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Tutorial Session (5)

لا يكتب في هذا الهامش

Q1 Suppose that $X_1, X_2 \ldots$ are identically independent distributed (i.i.d) random variables where $Pr(X_k = 1) = Pr(X_k = -1) = \frac{1}{2}$ and $S_n = \sum_{k=1}^{n} X_k$

(a) Show that S_n is a martingale

(b) Show that $S_n^2 - n$ is a martingale

• Answer

(a) $\boxed{1}$ to show that $E|S_n| < \infty$

$|S_n| = |X_1 + \cdots + X_n|$

$\leq |X_1| + \cdots + |X_n| = \underbrace{1 + \cdots + 1}_{n \text{ times}} = n$

$|X_k| = 1, \; k = 1, 2, \ldots$

[illegible]

$\therefore E[|S_n|] < \infty$

$\boxed{2}$ To show that $E[S_{n+1} | S_1, \ldots, S_n] = S_n$

$E[S_{n+1} | S_1, \ldots, S_n]$

$= E[S_n + X_{n+1} | S_1, \ldots, S_n]$

$= E[S_n | S_1, \ldots, S_n] + E[X_{n+1} | S_1, \ldots, S_n]$

$= S_n + E[X_{n+1}]$ as S_n is determined by $S_1, \ldots, S_n$ and X_{n+1} is independent of $S_1, \ldots, S_n$

$\therefore E[X_{n+1}] = (1)\, pr(X_{n+1} = 1) + (-1)\, pr(X_{n+1} = -1)$

$= 1(\frac{1}{2}) + (-1) \cdot (\frac{1}{2}) = 0$

$\therefore E[S_{n+1} | S_1, \ldots, S_n] = S_n + 0 = S_n$

$\therefore$ From $\boxed{1}$ and $\boxed{2}$, S_n is a martingale.

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(b) let $Y_n = S_n^2 - n$ where $S_n = \sum_{K=1}^{n} X_K$

$\boxed{1}$ To show that $E|Y_n| < \infty$

$$|Y_n| = |S_n^2 - n| \leq S_n^2 + n$$
$$\leq (|X_1| + \dots + |X_n|)^2 + n = n^2 + n$$
$$\underbrace{(1 + 1 + \dots + 1)^2}_{n \text{ times}}$$

$\therefore E[|Y_n|] < \infty$

$\boxed{2}$ To show that $E[Y_{n+1} | Y_1, \dots, Y_n] = Y_n$

$$E[Y_{n+1} | Y_1, \dots, Y_n] = E[S_{n+1}^2 - (n+1) | Y_1, \dots, Y_n]$$
$$= E[(S_n + X_{n+1})^2 - (n+1) | Y_1, \dots, Y_n]$$
$$= E[S_n^2 | Y_1, \dots, Y_n] + E[X_{n+1}^2 | Y_1, \dots, Y_n]$$
$$+ 2E[S_n X_{n+1} | Y_1, \dots, Y_n] - (n+1)$$

$$\therefore E[Y_{n+1} | Y_1, \dots, Y_n] = S_n^2 + E[X_{n+1}^2] + 2 S_n E[X_{n+1}] - n - 1$$

as S_n, S_n^2 are determined by $Y_1, \dots, Y_n$

and X_{n+1}, X_{n+1}^2 are independent of $Y_1, \dots, Y_n$

$$\therefore E[Y_{n+1} | Y_1, \dots, Y_n] = S_n^2 \cancel{+ 1} + \cancel{2 S_n (0)} - n \cancel{- 1}$$
$$= S_n^2 - n = Y_n$$

where $E[X_{n+1}] = 0$ as we proved before and

$$E[X_{n+1}^2] = 1^2 \, pr(X_{n+1} = 1) + (-1)^2 \, pr(X_{n+1} = -1)$$
$$= 1 \cdot (\tfrac{1}{2}) + 1 \cdot (\tfrac{1}{2}) = 1$$

$\therefore$ From $\boxed{1}$ and $\boxed{2}$, $Y_n = S_n^2 - n$ is a martingale. #