var(\xi_k) = \sigma^2$

and $\because$ $E(N+1) = E(N) + 1 = \nu + 1$,

$Var(N+1) = Var(N) + Var(1) = \tau^2 + 0 = \tau^2$

$\therefore$ For the random sum $X = \xi_0 + \xi_1 + \cdots + \xi_N$

$E(X) = \mu(\nu+1)$,

$Var(X) = (\nu+1)\sigma^2 + \mu^2\tau^2$

#

لا يكتب في
هذا الهامش

2

pb. 2.3.3 p. 65

Suppose that $\xi_1, \xi_2, \ldots$ are independent and identically distributed with $pr(\xi_k = \pm 1) = \frac{1}{2}$ let N be independent of $\xi_1, \xi_2, \ldots$ and follow the geometric probability mass function

$$p(k) = \alpha(1-\alpha)^k \quad \text{for } k = 0, 1, \ldots$$

where $0<\alpha<1$. Form the random sum $Z = \xi_1 + \xi_2 + \cdots + \xi_N$.
Determine the mean and Variance of Z.

Ans: we have

ξ_k	-1	1
$pr(\xi_k)$	$\frac{1}{2}$	$\frac{1}{2}$

$$\because \mu = E(\xi_k) = \sum \xi_k \, pr(\xi_k)$$

$$\therefore \mu = -1\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{2}\right) = 0$$

$$E(\xi_k^2) = (-1)^2\left(\tfrac{1}{2}\right) + 1^2\left(\tfrac{1}{2}\right) = 1$$

and $\sigma^2 = Var(\xi_k) = E(\xi_k^2) - [E(\xi_k)]^2$

$$= 1 - 0 = 1$$

$\because N \sim geom(\alpha)$, $P_N(k) = \alpha(1-\alpha)^k$, $k = 0, 1, \ldots$

$$\therefore \nu = E(N) = \frac{1-\alpha}{\alpha}, \quad \tau^2 = Var(N) = \frac{1-\alpha}{\alpha^2}$$

$$\because Z = \xi_1 + \xi_2 + \cdots + \xi_N$$

$$\therefore E(Z) = \mu\nu = 0\left(\frac{1-\alpha}{\alpha}\right) = 0$$

and $Var(Z) = \nu\sigma^2 + \mu^2\tau^2$

$$= \frac{1-\alpha}{\alpha}(1) + 0\left(\frac{1-\alpha}{\alpha^2}\right)$$

$$\therefore Var(Z) = \frac{1-\alpha}{\alpha}$$

$X \sim geom(p)$
$\Rightarrow pr(X=k) = p(1-p)^k$, $k = 0, 1, \ldots$
$\Rightarrow E(X) = \frac{1-p}{p}$, $Var(X) = \frac{1-p}{p^2}$

#

3

pb 2.4.3 p. 70

Let X have a poisson distribution with parameter $\lambda > 0$. Suppose λ itself is random, following an exponential density with parameter θ.

(a) What is the marginal distribution of X?

(b) Determine the conditional density for λ given $X = k$.

Ans:

(a)
$$pr(X=k) = \int_0^\infty pr(k|\lambda)\, f(\lambda)\, d\lambda$$
$$= \int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!}\, \theta\, e^{-\theta\lambda}\, d\lambda$$
$$= \frac{\theta}{k!}\int_0^\infty \lambda^k e^{-(1+\theta)\lambda}\, d\lambda$$
$$pr(X=k) = \frac{\theta}{k!}\int_0^\infty \left(\frac{u}{1+\theta}\right)^k e^{-u}\, \frac{du}{1+\theta}, \quad u = (1+\theta)\lambda$$
$$= \frac{\theta}{k!}\left(\frac{1}{1+\theta}\right)^k \frac{1}{1+\theta}\int_0^\infty u^k e^{-u}\, du$$
$$= \frac{\theta}{k!}\left(\frac{1}{1+\theta}\right)^k \left(\frac{1}{1+\theta}\right)\Gamma(k+1)$$
$$= \frac{\theta}{\cancel{k!}}\left(\frac{1}{1+\theta}\right)^k \left(\frac{1}{1+\theta}\right)\cancel{k!}$$
$$pr(X=k) = \left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)^k, \quad k = 0, 1, \ldots$$

$X \sim geom(p)$
$\Rightarrow pr(X=k) = p(1-p)^k, \ k = 0, 1, 2, \ldots$

which is the prob. mass fn for geometric distn with parameter $\frac{\theta}{1+\theta}$

$$\left\{\begin{array}{l} p \to \frac{\theta}{1+\theta} \\ 1-p \to 1 - \frac{\theta}{1+\theta} = \frac{1+\cancel{\theta}-\cancel{\theta}}{1+\theta} = \frac{1}{1+\theta} \end{array}\right.$$

i.e. $X \sim geom\left(\frac{\theta}{1+\theta}\right)$ #

(b) $\because$
$$P_{\lambda,X}(\lambda,k) = f_{\lambda|X}(\lambda|k)\, P_X(k)$$
$$= P_{X|\lambda}(k|\lambda)\, f(\lambda)$$

$$\therefore f_{\lambda|X}(\lambda|k) = \frac{P_{X|\lambda}(k|\lambda)\, f(\lambda)}{P_X(k)}$$

$$= \frac{\frac{e^{-\lambda}\lambda^k}{k!} \cdot \theta e^{-\theta\lambda}}{\left(\frac{\theta}{1+\theta}\right)\left(\frac{1}{1+\theta}\right)^k}$$

$$\therefore f_{\lambda|X}(\lambda|k) = \frac{(1+\theta)^{k+1}\lambda^k e^{-(1+\theta)\lambda}}{k!}$$

$$\therefore f_{\lambda|X}(\lambda|k) = \frac{(1+\theta)^{k+1}}{\Gamma(k+1)}\lambda^k e^{-(1+\theta)\lambda}$$

which is the gamma prob. density fn with parameters $(k+1, 1+\theta)$

Remember

$X \sim gamma(n,\lambda)$

$\Rightarrow f(x) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}, \quad x \geq 0$

where $\Gamma(n) = (n-1)!$, $n = 1, 2, \dots$

Not that

$x \rightarrow \lambda$
$\lambda \rightarrow 1+\theta$
$n \rightarrow k+1$

for gamma pdf

---

pb 2.4.7 p.71

Suppose that X and Y are independent random variables, each having the same exponential distribution with parameter α. What is the conditional probability density function for X, given that $Z = X + Y = z$?

Ans: $f_{X|Z}(x|z) = \frac{f_{X,Z}(x,z)}{f(z)}$ (1)

$\because X \sim exp(\alpha)$, $Y \sim exp(\alpha)$

and $Z = X + Y$

$\therefore Z \sim gamma(2, \alpha)$

$\therefore f(z) = \frac{\alpha^2}{\Gamma(2)} z e^{-\alpha z}$

$\therefore f(z) = \alpha^2 z e^{-\alpha z}$ (2)

$\because$ X and Y are independent r.v.s

$\therefore f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$\Rightarrow$

$\therefore f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$= \alpha^2 e^{-\alpha(x+y)}$

$\therefore f_{X,Y}(x,y) = \alpha^2 e^{-\alpha z}$

which equals the density fn $f_{X,Z}(x,z)$ for $0 \leq x \leq z$

$\therefore f_{X,Z}(x,z) = \alpha^2 e^{-\alpha z}$ (3)

Substitute (2) and (3) in (1), we get

$$f_{X|Z}(x|z) = \frac{\cancel{\alpha^2 e^{-\alpha z}}}{\cancel{\alpha^2} z \cancel{e^{-\alpha z}}} = \frac{1}{z}, \quad 0 \leq x \leq z$$

$\therefore X|Z$ is uniformly distributed over $[0, z]$.

#