Logic is the hygiene the mathematician practices to keep his ideas healthy and strong

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CHAPTER I

THE FOUNDATIONS: LOGIC AND PROOFS

1 Propositional Logic

Our discussion begins with an introduction to the basic building blocks of logic propositions.

Definition 1.1

A proposition is a declarative sentence (that is, a sentence that declares a fact). The only statements that are considered are propositions, which contain no variables. Propositions are either true or false, but not both.

Example 1 :
All the following declarative sentences are propositions.

1. Any integer is odd or even.
2. $1 + 1 = 2$.
3. $2 + 2 = 3$.

Examples of non-propositions:

Example 2 :

1. What time is it?
2. $x + 1 = 2$, (may be true, may not be true, it depends on the value of x.)

3. $x.0 = 0$, (always true, but it's still not a proposition because of the variable.)

4. $x.0 = 1$, (always false, but not a proposition because of the variable.)

The truth value of a proposition is true, denoted by T, if it is a true proposition, and the truth value of a proposition is false, denoted by F, if it is a false proposition. The area of logic that deals with propositions is called the propositional calculus or propositional logic.
We will use letters such as $p, q, r, s, \ldots$ or $A, B, C, D, \ldots$ to represent propositions. The letters are called logical variables.

Propositions can be constructed from other propositions using logical connectives

1. Negation: $\neg$ (not alternatively $-$),
2. Conjunction $\wedge$ (and),
3. Disjunction $\vee$ (or),
4. Implication $\rightarrow$
5. Biconditional $\leftrightarrow$

The Negation of a Proposition

Definition 1.2

Let p be a proposition. The negation of p, denoted by $\neg p$ (also denoted by $\bar{p}$), is the statement "It is not the case that p." The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Example 3 :
The negation of the proposition "Badr's PC runs Linux "
The negation is:
"It is not the case that Badr's PC runs Linux." This negation can be more simply expressed as "Badr's PC does not run Linux."

The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

The conjunction of Propositions

Definition 1.3

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

The truth table for the conjunction of two propositions.

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

In this case, we will say that the compound propositions $p \wedge q$ and $q \wedge p$ are equivalent propositions. We also say that the operator $\wedge$ is commutative.

The disjunction of Propositions

Definition 1.4

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

The truth table for the disjunction of two propositions.

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

In this case, we will say that the compound propositions $p \vee q$ and $q \vee p$ are equivalent propositions. We also say that the operator $\vee$ is commutative.

The exclusive or of Propositions

Definition 1.5

Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

The truth table for the exclusive "or"

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

The conditional statement

Definition 1.6

Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and is true otherwise.

In the conditional statement $p \to q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).
The statement $p \to q$ is called a conditional statement because $p \to q$ asserts that q is true on the condition that p holds. A conditional statement is also called an implication.

When $p \to q$, p is called a sufficient condition for q , q is a necessary condition for p.

The statement $p \to q$ is true when both p and q are true and when p is false (no matter what truth value q has). Conditional statements play such an essential role in mathematical reasoning.

Terminology is used to express $p \to q$.

"if p, then q"	" p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless $\neg p$"	

The truth table for the conditional statement $p \to q$ of two propositions.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse, Contrapositive and Inverse

We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names.

1. The proposition $q \to p$ is called the **converse** of $p \to q$.
2. The proposition $\neg q \to \neg p$ is called the **contrapositive** of $p \to q$.
3. The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

We will see that of these three conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$.

Exercise 1 :
What are the contrapositive, the converse, and the inverse of the conditional statement
"The home team wins whenever it is raining".
Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as:
If it is raining, then the home team wins. Consequently, the **contrapositive** is:
If the home team does not win, then it is not raining.
The **converse** is : If the home team wins, then it is raining.
The **inverse** is : If it is not raining, then the home team does not win.
Only the contrapositive is equivalent to the original statement.
Biconditional Statements

Definition 1.7

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q". The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

The biconditional $p \leftrightarrow q$ is true when the conditional statements $p \to q$ and $q \to p$ are true, and is false otherwise.
That is why we use the words "if and only if" to express this logical connective and why it is symbolically written by combining the symbols $\to$ and $\leftarrow$.
The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation "iff" for "if and only if."
There are some other common ways to express $p \leftrightarrow q$:
"p is necessary and sufficient for q"
"if p then q", and conversely if "q then p".

The Truth Table for the Biconditional $p \leftrightarrow q$ of Two Propositions..

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Tables of Compound Propositions We have now introduced four important logical connectives: **conjunctions**, **disjunctions**, **conditional statements**, and **biconditional statements** as well as negations. We can use these connectives to build up complicated compound propositions involving any number of propositional variables. We can use truth tables to determine the truth values of these compound propositions.

Example 4 :

Construction of the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$.

Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$					
p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

2 Logical Equivalences

Definition 2.1

1. A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology. You can think of a **tautology** as a rule of logic.

2. A compound proposition that is always false is called a **contradiction**. In other words, a contradiction is false for every assignment of truth values to its simple components.

3. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

We will use T to denote any tautology and F to denote any contradiction.

Logical Equivalences Compound propositions that have the same truth values in all possible cases are called logically equivalent. We can also define this notion as follows.

Definition 2.2

The compound propositions p and q are called logically equivalent and denoted $p \equiv q$ if $p \leftrightarrow q$ is a tautology.

Example 5 :
We can construct examples of tautologies and contradictions using just one propositional variable.
$p \vee \neg p$ is always true, it is a tautology and $p \wedge \neg p$ is always false, it is a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Theorem 2.3

[De Morgan's Laws]
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$.

Truth Table of $\neg(p \wedge q)$ and $\neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$									
p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F	T	F	F
T	F	F	T	T	F	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	T	T	F	T	T	F	T	T

Exercise 2 :
Prove that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Solution

Truth Table of $p \to q \equiv \neg p \vee q$				
p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Exercise 3 :
Prove that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.

Solution

Truth Table of $p \to q \equiv \neg q \to \neg p$					
p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Exercise 4 :
Prove that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. (This is the distributive law of disjunction over conjunction.)

Solution

Truth Table of $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$							
p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Exercise 5 :
Prove that $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are logically equivalent. (This is the distributive law of conjunction over disjunction.)

Solution

Truth Table of $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$							
p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

Logical equivalences involving conditional statements

1. $p \to q \equiv \neg p \vee q$

2. $p \to q \equiv \neg q \to \neg p$

3. $p \to (q \vee r) \equiv (p \wedge \neg q) \to r \equiv (p \wedge \neg r) \to q$.

4. $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$.

5. $p \to (q \to r) \equiv (p \wedge q) \to r$.

Logical equivalences involving biconditional statements

1. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$,

2. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$,

3. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$.

Exercise 6 :
Prove that $(p \wedge q) \to (p \vee q)$ is a tautology.
Solution

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Proof without truth table:

$$\begin{aligned}(p \wedge q) \to (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q)\\ &\equiv \neg p \vee \neg q \vee p \vee q\\ &\equiv T.\end{aligned}$$

Exercise 7 :
Prove that $(p \wedge q) \wedge (\neg p \vee \neg q)$ is a contradiction. **Solution**

p	q	$p \wedge q$	$\neg p \vee \neg q$	$(p \wedge q) \wedge (\neg p \vee \neg q)$
T	T	T	F	F
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

Proof without truth table:

$$\begin{aligned}(p \wedge q) \wedge (\neg p \vee \neg q) &\equiv (p \wedge q \wedge \neg p) \vee (p \wedge q \wedge \neg q)\\ &\equiv F.\end{aligned}$$

Exercise Proof without truth table that

1. $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent,

2. $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$,

Solution

1.

$$\begin{aligned}\neg(p \to q) &\equiv \neg(\neg p \vee q)\\ &\equiv \neg(\neg p) \wedge \neg q\\ &\equiv p \wedge \neg q\end{aligned}$$

2.

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q)\\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]\\
&\equiv \neg p \wedge (p \vee \neg q)\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)\\
&\equiv F \vee (\neg p \wedge \neg q) \equiv \neg p \wedge \neg q
\end{aligned}$$

3 Predicates and Quantifiers

Predicates

Definition 3.1

A predicate is a statement that contains variables (predicate variables), and they may be true or false depending on the values of these variables.

Example 6 :
If $P(x) =$ "x^2 is greater than x" is a predicate. It contains one predicate variable x. If we choose $x \in]0, 1]$, $P(x)$ is a proposition always false.
If we choose $x \in]1, +\infty[$, $P(x)$ is a proposition always true.

Definition 3.2

1. The domain of a predicate variable is the collection of all possible values that the variable may take.
2. The truth domain of a predicate variable is the set of the variable that takes the predicate true.

The Universal Quantifier

Definition 3.3

The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain."
The notation $\forall x;\ P(x)$ denotes the universal quantification of $P(x)$.
The symbol $\forall$ is called the **universal quantifier**.
An element for which $P(x)$ is false is called a **counterexample** of $\forall x;\ P(x)$.

Example 7 :
This is a formal way to say that for all values that a predicate variable x can take in a domain D, the predicate is true:

$$\forall x \in \mathbb{R}, \quad x^2 \geq 0$$

for all x belonging to the real numbers $x^2 \geq 0$.

Example 8 :

Let $p(x)$ be the predicate "$x > 0$".
If $D = \mathbb{N}$ the proposition "$\forall\ x : x > 0$" is true.
If $D = \mathbb{Z}$ the proposition "$\forall\ x : x > 0$" is false.
So, the universal set is important.
The Existential Quantifier

Definition 3.4

The existential quantification of $P(x)$ is the proposition
"There exists an element x in the domain such that $P(x)$".
We use the notation $\exists x;\ P(x)$ for the existential quantification of $P(x)$.
The symbol $\exists$ is called the existential quantifier.

Example 9 :

$$\exists x \in \mathbb{R}, \quad x^2 > x.$$

Truth value of quantified statements

1. $\forall x \in D,\ P(x)$ i.e. $P(x)$ is true for every $x \in D$.
   It is false whenever there is at least one x in D for which $P(x)$ is false.
   $\neg(\forall x \in D,\ P(x)) \equiv \exists x \in D,\ \neg(P(x))$ i.e. There is one $x \in D$ for which $P(x)$ is false.

   Example 10 :

   If $D = \{x_1, \ldots, x_n\}$, we have the following equivalence:

   $$(\forall x \in D,\ P(x)) \equiv (P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)).$$

   $$\neg(\forall x \in D,\ P(x)) \equiv \exists j \in \{1, \ldots, n\},\ \neg P(x_j).$$

2. $\exists x \in D,\ P(x)$. It is true when $P(x)$ is true for at least one x in D. It is false when $P(x)$ is false for all x in D.
   $\neg(\exists x \in D,\ P(x)) \equiv \forall x \in D,\ \neg(P(x))$ i.e. for all $x \in D$, $P(x)$ is false.

   Example 11 :

   If $D = \{x_1, \ldots, x_n\}$, we have the following equivalence:

   $$(\exists x \in D,\ P(x)) \equiv (P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)).$$

   $$\neg(\exists x \in D,\ P(x)) \equiv \forall j \in \{1, \ldots, n\},\ \neg P(x_j).$$

3. We can similarly assign truth values to combinations of predicates, or negation of combinations of predicates. The equivalence

   $$\neg(\forall x \in D,\ P(x) \wedge Q(x)) \equiv \exists x \in D,\ \neg(P(x) \wedge Q(x)),$$

   or

   $$\neg(\forall x \in D,\ P(x) \wedge Q(x)) \equiv \exists x \in D,\ \neg P(x) \vee \neg Q(x).$$

How To Determine Truth Value

1. **Method of exhaustion**
   Suppose that the domain D is finite and we want to show that $\forall x \in D$, $P(x)$ is true. Try all cases.
   For example, if $D = \{-1, 2, 3\}$, and $P(x) =$ "$x^2 - x - 1 \geq 0$", then just compute $x^2 - x + 1$ for all the values of $x \in D$ to conclude that this true or false.

2. **Method of case**
   Suppose you want to show that $(\exists x \in D,\ P(x))$ is true. For this, we just need to find an $x \in D$ for which $P(x)$ is true.

   Example 12 :

   To show that "$\exists x \in [-1,1],\ x^3 = 2x$" is true, take $x = 0$.
   Similarly, if to show that $\forall\ x \in [1, +\infty[, x^3 > 2x$ is false, it is enough to find one counterexample. Take $x = 1$.

3. **Method of logic derivation** This method consists of using logical steps to transform one logical expression into another.

Example 13 :
If $D = \{x_1, \ldots x_n\}$ and we want to know the truth value of $\exists x \in D,\ (P(x) \vee Q(x))$. This is equivalent to the following:

$$\begin{aligned}\exists x \in D,\ (P(x) \vee Q(x)) &\equiv \exists j \in \{1, \ldots, n\},\ P(x_j) \vee Q(x_j) \\ &\equiv (P(x_1) \vee Q(x_1)) \vee \ldots \vee (P(x_n) \vee Q(x_n)) \\ &\equiv (\exists x \in D,\ (P(x)) \vee (\exists x \in D,\ Q(x)).\end{aligned}$$

Examples

1. Let $P(x)$ be the statement "$x^2 - x + 1 > 0.$" What is the truth value of the quantification $\forall x\ P(x)$, where the domain consists of all real numbers?
   $P(x)$ is true for every real number x, because, $P(x) = (x - \frac{1}{2})^2 + \frac{3}{4}$.

2. Let $Q(x)$ be the statement "$x^2 - 3x + 1 > 0.$" What is the truth value of the quantification $\forall x;\ P(x)$, where the domain consists of all real numbers?
   $Q(x)$ is not true for every real number x, for instance, $Q(1)$ is false. That is, $x = 1$ is a counterexample for the statement $\forall x;\ Q(x)$. Thus $\forall x;\ Q(x)$ is false.

Examples Find the negations of the following statements

1. $\forall x;\ (\sin x \leq x)$,

2. $\exists x;\ (\sin x = x)$.

Solution

1. The negation of $\forall x;\ (\sin x \leq x)$ is the statement $\exists x;\ (\sin x > x)$.

2. The negation of $\exists x;\ (\sin x = x)$ is the statement $\forall x;\ (\sin x \neq x)$.

4 Proof Techniques

Proof Techniques

Definition 4.1

1. A proof is a chain of deductions that establishes the truth of a statement.

2. A theorem is a statement obtained by a correct deduction or a sequence of correct deductions (that is, using explicitly the allowed rules of inference) from logical axioms and, possibly, other results of the same type already established.

3. A lemma is a "helping theorem" or a result which is needed to prove a theorem.

Definition 4.2

4. A corollary is a result which follows directly from a theorem.

5. Less important theorems are sometimes called propositions.

Remarks 1 :

1. Many theorems assert that a property holds for all elements in a domain, such as the integers, the real numbers, or some of the discrete structures that we will study in this class.

2. Many theorems have the form: $\forall x \in D$; $(P(x) \rightarrow Q(x))$ is true. To prove, we show that for all arbitrary element c of the domain, $P(c) \rightarrow Q(c)$ is true. So, we must prove something of the form: $p \rightarrow q$ is true.
   We have two trivial cases
   - Trivial Proof: If we know that q is true, then $p \rightarrow q$ is true as well.
   - Vacuous Proof: If we know that p is false then $p \rightarrow q$ is true.

In which follows we will see how to apply the logic rules to justify different proof techniques. We will discuss four proof techniques: **Direct proof, proof by induction, proof by contradiction and proof by contradiction.**

Direct Proof A **direct proof** of a conditional statement $p \rightarrow q$ is constructed when the first step is the assumption that p is true, subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.
A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs. In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.

Example 14 :
Give a direct proof of "If n is an odd integer, then n^2 is odd."
Solution
Note that the statement is $\forall n$; $(P(n) \rightarrow Q(n))$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd."
We assume that the hypothesis of this conditional statement is true, namely, we assume that $n = 2k + 1$ for some $k \in \mathbb{N}$.
$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is also odd.

Example 15 :
Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a perfect square if there is an integer b such that $a = b^2$.)

Solution
We assume that the hypothesis of this conditional statement is true, namely, we assume that $m = p^2$ and $n = q^2$. Hence, $mn = p^2q^2 = (pq)^2$, then mn is also a perfect square.

Proof by Contraposition We need other methods of proving statements of the form $\forall x\ (P(x) \rightarrow Q(x))$. The method now is to do not start with the premises and end with the conclusion. This method is called **a indirect proof**.
A useful type of indirect proof is known as **proof by contraposition**. This method consists to use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$. We take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, to show that $\neg p$ is true.

Example 16 :

1. Prove $3n + 2$ is odd, then n is odd.

2. Prove that if $n = pq$, then $p \leq \sqrt{n}$ or $q \leq \sqrt{n}$. $(n, p, q \in \mathbb{N})$.

Solutions

1. If $n = 2k$ for some integer k, then $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$ is even.

2. We assume that the statement $(p \le \sqrt{n}) \vee (q \le \sqrt{n})$ is false, which means that both $p > \sqrt{n}$ and $q > \sqrt{n}$ are false. Then $n = pq > \sqrt{n}\sqrt{n} = n$ which is false.

Proof by Contradiction Suppose we want to prove that a statement p is true. Furthermore, suppose that we can find a contradiction q such that $\neg p \to q$ is true. We conclude that $\neg p$ is false, which means that p is true.

Example 17 :
$\sqrt{2}$ is irrational.
Assume that $\sqrt{2}$ is rational. There exist integers p and q with $\sqrt{2} = \frac{p}{q}$, where $q \neq 0$ and p and q have no common factors. When both sides of this equation are squared, it follows that $2q^2 = p^2$. Hence p^2 is even, let $p = 2r$ for some integer r. Thus, $2q^2 = 4r^2$ and $q^2 = 2r^2$ and q is even, which is impossible. Then $\sqrt{2}$ is irrational.

Example 18 :
Prove by contradiction that "if $3n + 2$ is odd, then n is odd."
Let p be "$3n + 2$ is odd" and q be "n is odd." To construct a proof by contradiction, assume that both p and $\neg q$ are true. That is, assume that $3n+2$ is odd and n is even. Then there is $k \in \mathbb{N}$ such that $n = 2k$. This implies that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. We have a contradiction. This completes the proof by contradiction.

Example 19 :
Prove that there is no largest prime number.
Assume that there is a largest prime number. Hence, we can list all the primes $p_1, \dots, p_n$. Let

$$m = p_1.p_2 \dots .p_n - 1.$$

None of the prime numbers on the list divides m. Therefore, either m is prime or there is a smaller prime that divides m. This contradicts the assumption that there is a largest prime. Therefore, there is no largest prime.

Proof of Equivalence To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \to q$ and $q \to p$ are both true. The validity of this approach is based on the tautology $(p \leftrightarrow q) \leftrightarrow (p \to q) \wedge (q \to p)$.

Example 20 :
Prove that n is odd if and only if n^2 is odd."
Let p is "n is odd" and q is "n^2 is odd." To prove this result, we need to show that $p \to q$ and $q \to p$ are true.
We have already shown that $p \to q$ is true.
We prove that $q \to p$ by contraposition.
If n is odd, then $n = 2k - 1$ for some integer k.
$n^2 = 4k^2 - 4k + 1 = 2(2k^2 - 2k) + 1$, which proves that q is false.

Proof by cases A proof by cases of a mathematical statement should include the following:

1. Determine all possible cases which must be considered in order to prove the mathematical statement,
2. write the proof for each case.

Example 21 :
Prove that 3 does not divide the numbers $n^2 + 2n - 1$ for all $n \in \mathbb{N}$.
Case 1: Assume that $n = 3k$, then $n^2 + 2n - 1 = 3(3k^2 + 2k) - 1$ and 3 does not divide $n^2 + 2n - 1$.
Case 2: Assume that $n = 3k + 1$, then $n^2 + 2n - 1 = 3(3k^2 + 4k) + 2$ and 3 does not divide $n^2 + 2n - 1$.
Case 3: Assume that $n = 3k + 2$, then $n^2 + 2n - 1 = 3(3k^2 + 6k + 2) + 1$ and 3 does not divide $n^2 + 2n - 1$.

Example 22 :
Prove that $x + |x - 1| \geq 1$ for all real numbers x.
Case1: Assume that $x \geq 1$. Then $x - 1 \geq 0$ and $|x - 1| = x - 1$, so that $x + |x - 1| = x + (x - 1) = 2x - 1 \geq 2 - 1 = 1$, i.e., $x + |x - 1| \geq 1$.
Case2: Assume that $x < 1$. Then $x - 1 < 0$ and $|x - 1| = -(x - 1) = 1 - x$, so that $x + |x - 1| = x + (1 - x) = 1$, i.e., $x + |x - 1| \geq 1$.
Thus, for all possible cases, it has been proven that $x + |x - 1| \geq 1$.

Exercise 8 :

1. If x is a real number, then $|x+3| - \ x > 2$.
2. If x is a real number, then $|x-1| + |x+5| \ \geq 6$.
3. The expression $2m^2 - 1$ is odd for all integers m .
4. If n is an even integer, then $n = 4k$ or $n = 4k-2$ for some integer k .
5. If a and b are real numbers, then $|\ |a| - |b|\ | \ \leq \ |a-b|$.

Proof by working backward A proof by working backward of a mathematical statement should include the following.

1. Begin with the final result, which must be proven true,
2. work backward step-by-step, writing equivalent statements, until a connection with the assumptions of the problem.

Example 23 :
Prove that $x^2 + y^2 \geq 2xy$ for all real numbers x and y.

$x^2 + y^2 \geq 2xy$ is true if and only if $x^2 + y^2 - 2xy \geq 0$ is true if and only if $(x-y)^2 \geq 0$ which is true. Since the last statement is true, all of the equivalent statements are true. In particular, $x^2 + y^2 \geq 2xy$.

Exercise 9 :
Prove the following:

1. The expression $x + \dfrac{9}{x} \geq 6$ for all real numbers $x > 0$.
2. If $n^3 + 5n + 6$ is divisible by 3 for some integer n, then $(n+1)^3 + 5(n+1) + 6$ is divisible by 3.
3. The expression $\dfrac{x^4}{4} + (x+1)^3 > \dfrac{(x+1)^4}{4}$ for all real numbers $x \geq -1$.
4. There is a fixed positive integer N for which $\dfrac{3}{n} < \dfrac{n-4}{n+10}$ for all integers $n \geq N$.

5 Mathematical Induction

Mathematical Induction This proof technique is to prove statements of the form $\forall n \in \mathbb{N}, P(n)$. We have two steps to do the proof:

1. We show that $P(1)$ is true. (Basis step)
2. Assume that $P(n)$ is true for some $n \geq 1$ (induction hypothesis) and prove that $P(n+1)$ is then true. (Inductive step)

Example 24 :
$S_n = \displaystyle\sum_{k=1}^{n} (2k-1)^2 = \frac{n(4n^2-1)}{3}$.

$S_1 = 1$ and assume that $S_n = \dfrac{n(4n^2-1)}{3}$ for some $n \in \mathbb{N}$.

$$\begin{aligned} S_{n+1} &= S_n + (2n+1)^2 = \frac{n(4n^2-1)}{3} + (2n+1)^2 \\ &= (2n+1)\frac{n(2n-1) + 3(2n+1)}{3} \\ &= \frac{(n+1)(4(n+1)^2-1)}{3}. \end{aligned}$$

Example 25 :
$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.
$S_1 = 1$ and assume that $S_n = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ for some $n \in \mathbb{N}$.

$$\begin{aligned} S_{n+1} &= S_n + (n+1)^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + (n+1)^4 \\ &= (n+1)\frac{n(2n+1)(3n^2+3n-1)+30(n+1)^3}{30} \\ &= \frac{(n+1)(n+2)(2n+3)(3(n+1)^2+3(n+1)-1)}{30}. \end{aligned}$$

Example 26 :
$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.
$S_1 = \frac{1}{2}$ and assume that $S_n = \frac{n}{n+1}$ for some $n \in \mathbb{N}$.

$$\begin{aligned} S_{n+1} &= S_n + \frac{1}{(n+1)(n+2)} \\ &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= \frac{1}{n+1}(n + \frac{1}{n+2}) = \frac{n+1}{n+2}. \end{aligned}$$

Example 27 :
Let $(a_n)_n$ be a sequence defined by $a_0 \in (0,1]$, and for all $n \geq 0$, $a_{n+1} = \frac{2a_n + a_n^2}{4}$.
Prove that $\forall n \geq 0$, $a_n \in (0,1]$ and $a_n \leq a_{n+1}$.
Base step: $a_0 \in (0,1]$.
Inductive step:
Assume that $a_n \in (0,1]$. Then $0 < a_{n+1} = \frac{2a_n + a_n^2}{4} \leq \frac{3}{4} \leq 1$.
Then $a_n \in (0,1]$ for all $n \geq 0$.
Base step: $a_1 = \frac{2a_0 + a_0^2}{4} \leq = \frac{2a_0 + a_0}{4} = \frac{3}{4}a_0 \leq a_0$.
Inductive step:
Assume that $a_n \leq a_{n-1}$. Then $a_{n+1} = \frac{2a_n + a_n^2}{4} \leq \frac{3}{4}a_n \leq a_n$.
Then $a_{n+1} \leq a_n$ for all $n \geq 0$.

Strong Induction We want to prove $\forall n \in \mathbb{N}$, $(P(n))$ is true.
Basis step: show $P(1)$ is true.
Inductive step: show $(P(1) \wedge \ldots \wedge P(n)) \rightarrow P(n+1)$ for all $n \in \mathbb{N}$. (Assume n is arbitrary and $P(1), \ldots, P(n)$ are true. Show $P(n+1)$ is true.)

Example 28 :
Let $(a_n)_n$ be a sequence defined by $a_1 = 2$, $a_2 = 4$ and for all $n \geq 3$, $a_n = a_{n-1} + 2a_{n-2}$.
Prove that $\forall n \geq 1$, $a_n = 2^n$.
Base step: $a_1 = 2 = 2^1$, $a_2 = 4 = 2^2$.
Inductive step:
Assume for $n \geq 3$ fixed but random that: $a_{n-1} = 2^{n-1}$ and $a_{n-2} = 2^{n-2}$.
Then $a_n = a_{n-1} + 2a_{n-2} = 2^{n-1} + 2^{n-1} = 2^n$.

6 Exercises

Exercise 10 :

For each of the following decide whether it is a proposition or not, and if it is, indicate whether it is true or false.

1. 15 is a positive number,
2. The Earth is flat,
3. $x^2 \geq 0$,
4. The next sentence is true,
5. The previous sentence is false.

Exercise 11 :
If p and q are false and r and s are true what are the truth-values of the following propositions?

1. $(\neg p) \wedge (r \vee (\neg q))$,
2. $(r \wedge (\neg s)) \vee (p \vee (\neg(\neg(q)))$.

Exercise 12 :
Which of the following propositional forms are tautologies?

1. $(p \vee q) \rightarrow (q \vee p)$,
2. $p \rightarrow ((p \vee q) \vee r)$,
3. $p \rightarrow (q \rightarrow (q \rightarrow p))$,
4. $((p \rightarrow q) \leftrightarrow q) \rightarrow p$,
5. $(p \wedge q) \rightarrow (p \vee r)$,
6. $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$.

Exercise 13 :
Prove that

1. $\neg(p \rightarrow q) \equiv p \wedge \neg q$,
2. $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.

Exercise 14 :
Prove that the following propositional forms are equivalent.

1. $\neg(p \wedge q)$ and $(\neg p) \vee (\neg q)$
2. $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$.

Exercise 15 :
Give the truth tables of the following propositional forms:

1. $(\neg p) \vee (q \wedge (\neg r))$,
2. $(p \vee (\neg q)) \wedge ((\neg p) \wedge q)$,
3. $((\neg p) \vee (\neg q)) \vee (p \wedge q)$,
4. $[(p \wedge (\neg q)) \vee (q \wedge (\neg r))] \vee (r \wedge (\neg p))$.

   Which of these are tautologies and which are contradictions?

Exercise 16 :
Rewrite $p \leftrightarrow q$ and $(p \to q) \to q$ without $\to$.

Exercise 17 :
Prove the following equivalence without truth table:

1. $q \vee ((\neg p) \wedge (\neg q)) \equiv \neg(p \wedge (\neg q))$,
2. $(p \vee q \vee r) \wedge (p \vee (\neg q) \vee r) \equiv p \vee r$,
3. $p \wedge p \equiv p$, (Hint: $p \equiv p \wedge T$ and use another law on T.)
4. $p \vee p \equiv p$, (Hint: $p \equiv p \vee F$ and use another law on F.)
5. $(p \wedge q) \wedge (p \vee p) \equiv p \wedge q$. (Hint: use a distributive law on the left hand side and then apply results from earlier parts of this question.)
6. $r \to (p \to q) \equiv (p \wedge r) \to q$,
7. $\neg q \to \neg p \equiv p \to q$.

Exercise 18 :
Show that $p \wedge (q \vee r)$ and $(p \wedge q) \vee r$ are not equivalent and that $(p \wedge (q \vee r)) \to ((p \wedge q) \vee r)$ is a tautology.

Exercise 19 :
Prove that

1. $p \to q \equiv (p \wedge (\neg q)) \to F$.
2. Find a form equivalent to $p \leftrightarrow q$ which only uses the $\neg$, $\wedge$ and $\vee$ connectives.

Exercise 20 :
Prove the following logical equivalences involving conditional statements without using truth table.

1. $\neg p \to q \equiv p \vee q$
2. $\neg(p \to \neg q) \equiv p \wedge q$
3. $\neg(p \to q) \equiv p \wedge \neg q$
4. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
5. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
6. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
7. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Exercise 21 :
Prove the following logical equivalences involving biconditional statements.

1. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$,
2. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$,
3. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$,
4. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$.

$$
\begin{aligned}
\neg(p \leftrightarrow q) &\equiv \neg\left((p \rightarrow q) \wedge (q \rightarrow p)\right) \\
&\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee p)) \\
&\equiv (p \wedge \neg q) \vee (q \wedge \neg p)) \\
&\equiv ((p \wedge \neg q) \vee q) \wedge ((p \wedge \neg q) \vee \neg p) \\
&\equiv ((p \wedge q) \vee (\neg q \wedge q)) \wedge ((p \wedge \neg p) \vee (\neg q \wedge \neg p)) \\
&\equiv (p \wedge q) \wedge (\neg q \wedge \neg p) \\
&\equiv p \leftrightarrow \neg q.
\end{aligned}
$$

Exercise 22 :
Prove that $((\neg r) \wedge p \wedge q) \vee ((\neg r) \wedge (\neg p) \wedge q) \equiv (\neg r) \wedge q$.

Exercise 23 :
In the following predicates find examples of values for the variables that make the resulting proposition true.

1. $X = \mathbb{R}$
   $p(x, y)$: $x + y = 0$,
2. $X = \{1, 2, 3, 4, 5\}$
   $p(x, y, z)$: $x^2 + y^2 = z^2$,
3. $X = \mathbb{Z}$
   $p(x, y)$: $(x > y) \wedge (x^2 < y^2)$,
4. $X = \mathbb{N}$,
   $p(x, y)$: $(x + y = 0) \vee (xy \leq 1)$.

Exercise 24 :
Let $X = \mathbb{R}$ and $P(x, y)$: $xy = 4$, and $Q(x, y)$: $x > y$.
Write out the following propositions as mathematical statements. Indicate, with reasons, which are true and which are false. If false, give a counterexample.

1. $P(8, \frac{1}{2})$,
2. $\exists y : P(2, y)$,
3. $\forall\ x : Q(x, y)$,
4. $\exists x, y$: $[P(x, y) \rightarrow Q(x, y)]$,

Exercise 25 :
Negate and simplify the following

1. $\forall x$: $x^2 > 0$,
2. $\exists\ x$: $2x = 1$,
3. $\forall\ x$: $P(x) \vee Q(x)$,

Exercise 26 :
Prove by induction that

1. for all $n \in \mathbb{N}$, $n < 2^n$,
2. for all integers $n \geq 4$, $2^n < n!$,
3. for all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.

Exercise 27 :
Use induction to prove:

1. For all $n \in \mathbb{N}$,
$$\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2).$$
2. For all $n \in \mathbb{N}$,
$$\sum_{k=1}^{n} \frac{4}{k(k+1)(k+2)} = 1 - \frac{2}{(n+1)(n+2)}.$$
3. For all $n \in \mathbb{N}$,
$$\sum_{k=1}^{n} k^3 = \frac{1}{4}[n(n+1)]^2.$$

Exercise 28 :

4. If $u_k = 2u_{k-1} + 1$ for all $k \in \mathbb{N}$, show that
$$u_n = 2^{n-1}(u_1 + 1) - 1.$$
Given $u_1 = 1$, evaluate $\sum_{k=1}^{n} u_k$.
5. $3^{2n} + 7$ is divisible by 8 for all $n \in \mathbb{N}$.
6. $(3n+1)7^n - 1$ is divisible by 9 for all $n \in \mathbb{N}$.
7. $2^{n-1} \leq n!$ for all $n \in \mathbb{N}$.
8. $(n+1)^n \leq n^{n+1}$ for all $n \geq 2$.
9. $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$ and $x > -1$.

Exercise 29 :

1. $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
for any $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$. Here $\binom{n}{k}$ is the *binomial coefficient,* defined by
$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$
where $k! = k(k-1)(k-2)\cdots(2)(1)$ for any natural number k, and $0! = 1$.

Exercise 30 :
Prove that the numbers $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, $\sqrt{p}$ are all not rational, where p is a prime number.

Exercise 31 :
Let $(u_n)_n$ be the sequence defined by: $u_0 = u_1 = -1$ and $u_{n+1} = 5u_n - 6u_{n-1}$ for all $n \in \mathbb{N}$.
Prove that $u_n = 3^n - 2^{n+1}$ for all $n \in \mathbb{N}$.

Exercise 32 :
Let $(u_n)_n$ be the sequence defined by: $u_0 = 2$, $u_1 = 3$ and $u_{n+1} = 3u_n - 2u_{n-1}$ for all $n \in \mathbb{N}$.
Prove that $u_n = 1 + 2^n$ for all $n \in \mathbb{N}$.

Exercise 33 :

Let $(u_n)_n$ be the sequence defined by: $u_0 = 1$, $u_1 = 4$ and $u_{n+1} = 2u_n - u_{n-1}$ for all $n \in \mathbb{N}$. Prove that $u_n = 3n + 1$ for all $n \in \mathbb{N}$.

Exercise 34 :
Let $(u_n)_n$ be the sequence defined by: $u_0 = u_1 = 1$ and $u_{n+1} = u_n + \frac{2}{n+1}u_{n-1}$ for all $n \in \mathbb{N}$. Prove that $1 \leq u_n \leq n^2$ for all $n \in \mathbb{N}$.

Exercise 35 :
Let $(u_n)_n$ be the sequence defined by: $u_0 = 1$ and $u_{n+1} = \sum_{k=0}^{n} u_k$ for all $n \geq 0$.

Prove that $u_n = 2^{n-1}$ for all $n \in \mathbb{N}$.

Exercise 36 :
Prove by induction that

1. $2^{n+1} > n$ for all $n \in \mathbb{N}$.
2. $n! \geq n^2$.
3. $\sum_{k=0}^{n} k2^k = (n-1)2^{n+1} + 1$.
4. $\forall n \geq 15,\ \frac{3^n}{n!} \leq \frac{1}{2^n}$.

Exercise 37 :
A set of measuring cups includes a 4−cup, 9−cup, 11−cup, and 14−cup measure. Show that this set can be used to measure out any number of cups greater than or equal to 11.

Exercise 38 :
Prove by induction that

1. $7^n - 1$ is divisible by 6, for all $n \geq 1$.
2. $11^n - 6$ is divisible by 5, for all $n \geq 1$.
3. $6.7^n - 2.3^n$ is divisible by 4, for all $n \geq 1$.
4. $\frac{1}{2} + \frac{2}{3} + \ldots + \frac{n}{n+1} < \frac{n^2}{n+1}$, for all $n \geq 2$.
5. $1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{2^n} = \sum_{k=1}^{2^n} \frac{1}{k} > 1 + \frac{n}{2}$, for all $n \geq 1$.
6. $1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)$, for all $n \geq 1$.

Exercise 39 :
Use mathematical induction to show that postage of 4 SAR or more can be achieved by using only 2 SAR and 5 SAR stamps.

Exercise 40 :
Prove by induction that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 2^n \\ 2^n & 2^n \end{pmatrix}$.

Solution to Exercise 1:

1. 15 is a positive number, (Is a proposition and true)
2. The Earth is flat, (Is a proposition and false)
3. $x^2 \geq 0$, (Is not a proposition)

4. The next sentence is true, (Is not a proposition)

5. The previous sentence is false. (Is not a proposition).

Solution to Exercise 2:

1. $(\neg p) \wedge (r \vee (\neg q))$ is true,

2. $(r \wedge (\neg s)) \vee (p \vee (\neg(\neg(q)))$ is true.

Solution to Exercise 3:

1. $(p \vee q) \rightarrow (q \vee p)$ is a tautology.

2. If p is true, then $(p \vee q) \vee r$ is also true. Then $p \rightarrow ((p \vee q) \vee r)$ is a tautology.

3. $p \rightarrow (q \rightarrow (q \rightarrow p))$. If p is true and q is false, then $q \rightarrow p$ is true and $q \rightarrow (q \rightarrow p)$ is also true. Then $p \rightarrow (q \rightarrow (q \rightarrow p))$ is a tautology.

4. $((p \rightarrow q) \leftrightarrow q) \rightarrow p$. If p is false and q is true, then $p \rightarrow q$ is true, $(p \rightarrow q) \leftrightarrow q$ is true and $((p \rightarrow q) \leftrightarrow q) \rightarrow p$ is false. Then $p \rightarrow (q \rightarrow (q \rightarrow p))$ is not a tautology.

5. $(p \wedge q) \rightarrow (p \vee r)$ is a tautology.

6. $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$ is not a tautology.

Solution to Exercise 4:

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$p \wedge \neg q$	$\neg p \wedge q$	$\neg(p \vee (\neg p \wedge q))$	$\neg p \wedge \neg q$
T	T	T	T	T	F	F	F
T	F	F	F	F	F	T	T
F	T	T	F	F	F	T	T
F	F	T	F	F	T	F	F

Proof without truth table:

$$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q.$$

$$\begin{aligned} \neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q)) \\ &\equiv \neg p \wedge (p \vee \neg q) \\ &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \\ &\equiv \neg p \wedge \neg q. \end{aligned}$$

Solution to Exercise 5:

1.

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$(\neg p) \vee (\neg q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

2.

p	q	r	$p \vee q$	$p \vee r$	$q \wedge r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

Solution to Exercise 6:

1.

Truth Table of $(\neg p) \vee (q \wedge (\neg r))$						
p	q	r	$\neg p$	$\neg r$	$q \wedge (\neg r)$	$(\neg p) \vee (q \wedge (\neg r))$
T	T	T	F	F	F	T
T	T	F	F	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	F	F
F	T	T	T	F	F	T
F	T	F	T	T	F	T
F	F	T	T	F	F	T
F	F	F	T	T	F	F

2.

Truth Table of $(p \vee (\neg q)) \wedge ((\neg p) \wedge q)$							
p	q	r	$\neg q$	$p \vee (\neg q)$	$\neg p$	$(\neg p) \wedge q$	$(p \vee (\neg q)) \wedge ((\neg p) \wedge q)$
T	T	T	T	T	F	F	F
T	T	F	T	T	F	F	F
T	F	T	T	T	F	F	F
T	F	F	F	T	F	F	F
F	T	T	T	F	T	T	F
F	T	F	T	T	T	T	T
F	F	T	T	T	T	F	F
F	F	F	F	F	T	F	F

3.

Truth Table of $((\neg p) \vee (\neg q)) \vee (p \wedge q)$						
p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \vee \neg q$	$((\neg p) \vee (\neg q)) \vee (p \wedge q)$
T	T	F	F	T	F	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

4. Let $S = [(p \wedge (\neg q)) \vee (q \wedge (\neg r))] \vee (r \wedge (\neg p))$

Truth Table of $[(p \wedge (\neg q)) \vee (q \wedge (\neg r))] \vee (r \wedge (\neg p))$							
p	q	r	$\neg q$	$p \wedge (\neg q)$	$r \wedge \neg p$	$(\neg r) \wedge q$	Q
T	T	T	T	F	F	F	F
T	T	F	T	F	T	T	T
T	F	T	T	T	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	F	T	F	T
F	T	F	T	F	F	T	T
F	F	T	T	F	T	F	T
F	F	F	F	F	F	F	F

Solution to Exercise 7:

$$\begin{aligned} p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) \\ &\equiv ((\neg p) \vee q) \wedge ((\neg q) \vee p). \end{aligned}$$

$$\begin{aligned} (p \to q) \to q &\equiv ((\neg(p \to q)) \vee q) \\ &\equiv ((p \wedge (\neg q)) \vee q) \\ &\equiv ((p \vee q) \wedge ((\neg q) \vee q)) \\ &\equiv (p \vee q) \wedge I \\ &\equiv p \vee q. \end{aligned}$$

Solution to Exercise 8:

1. $q \vee ((\neg p) \wedge (\neg q)) \equiv (q \vee (\neg p)) \wedge (q \vee (\neg q)) \equiv (q \vee (\neg p)) \equiv \neg(p \wedge (\neg q))$,

2. $$\begin{aligned} (p \vee q \vee r) \wedge (p \vee (\neg q) \vee r) &\equiv ((p \vee r) \vee q) \wedge ((p \vee r) \vee (\neg q)) \\ &\equiv (p \vee r) \vee ((p \vee r) \vee (\neg q)) \vee ((p \vee r) \vee q) \vee (q \vee (\neg q)) \\ &\equiv p \vee r, \end{aligned}$$

3. $p \equiv p \wedge T \equiv p \wedge (p \vee \neg p) \equiv (p \wedge p) \vee (p \wedge \neg p) \equiv p \wedge p$.

4. $p \equiv p \vee F \equiv p \vee (p \wedge (\neg p)) \equiv (p \vee p) \wedge (p \vee (\neg p)) \equiv p \vee p$.

5. $(p \wedge q) \wedge (p \vee q) \equiv (p \wedge q \wedge p) \vee (p \wedge q \wedge q) \equiv (p \wedge q) \vee (p \wedge q) \equiv p \wedge q$.

6. $r \to (p \to q) \equiv \neg r \vee (\neg p \vee q) \equiv (\neg r \vee \neg p) \vee q \equiv \neg(p \wedge r) \vee q \equiv (p \wedge r) \to q$,

7. $\neg q \to \neg p \equiv q \vee \neg p \equiv \neg p \vee q \equiv p \to q$.

Solution to Exercise 9:
If r is true and p is false, then $p \wedge (q \vee r)$ is false and $(p \wedge q) \vee r$ is true.

$$\begin{aligned} (p \wedge (q \vee r)) \to ((p \wedge q) \vee r) &\equiv \neg(p \wedge (q \vee r)) \vee ((p \wedge q) \vee r) \\ &\equiv (\neg p \vee (\neg q \wedge \neg r)) \vee ((p \wedge q) \vee r) \\ &\equiv (\neg p \vee (p \wedge q)) \vee ((\neg q \wedge \neg r) \vee r) \\ &\equiv (\neg p \vee p) \wedge (\neg p \vee q)) \vee ((\neg q \vee r) \wedge (\neg r \vee r)) \\ &\equiv (\neg p \vee q) \vee (\neg q \vee r) \\ &\equiv (\neg q \vee q) \vee (\neg p \vee r) \\ &\equiv T \vee (\neg p \vee r) \equiv T. \end{aligned}$$

Solution to Exercise 10:

1. $(p \wedge (\neg q)) \to F \equiv \neg(p \wedge (\neg q)) \vee F \equiv \neg(p \wedge (\neg q)) \equiv \neg p \vee q \equiv p \to q$.

2. $$\begin{aligned} p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) \\ &\equiv (\neg p \vee q) \wedge (\neg q \vee p). \end{aligned}$$

Solution to Exercise 11:

1. $\neg p \to q \equiv \neg(\neg) \vee q \equiv p \vee q$.

2. $\neg(p \to \neg q) \equiv \neg(\neg p \vee \neg q) \equiv p \wedge q$.

3. $\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$.

4. $$\begin{aligned} (p \to q) \wedge (p \to r) &\equiv (\neg p \vee q) \wedge (\neg p \vee r) \\ &\equiv \neg p \wedge (\neg p \vee r) \vee (q \wedge (\neg p \vee r)) \\ &\equiv \neg p \vee ((q \wedge \neg p) \vee (q \wedge r)) \\ &\equiv (\neg p \vee (q \wedge \neg p)) \vee (q \wedge r) \\ &\equiv \neg p \vee (q \wedge r) \\ &\equiv p \to (q \wedge r) \end{aligned}$$

5.

$$
\begin{aligned}
(p \to r) \wedge (q \to r) &\equiv (\neg p \vee r) \wedge (\neg q \vee r) \\
&\equiv (\neg p \wedge (\neg q \vee r)) \vee (r \wedge (\neg q \vee r)) \\
&\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge r)) \vee (\neg q \vee r) \vee r \\
&\equiv (\neg p \wedge \neg q) \vee r \\
&\equiv (p \vee q) \to r
\end{aligned}
$$

6.

$$
\begin{aligned}
(p \to q) \vee (p \to r) &\equiv \neg p \vee q \neg p \vee r \\
&\equiv \neg p \vee (q \vee r) \\
&\equiv p \to (q \vee r)
\end{aligned}
$$

7.

$$
\begin{aligned}
(p \to r) \vee (q \to r) &\equiv \neg p \vee r \vee \neg q \vee r \\
&\equiv (\neg p \vee \neg q) \vee r \\
&\equiv (p \wedge q) \to r.
\end{aligned}
$$

Solution to Exercise 12:

1.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

2.

p	q	$p \to q$	$\neg p \leftrightarrow \neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$\neg(p \leftrightarrow q)$	$p \leftrightarrow \neg q$
T	T	T	T	T	F	F	F
T	F	F	F	F	F	T	T
F	T	T	F	F	F	T	T
F	F	T	F	F	T	F	F

3.

$$
\begin{aligned}
p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) \\
&\equiv (\neg p \vee q) \wedge (\neg q \vee p) \\
&\equiv (\neg p \wedge (\neg q \vee p)) \vee (q \wedge (\neg q \vee p)) \\
&\equiv (\neg p \wedge \neg q) \vee (q \wedge p)
\end{aligned}
$$

Solution to Exercise 13:

$$
\begin{aligned}
((\neg r) \wedge p \wedge q) \vee ((\neg r) \wedge (\neg p) \wedge q) &\equiv ((\neg r) \wedge (p \wedge q)) \vee ((\neg r) \wedge ((\neg p) \wedge q)) \\
&\equiv (\neg r) \wedge ((p \wedge q) \vee ((\neg p) \wedge q)) \\
&\equiv (\neg r) \wedge ((q \wedge (\neg p)) \vee (q \wedge (\neg p))) \\
&\equiv (\neg r) \wedge (q \wedge (p \vee (\neg p))) \\
&\equiv (\neg r) \wedge (q \wedge T) \\
&\equiv (\neg r) \wedge q
\end{aligned}
$$

Solution to Exercise 14:

1. any $x \in \mathbb{R}$ and take $y = -x$,
2. We can take $x = 1, y = 2, z = 5$.
3. We can take $x \leq 0$ and $y = -1$
4. $p(x, y)$ is false for all $x, y \in \mathbb{N}$.

Solution to Exercise 15:

1. $P(8, \frac{1}{2})$ is true,
2. $\exists y : P(2, y)$ is true, we take $y = 2$
3. $\forall\ x :\ Q(x, y)$ is false, we can take $x = y$.
4. $\exists x, y$: $[P(x, y) \to Q(x, y)]$ is true, we can take $x = 4, y = 1$

Solution to Exercise 16:

1. $\exists x :\ x^2 \leq 0$ which is equivalent to $x = 0$.
2. $\forall\ x :\ 2x \neq 1$,
3. $\exists\ x :\ (\neg P(x)) \wedge \neg(Q(x))$,

Solution to Exercise 17:

1. For $n = 1$, $1 < 2^1$. We assume that $n < 2^n$ for some $n \in \mathbb{N}$. $n+1 < 2^n+1 < 2^n+2^n = 2^{n+1}$. Then for all $n \in \mathbb{N}$, $n < 2^n$,
2. For $n = 4$, $2^4 = 16 < 4! = 24$. We assume that $2^n < n!$ for some $n \in \mathbb{N}$. Then $2^{n+1} = 2.2^n < 2.n! < n!.(n+1) = (n+1)!$. Then for all integers $n \geq 4$, $2^n < n!$,
3. For $n = 1$, $1^3 - 1 = 0$ is divisible by 3. We assume that $n^3 - n$ is divisible by 3 for some $n \in \mathbb{N}$. Then $(n+1)^3 - (n+1) = n^3 - n + 3n^2 + 3n = 3(n^2 + n)$ is divisible by 3. Then for all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.

Solution to Exercise 18:

1. $1(1+1) = 2 = \frac{1}{3}1(1+1)(1+2)$, then the property is true for $n = 1$. Assume that $\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$.

$$\begin{aligned}
\sum_{k=1}^{n+1} k(k+1) &= \sum_{k=1}^{n} k(k+1) + (n+1)(n+2) \\
&= \frac{1}{3}n(n+1)(n+2) + (n+1)(n+2) \\
&= \frac{1}{3}(n+1)(n+2)(n+3).
\end{aligned}$$

Then $\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$ for all $n \in \mathbb{N}$,

2. The property is true for $n = 1$. Assume that $\sum_{k=1}^{n} \frac{4}{k(k+1)(k+2)} = 1 - \frac{2}{(n+1)(n+2)}$.

   We remark that for all $k \in \mathbb{N}$, $\frac{4}{k(k+1)(k+2)} = \frac{2}{k} - \frac{4}{k+1} + \frac{2}{k+2}$.

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{4}{k(k+1)(k+2)} &= 1 - \frac{2}{(n+1)(n+2)} + \frac{4}{(n+1)(n+2)(n+3)} \\
&= 1 - \frac{2}{(n+1)(n+2)} + \frac{2}{n+1} - \frac{4}{n+2} + \frac{2}{n+3} \\
&= 1 - \frac{2}{n+1} + \frac{2}{n+2} + \frac{2}{n+1} - \frac{4}{n+2} + \frac{2}{n+3} \\
&= 1 - \frac{2}{n+2} + \frac{2}{n+3} \\
&= 1 - \frac{2}{(n+2)(n+3)}.
\end{aligned}$$

   Then for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{4}{k(k+1)(k+2)} = 1 - \frac{2}{(n+1)(n+2)}.$$

3. The property is true for $n = 1$. Assume that $\sum_{k=1}^{n} k^3 = \frac{1}{4}[n(n+1)]^2$.

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \frac{1}{4}[n(n+1)]^2 + (n+1)^3 \\
&= (n+1)^2 \left(\frac{n^2+4n+4}{4} \right) \\
&= \frac{1}{4}[(n+1)(n+2)]^2.
\end{aligned}$$

   Then for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}[n(n+1)]^2.$$

4. The property is true for $n = 1$. Assume that $u_n = 2^{n-1}(u_1 + 1) - 1$.

$$u_{n+1} = 2u_n + 1 = 2^n(u_1 + 1) - 2 + 1 = 2^n(u_1 + 1) - 1.$$

   Then for all $n \in \mathbb{N}$, $u_n = 2^{n-1}(u_1 + 1) - 1$.

   If $u_1 = 1$,

$$\sum_{k=1}^{n} u_k = \sum_{k=1}^{n} 2^k - 1 = 2(2^n - 1) - n = 2^{n+1} - (n+2).$$

5. The property is true for $n = 1$. Assume that $3^{2n} + 7$ is divisible by 8, say $3^{2n} + 7 = 8k$.

   $3^{2n+2} + 7 = 9.3^{2n} + 7 = 9(8k - 7) + 7 = 72k - 56 = 8(9k - 7)$. Then for all $n \in \mathbb{N}$ $3^{2n} + 7$ is divisible by 8.

6. The property is true for $n=1$. Assume that $(3n+1)7^n - 1 = 9k$, $k \in \mathbb{N}$.

   $(3(n+1)+1)7^{n+1} - 1 = 7((9k+1) + 3.7^n - 1 = 9(7k) + 3(2 + 7^{n+1})$.

   We prove by induction that $2 + 7^n$ is divisible by 3.

   The property is true for $n=1$. Assume that $2 + 7^n = 3k$, then $2 + 7^{n+1} = 2 + 7(3k-2) = 3(7k) - 12 = 3(7k-4)$.

   Then for all $n \in \mathbb{N}$, $(3n+1)7^n - 1$ is divisible by 9.

7. The property is true for $n=1$. Assume that $2^{n-1} \leq n!$, then $2^n \leq 2.n! \leq (n+1)!$.

8. $2^n \leq n!$ is valid for $n=4$. Assume that $2^n \leq n!$ is valid a fixed $n \geq 4$, then $2^{n+1} = 2.2^n \leq 2.n! \leq (n+1)!$, which proves that $2^n \leq n!$ is valid for all $n \in \mathbb{N}$, $n \geq 4$.

9. The property is true for $n=1$. Assume that $(1+x)^n \geq 1+nx$, then $(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x$.

10.

$$\begin{aligned} \binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{n!(n-k+1)}{k!(n+1-k)!} + \frac{n!k}{k!(n+1-k)!} \\ &= \binom{n+1}{k}. \end{aligned}$$

The property is true for $n=1$. Assume that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, then

$$\begin{aligned} (x+y)^{n+1} &= (x+y) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1} \\ &= 1 + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} y^k + y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^{n+1-k} y^k \\ &= 1 + y^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) x^{n+1-k} y^k \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k. \end{aligned}$$

Solution to Exercise 19:
If $\sqrt{3} = \frac{p}{q}$ for p, q relatively prime integers, then $3q^2 = p^2$. 3 divides p^2 say $p = 3k$, then $q^2 = 3k^2$, which is absurd. Then $\sqrt{3}$ is not rational.
If $\sqrt{5} = \frac{p}{q}$ for p, q relatively prime integers, then $5q^2 = p^2$. 5 divides p^2 say $p = 5k$, then $q^2 = 5k^2$, which is absurd. Then $\sqrt{5}$ is not rational.
If $\sqrt{6} = \frac{p}{q}$ for p, q relatively prime integers, then $6q^2 = p^2$. 3 divides p^2 say $p = 3k$, then $2q^2 = 3k^2$, which is absurd. Then $\sqrt{6}$ is not rational.

Solution to Exercise 20:
If $\sqrt[3]{2} = \frac{p}{q}$ for p, q relatively prime integers, then $2q^3 = p^3$. 2 divides p^3 say $p = 2k$, then $q^3 = 4k^3$, which is absurd. Then $\sqrt[3]{2}$ is not rational.
If $\sqrt[3]{3} = \frac{p}{q}$ for p, q relatively prime integers, then $3q^3 = p^3$. 3 divides p^3 say $p = 3k$, then $q^3 = 9k^3$, which is absurd. Then $\sqrt[3]{3}$ is not rational.

If $\sqrt{p} = \frac{m}{n}$ for m, n relatively prime integers, then $pm^2 = n^2$. p divides n^2 say $n = pk$, then $n^2 = pk^2$, which is absurd. Then $\sqrt{p}$ is not rational.

Solution to Exercise 21:
There is an 11−cup measure. We can measure 12 cups by using the 4−cup measure three times. For 13, we can use the 4-cup measure combined with the 9-cup. And for 14, we can use the 14−cup. So if $P(n)$ is the predicate that n cups can be measured using the available cup sizes, we have established that $P(11)$, $P(12)$, $P(13)$, and $P(14)$ are all true.
Induction hypothesis. We assume $P(m)$ holds for all m such that $14 \leq m \leq n$.
$n + 1 \geq 15$. We can measure $(n + 1) - 4$ cups by the induction hypothesis, since $(n + 1) - 4$ is greater than or equal to 11 and is therefore a case already proven. But then we can measure $n + 1$ cups by measuring $(n + 1) - 4$ cups and then adding a single additional scoop using the 4−cup measure.

Solution to Exercise 22:

1. For $n = 1$, $7 - 1 = 6$ is divisible by 6.
   Assume that $7^n - 1$ is divisible by 6. There is $m \in \mathbb{N}$ such that $7^n - 1 = 6m$.
   $7^{n+1} - 1 = 7(6m + 1) - 1 = 42m + 6 = 6(7m + 1)$ is divisible by 6.

2. For $n = 1$, $11 - 6 = 6$ is divisible by 5.
   Assume that $11^n - 6$ is divisible by 5. There is $m \in \mathbb{N}$ such that $11^n - 6 = 5m$.
   $11^{n+1} - 6 = 11(5m + 6) - 6 = 55m + 60 = 5(11m + 12)$ is divisible by 5.

3. For $n = 1$, $6.7 - 2.3 = 36$ is divisible by 4.
   Assume that $6.7^n - 2.3^n$ is divisible by 4. There is $m \in \mathbb{N}$ such that $6.7^n - 2.3^n = 4m$.
   $6.7^{n+1} - 2.3^{n+1} = 7(4m + 2.3^n) - 2.3^{n+1} = 4.7m + 14.3^n - 6.3^n = 4.7m + 8.3^n$ is divisible by 4.

4. For $n = 2$, $\frac{1}{2} + \frac{2}{3} = \frac{5}{6} < \frac{4}{3}$.
   Assume that $\frac{1}{2} + \frac{2}{3} + \ldots + \frac{n}{n+1} < \frac{n^2}{n+1}$.
   $\frac{1}{2} + \frac{2}{3} + \ldots + \frac{n}{n+1} + \frac{n+1}{n+2} < \frac{n^2}{n+1} + \frac{n+1}{n+2}$.
   Moreover $\frac{n^2}{n+1} + \frac{n+1}{n+2} - \frac{(n+1)^2}{n+2} = n^2 n + 1 - n(n+1)n + 2 = -n(n+1)(n+2) < 0$.
   Then $\frac{1}{2} + \frac{2}{3} + \ldots + \frac{n}{n+1} < \frac{n^2}{n+1}$, for all $n \geq 2$.

5. Let $S_n = \sum_{k=1}^{2^n} \frac{1}{k}$.
   For $n = 1$, $S_1 = 1 + 1 > 1 + \frac{1}{2}$.
   Assume that $S_n > 1 + \frac{n}{2}$.
   $S_{n+1} = S_n + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} > S_n + \frac{2^n}{2^{n+1}} > 1 + \frac{n}{2} + \frac{1}{2} = 1 + \frac{n+1}{2}$. Then $S_n > 1 + \frac{n}{2}$ for all $n \geq 1$.

6. Let $T_n = \sum_{k=1}^{n} \frac{1}{k}$.
   For $n = 1$, $T_1 = 1 + 1 > \ln 2$.
   Assume that $T_n > \ln(n + 1)$.
   $T_{n+1} = T_n + \frac{1}{n+1} > \ln(n+1) + \frac{1}{n+1}$.
   $\ln(n+1) + \frac{1}{n+1} - \ln(n+2) = \frac{1}{n+1} - \ln(1 + \frac{1}{n+1}) > 0$ since $x - \ln(1 + x) > 0$, for all $0 < x < 1$. Then $T_n > \ln(n + 1)$ for all $n \geq 1$.

Solution to Exercise 23:
Consider the Inductive Step, where we want to prove that n−SAR postage can be achieved using only 2−SAR and 5−SAR stamps.

The Basis Step: for $n = 4$ we take 2 stamps of $2-$SAR and for $n = 5$, we take one stamp of $5-$SAR.
Assume that the case $n \geq 6$ and the postage of $k-$SAR or more can be achieved by using only $2-$SAR and $5-$SAR stamps for $4 \leq n$.
By the inductive assumption, we can make postage of $n - 2$SAR. We add a $2-$SAR stamp to make $n-$SAR postage. The Inductive Step is complete

Solution to Exercise 24:
The result is true for $n = 1$. We assume that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 2^n \\ 2^n & 2^n \end{pmatrix}$ for some $n \geq 1$.
$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^n & 2^n \\ 2^n & 2^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{n+1} & 2^{n+1} \\ 2^{n+1} & 2^{n+1} \end{pmatrix}.$

CHAPTER II

RELATIONS

1 Binary Relations

The topic of this chapter is relations, it is about having 2 sets, and connecting related elements from one set to another. There is three important type of relations:
functions, equivalence relations and order relations. In this chapter, equivalence and order relations are only considered.

Definition 1.1

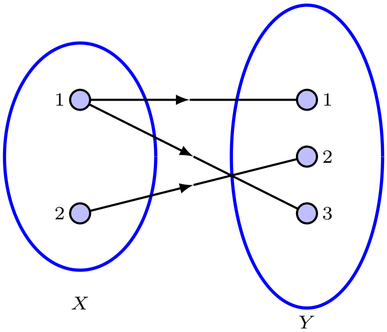
Let X and Y be two sets. A binary relation R from X to Y is a subset of the Cartesian product $X \times Y$. Given $x, y \in X \times Y$, we say that x is related to y by R, also written (xRy) if and only if $(x, y) \in R$.

Definition 1.2

Let R be a binary relation from X to Y.
the set $D(R) = \{x \in X;\ (x, y) \in R\}$ is called the domain of the relation. The set $R(R) = \{y \in Y;\ (x, y) \in R\}$ is called the range of the relation.

Example 29 :

Let $X = \{1, 2\}$ and $Y = \{1, 2, 3\}$, and the relation is given by $(x, y) \in R \iff x - y$ is even.
$X \times Y = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$ and $R = \{(1, 1), (1, 3), (2, 2)\}$.
To illustrate this relation we use the following diagram:



Definition 1.3

A relation on a set X is a relation from X to X. In other words, a relation on a set X is a subset of $X \times X$. (Relation of the same set is called also homogeneous relation)

Example 30 :
Let $X = \{1, 2, 3, 4\}$ and $R = \{(a, b);\ a \text{ divides } b\}$. Then

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Definition 1.4

Let R be a relation from the set X to the set Y.The inverse relation R^{-1} from Y to X is defined by:

$$R^{-1} = \{(y, x) \in Y \times X,\ (x, y) \in R\}.$$

(The inverse relation R^{-1} is also called the transpose or the converse relation of R and denoted also R^T).

Examples

1. Consider the sets $X = \{2, 3, 4\}$, $Y = \{2, 6, 8\}$, with the relation $(x, y) \in R$ if and only if x divides y.

$$X \times Y = \{(2, 2), (2, 6), (2, 8), (3, 2), (3, 6), (3, 8), (4, 2), (4, 6), (4, 8)\},$$

$$R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\},$$

$$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}.$$

   $(y, x) \in R^{-1}$ if and only if y is a multiple of x.

2. The **identity relation** defined on a set X is defined by $I = \{(x, x);\ x \in X\}$.

3. The **universal relation** R from X to Y is defined by $R = X \times Y$.

4. Let $X = \mathbb{Z}$ and R the relation defined by: $mRn \iff m^2 - n^2 = m - n$. Since $m^2 - n^2 = (m - n)(m + n)$, then $mRn \iff m = n$ or $m + n = 1$. Then

$$R = \{(m, m), (m, 1 - m);\ m \in \mathbb{Z}\}.$$

Boolean matrix of relation If $X = \{x_1, \dots, x_n\}$ and $X = \{y_1, \dots, y_m\}$ are finite sets and R a binary relation from X to Y, we represent the relation R by the following matrix: (called the Boolean matrix of R)

$$M_R = \begin{pmatrix} x_1Ry_1 & x_1Ry_2 & \dots & x_1Ry_m \\ x_2Ry_1 & x_2Ry_2 & \dots & x_2Ry_m \\ \vdots & \vdots & & \vdots \\ x_nRy_1 & \dots & \dots & x_nRy_m \end{pmatrix},$$

where $x_jRy_k = 1$ if $(x_j, y_k) \in R$ and 0 otherwise.

For example if $X = \{2, 3, 4\}$, $Y = \{2, 6, 8\}$, and the relation R defined by: $(x, y) \in R$ if and only if x divides y.
The relation R is represented by the following matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix which represents R^{-1} is the transpose of this matrix.

Definition 1.5

Let R, S be two relations in $X \times Y$. The relations $R \cup S$ and $R \cap S$ are called respectively the union and the intersection of these relations.

Example 31 :
Let R_1 and R_2 the relations on the set $X = \{a, b, c\}$ represented respectively by the matrices

$$M_{R_1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M_{R_2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

$R_1 = \{(a, a), (a, c), (b, a), (c, b)\}$, $R_2 = \{(a, a), (a, c), (b, b), (b, c), (c, a)\}$.
$R_1 \cap R_2 = \{(a, a), (a, c)\}$, $R_1 \cup R_2 = \{(a, a), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b)\}$. $R_1 - R_2 = \{(b, a), (c, b)\}$, $R_2 - R_1 = \{(b, b), (b, c), (c, a)\}$.

The matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$ are respectively:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

1.1 Composition of Relations

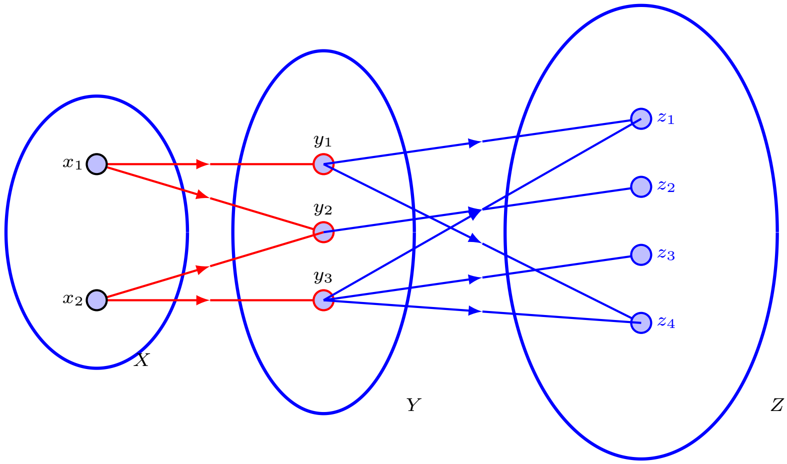
Definition 1.6

Given two relations $R \in X \times Y$ and $S \in Y \times Z$, the composition of R and S is the relation on $X \times Z$ defined by:

$$S \circ R = \{(x, z) \in X \times Z,\ \exists y \in Y, xRy, ySz\}.$$

Example 32 :

$X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, $Z = \{z_1, z_2, z_3, z_4\}$, $R = \{(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_3)\}$,
$S = \{(y_1, z_1), (y_1, z_4), (y_2, z_2), (y_3, z_1), (y_3, z_3), (y_3, z_4)\}$, $S \circ R = \{(x_1, z_1), (x_1, z_2), (x_1, z_4), (x_2, z_1), (x_2, z_2), (x_2, z_3), ($



The matrices of the relations R and S are respectively

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad M_S = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

The matrix representing $S \circ R$ is:

$$M_{S \circ R} = M_R . M_S = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The product of matrices is the Boolean product defined as the following: if $A = (a_{j,k})$ is a Boolean matrix of degree (m, n) and $B = (b_{j,k})$ is a Boolean matrix of degree (n, p), $A.B = (c_{j,k})$ is the Boolean matrix of degree (m, p) defined by:

$$C_{j,k} = \max\{a_{j,1}b_{1,k}, a_{j,2}b_{2,k}, \dots, a_{j,n}b_{n,k}\}.$$

Example 33 :
Let R be the relation from the set of names to the set of telephone numbers and let S be the relation from the set of telephone numbers to the set of telephone bills. The relations R and S are defined by the below tables.
Then the relation $S \circ R$ is a relation from the set of names to the set of telephone bills.

Table of the relation R	
Ali	104105106, 105325118, 104175100
Ahmed	105315307, 104137116, 107325112
Salah	107107121
Salem	104271216, 105145146

Table of the relation S	
104105106	735
105325118	245
104175100	535
105315307	250
104137116	1250
107325112	275
107107121	2455
104271216	445
105145146	1215

Table of the relation $S \circ R$	
Ali	1515
Ahmed	1775
Salah	2455
Salem	1660

Theorem 1.7

Let R_1 be a relation from X to Y and R_2 a relation from Y to Z. Then $(R_2 \circ R_1)^{-1} = R_1^{-1} \circ R_2^{-1}$.

Proof:

$$R_2 \circ R_1 = \{(x,z) \in X \times Z;\ \exists y \in Y,\ (x,y) \in R_1,\ (y,z) \in R_2\}$$

$$\begin{aligned}
(R_2 \circ R_1)^{-1} &= \{(z,x) \in Z \times X;\ \exists y \in Y,\ (x,y) \in R_1,\ (y,z) \in R_2\} \\
&= \{(z,x) \in Z \times X;\ \exists y \in Y,\ (y,x) \in R_1^{-1},\ (z,y) \in R_2^{-1}\} \\
&= R_1^{-1} \circ R_2^{-1}
\end{aligned}$$

Definition 1.8

Let R be a relation on the set X. The powers R^n, $n \in \mathbb{N}$ are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Example 34 :
If $X = \{1,2,3,4\}$ and $R = \{(1,2),(1,3),(2,1),(3,4)\}$. Then $R^2 = \{(1,1),(1,4),(2,2),(2,3)\}$, $R^3 = \{(1,2),(1,3),(2,1),(2,4)\}$.

$$M_R = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad M_{R^2} = M_R^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1.2 Representing Relations Using Digraphs

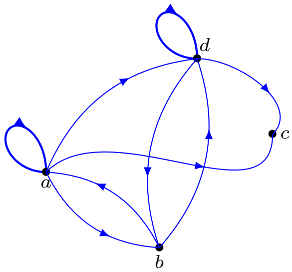
We have shown that a relation can be represented by listing all of its ordered pairs or by using a Boolean matrix. There is another representation. Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow. We use such pictorial representations when we think of relations on a finite set as directed graphs, or digraphs.

Definition 1.9

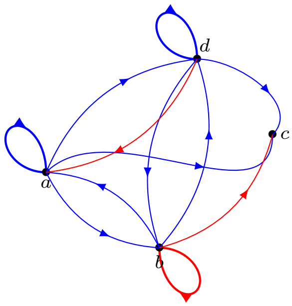
A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a,b), and the vertex b is called the terminal vertex of this edge.

When a relation R is defined on a set X, the arrow diagram of the relation can be modified so that it becomes a directed graph.
Instead of representing X as two separate sets of points, represent X only once, and draw an arrow from each point of X to each $R-$related point.
If a point is related to itself, a loop is drawn that extends out from the point and goes back to it.

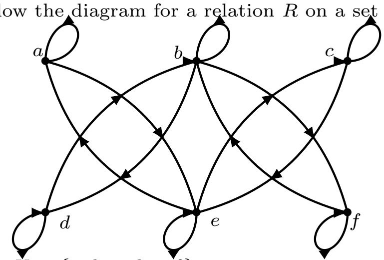
Example 35 :
Let $X = \{a,b,c,d\}$ and $R = \{(a,a),(a,b),(a,d),(b,a),(b,d),(d,d),(d,b),(d,c)\}$



The digraph of the relation R^2



Example 36 :
Below the diagram for a relation R on a set X.



$X = \{a, b, c, d, e, f\}$,
$R = \{(a,a), (a,e), (b,b), (b,d), (b,f), (c,c), (c,e), (d,b), (d,d), (e,a),$
$(e,c), (e,e), (f,b), (f,f)\}$

2 Equivalence Relation

Definition 2.1

A relation R on a set X is called reflexive if every element of X is related to itself: $\forall x \in X$, xRx.
If X is finite, R is reflexive if and only if $I \subset R$.

Example 37 :
If $X = \mathbb{Z}$ and the relation R is defined by $xRy \iff x - y \equiv 0[3]$. This relation is reflexive.

Definition 2.2

A relation R on a set X is called symmetric if $(x, y) \in R$ then $(y, x) \in R$: $\forall x, y \in X$, $xRy \Rightarrow yRx$.
R is symmetric if and only if $R^{-1} = R^T = R$.

Example 38 :
If $X = \mathbb{Z}$ and the relation R is defined by: $xRy \iff x - y \equiv 0[3]$. This relation is symmetric.

Definition 2.3

A relation R on a set X is called transitive if: $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$: $\forall x, y, z \in X$, $xRy \wedge yRz \Rightarrow xRz$.

Remarks 2 :

1. A relation R on a finite set X is reflexive if and only if the diagonal relation on X is a subset in R. (The diagonal relation on X is the relation $I = \{(x, x);\ x \in X\}$).

2. A relation R on a set X is symmetric if and only if $R^{-1} = R$.

3. A relation R on a set A is transitive if and only if $R^n \subset R$ for all $n \geq 2$.

4. A relation R on a set A is transitive if and only if $R^2 \subset R$.

Definition 2.4

A relation R on a set X is called an equivalence relation if R is reflexive, symmetric and transitive.
The equivalence class of a in X is

$$[a] = \{x \in X, aRx\}.$$

R is an equivalence relation if and only if $R^2 \subset R \subset R^T$ and $I \subset R$.

Example 39 :
The relation $\equiv [\text{mod } n]$ is an equivalence relation on $\mathbb{Z}$.
• It is reflexive: $x \equiv x[\text{mod } n]$ is always true.
• It is symmetric: $x \equiv y[\text{mod } n]$ means that $x = qn + y$ for some integer q, thus $y = -qn + x$ and $y \equiv x[\text{mod } n]$.
• It is transitive: if $x \equiv y[\text{mod } n]$ and $y \equiv z[\text{mod } n]$ then we have $x = qn + y$ and $y = rn + z$ thus $x = qn + y = n(q + r) + z$ and $x \equiv z[\text{mod } n]$.
The equivalence class of 0 is the multiples of n:

$$[0] = \{kn,\ k \in \mathbb{Z}\}.$$

Theorem 2.5

Let R be an equivalence relation on a set X. These statements for elements a and b of X are equivalent:

1. aRb
2. $[a] = [b]$
3. $[a] \cap [b] \neq \emptyset$.

Definition 2.6

A collection of subsets X_j, $j \in I$ (where I is an index set) forms a partition of X if $X_j \neq \emptyset$ for $j \in I$, $X_j \cap X_k = \emptyset$ and $\cup_{j \in I} X_j = X$.

Theorem 2.7

The equivalence classes of an equivalence relation R on a set X form a partition of X. Conversely, given a partition $\{A_j;\ j \in I\}$ of the set X, there is an equivalence relation R that has the sets A_j , $j \in I$ as its equivalence classes.

Definition 2.8

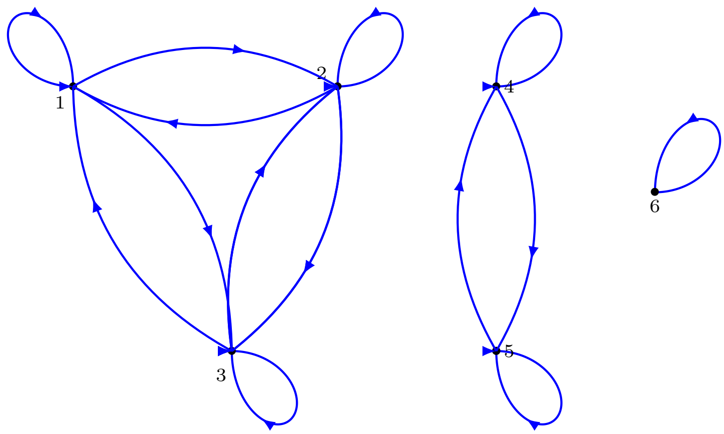
Given $(X_j)_{j \in I}$ a partition of X. The equivalence relation R on X related to this partition is the relation defined by:

$$xRy \iff \exists j \in I;\ x, y \in X_j.$$

Example 40 :
Let $X = \{0, 1, 2, 3, 4, 5\}$ and the partition of X: $\{0, 3, 4\}$, $\{1, 5\}$, $\{2\}$. The equivalence relation R induced by this partition is

$$\begin{aligned} R &= \{(0,0), (0,3), (3,0), (0,4), (4,0), (3,3), (3,4), (4,3), \\ &\quad (4,4), (1,1), (1,5), (5,1), (5,5), (2,2)\}. \end{aligned}$$

Example 41 :
Let R be the relation produced by the partition $X_1 = \{1, 2, 3\}$, $X_2 = \{4, 5\}$ and $X_3 = \{6\}$ of $X = \{1, 2, 3, 4, 5, 6\}$.
Give its digraph



3 Partial Order Relations

Definition 3.1

A relation R on a set X is antisymmetric if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$: $\forall x, y \in X,\ xRy \wedge yRx \Rightarrow x = y$.
R is antisymmetric if and only if $R \cap R^{-1} \subset I$.

Example 42 :
If $X = \mathbb{Z}$ and the relation R is defined by: $xRy \iff x \leq y$. This relation is antisymmetric

Definition 3.2

A relation R on a set X is a partial order if R is reflexive, antisymmetric and transitive. A set X together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (X, R).

Example 43 :
If $X = \mathbb{Z}$ and the relation R is defined by: $xRy \iff x \leq y$. This relation is a partial order.

Example 44 :
Suppose that a relation R on a set is represented by the following matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The relation R is reflexive because all the diagonal elements of this matrix are equal to 1.
The relation R is symmetric because $R^T = R$.
The relation R is not transitive and not antisymmetric.

Theorem 3.3

Let R be a partial order relation on X, then R^{-1} is also a partial order relation on X.

Definition 3.4

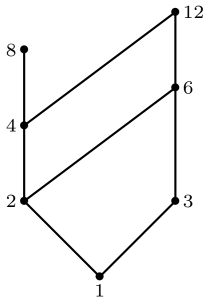
Let $(X, \leq)$ be a partial ordering set. The elements $a, b \in X$ are called comparable if either $a \leq b$ or $b \leq a$.
If neither $a \leq b$ nor $b \leq a$, we say that a and b are incomparable.
If any two elements in X are comparable, we say that the ordered set $(X, \leq)$ is total or linearly ordered set and the relation $\leq$ is called a total order or a linear order. A totally ordered set is also called a chain.

Example 45 :
Let $(\mathbb{N}, R)$ be the ordered set defined by the relation nRm if n divides m. The integers 3 and 9 comparable but 2 and 3 are not.

4 Hasse Diagrams

Let (X, R) be a finite poset. Many edges in the directed graph for a finite poset do not have to be shown because they must be present. The relation is reflexive, we do not have to show these loops because they must be present.
The relation is transitive, we do not have to show those edges that must be present because of transitivity.
Finally, draw the remaining edges upward and drop all arrows.

Example 46 :
The Hasse diagram representing the partial ordering $\{(a, b);\ a \text{ divides } b\}$ on the set $X = \{1, 2, 3, 4, 6, 8, 12\}$.



To obtain a Hasse diagram, proceed as follows:
Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward. Then eliminate

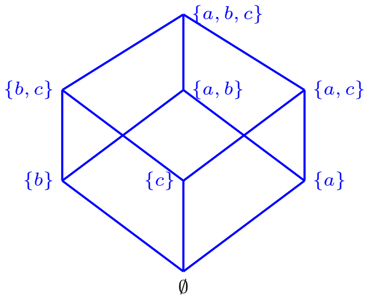
1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.

Example 47 :

Consider the relation, $\subset$, on the set $\mathcal{P}(a, b, c)$. That is, for all sets U and V in $\mathcal{P}(a, b, c)$,

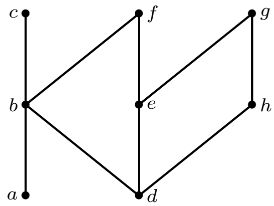
$$U \subset V \iff \forall x \in U, \; x \in V.$$

The Hasse diagram for this relation is:



Example 48 :

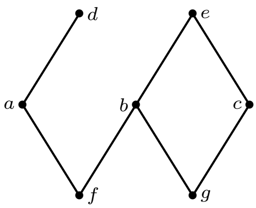
A partial order R on a set X with the following Hasse diagram. List the elements of R.



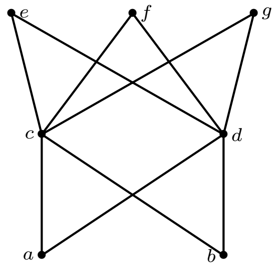
$$\begin{aligned} R \quad = \quad & \{(a,a),(a,b),(a,c),(a,f),(d,d),(d,b),(d,e),(d,h),(d,c), \\ & (d,f),(d,g),(b,b),(b,c),(b,f),(e,e),(e,f),(e,g), \\ & (h,h),(h,g),(c,c),(f,f),(g,g)\} \end{aligned}$$

Example 49 :
Draw a Hasse diagram of the partial order relation R on $X = \{a, b, c, d, e, f, g\}$ given by

$$\begin{aligned} R \quad = \quad & \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(g,g),(a,d),(b,e), \\ & (c,e),(f,a),(f,b),(f,d),(f,e),(g,b),(g,c),(g,e)\}\,. \end{aligned}$$

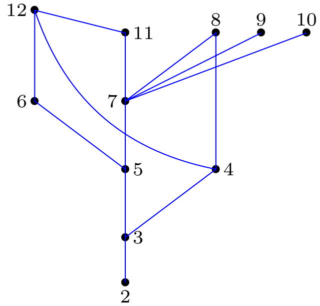


Example 50 :
Example of a poset $(X, \leq)$ which hass the following Hasse diagram.

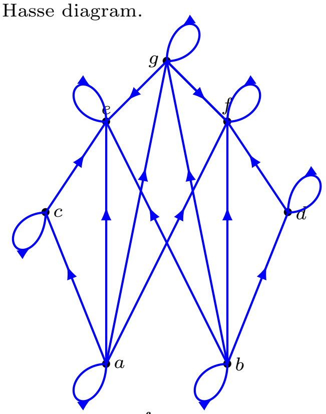


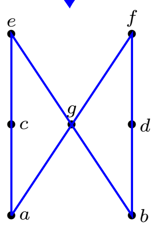
We take the relation R (divides), $a = 2, b = 3, c = 12, d = 18, e = 180, f = 252, g = 396$.

Example 51 :
Let $X = \{n \in \mathbb{N};\ 2 \leq n \leq 12\}$. A partial order relation R on X is defined by mRn if and only if either (m divides n) or (m is prime and $m < n$).
The Hasse diagram



Example 52 :
A partial order relation R on $X = \{a, b, c, d, e, f, g\}$ with the following directed graph. Draw its Hasse diagram.

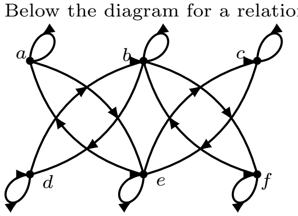




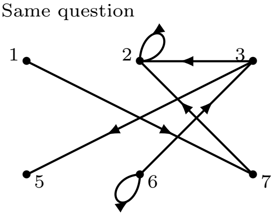
5 Exercises on Chapter 2

Exercise 41 :

1. Below the diagram for a relation R on a set X. Write the sets X and R.



2. Same question



Solution to Exercise 25:

1. $X = \{a, b, c, d, e, f\}$,
   $R = \{(a, a), (a, e), (b, b), (b, d), (b, f), (c, c), (c, e),$
   $(d, b), (d, d), (e, a), (e, c), (e, e), (f, b), (f, f)\}$

2. $X = \{1, 2, 3, 5, 6, 7\}$,
   $R = \{(1, 7), (2, 2), (3, 2), (3, 5), (6, 3), (6, 6), (7, 2)\}$

Exercise 42 :
Consider the relation $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a)\}$ on the set $X = \{a, b, c, d\}$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why?

Solution to Exercise 26:
R is reflexive, symmetric and transitive.

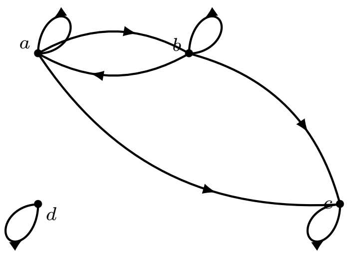
Exercise 43 :
Let $A = \{a, b, c, d\}$ and let R be the relation given by

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (b, c), (a, c)\}.$$

1. Draw the digraph of R.

2. Is R reflexive, symmetric, transitive? In each case give a counter example if the answer is no.

Solution to Exercise 27:

1. 

2. R is reflexive,
   R is not symmetric, $(b,c) \in R$ but $(c,b) \notin R$.

   R is transitive.

Exercise 44 :
Let $A = \mathbb{Z}$ and let $R = \{(x,y): \ x, y \in A, \ x^2 = y^2\}$. Prove that R is an equivalence relation.

Solution to Exercise 28:
Since $m^2 = m^2$, then $(m,m) \in R$.
$(m,n) \in R \iff m^2 = n^2 \iff n^2 = m^2 \iff (n,m) \in R$.
If $(m,n) \in R$ and $(n,p) \in R$, then $m^2 = n^2 = p^2$. It results that $(m,p) \in R$.

Exercise 45 :
Let $A = \mathbb{Z}$ and let $R = \{(x,y): \ xy + y^2 = x^2 + 1\}$.

1. Which of the following are true? $0R0$, $1R1$, $0R1$, $1R0$, $1R(-2)$, $0R(-2)$, $3R2$.

2. Show that R is not reflexive, not symmetric and not transitive. (Give counterexamples.)

Solution to Exercise 29:

1. $1R1$, $0R1$, $1R(-2)$, $3R2$ are true.

2. R is not reflexive since $(0,0) \notin R$,
   R is not symmetric since $(0,1) \in R$ but $(1,0) \notin R$,
   R is not transitive since $0R1$ and $1R(-2)$, but $(0,-2) \notin R$.

Exercise 46 :
On $\mathbb{Z}$ we define the relations, R and S as follows:
$xRy \iff x + y$ is even and $xSy \iff x - y$ is even.
The relations R and S are they equal?

Solution to Exercise 30:
$xRy \iff x+y = 2k \iff x+y-2y = 2k-2y$. Then $xSy \iff x-y = 2(k-y) = 2m \iff xRy$.

Exercise 47 :
Define a relation R on $\mathbb{Z}$ by declaring that xRy if and only if $x^2 \equiv y^2 \pmod 4$.
Prove that R is reflexive, symmetric and transitive.

Solution to Exercise 31:
Since $x^2 \equiv x^2 \pmod 4$, then R is reflexive.
If $x^2 \equiv y^2 \pmod 4$, then there is $m \in \mathbb{Z}$ such that $x^2 - y^2 = 4m$ and thus $y^2 \equiv x^2 \pmod 4$. The relation R is symmetric.
If $x^2 \equiv y^2 \pmod 4$ and $y^2 \equiv z^2 \pmod 4$, there is $m, n \in \mathbb{Z}$ such that $x^2 - y^2 = 4m$ and $z^2 - y^2 = 4n$. Then $z^2 - x^2 = 4(m-n)$ and the relation R is transitive.

Exercise 48 :
Is the following relations are reflexive, antisymmetric or transitive?

1. $X = \mathbb{R}$ and xRy if $|x| = |y|$,

2. $X = \mathbb{R}$ and xRy if $\sin^2 x + \cos^2 y = 1$.

3. $X = \mathbb{N}$ and xRy if there is p and q in $\mathbb{Z}$ such that $y = px^q$.

4. $X = \mathbb{R}^2$ and xRy if $\|x - y\| \leq 1$.

Solution to Exercise 32:

1. R is an equivalence relation.

2. R is an equivalence relation.

3. R is reflexive, not symmetric since $3R9$ but $(9,3) \notin R$,
   R is transitive. (If $y = px^q$ and $z = ry^s$, then $z = rp^s x^{sq}$)

4. R is reflexive and symmetric but not transitive.

Exercise 49 :
Let $X = \{2, 3, 4, 5, 6, 7, 8\}$ and R a relation over X. Draw the directed graph of R, after realizing that $xRy \iff x - y = 3n$ for some $n \in \mathbb{Z}$. Check that R is an equivalence relation.

Solution to Exercise 33:
The relation is:

$$\begin{aligned} R &= \{(2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (8,5), (8,2), \\ &\qquad (7,4), (6,3), (5,2)(5,8), (2,8), (4,7), (3,6), (2,5)\} \end{aligned}$$

Exercise 50 :
Let $X = \{0, 1, 2, 3\}$ and R a relation over X

$$R = \{(0,0), (0,1), (0,3), (1,1), (1,0), (2,3), (3,3)\}$$

Draw the directed graph of R. Check whether R is an equivalence relation. Give a counterexample in each case in which the relation does not satisfy one of the properties of being an equivalence relation.

Solution to Exercise 34:
R is not reflexive because $(2,2) \notin R$. It is not symmetric because $(3,2) \notin R$. It is not transitive because $(1,0)$ and $(0,3)$ are in R but $(1,3) \notin R$.

Exercise 51 :
Let $X = \{a, b, c\}$ and 2^X be the power set of X. A relation R is defined on 2^X as follows: For all $A, B \in 2^X$, $(A, B) \in R$ if the number of elements in A equals the number of elements in B. Show that R is an equivalence relation.

Exercise 52 :
Let $X = \{a, b, c\}$ and 2^X be the power set of X. A relation R is defined on 2^X as follows: For all $A, B \in 2^X$, $(A, B) \in R$ if the number of elements in A is less than the number of elements in B. Show that R is not an equivalence relation.

Exercise 53 :
If R and S are reflexive, then $R \cap S$ is so. Explain why?

Exercise 54 :
If R and S are symmetric, then $R \cap S$ is so. Explain why?

Exercise 55 :
If R and S are transitive, then $R \cap S$ is so. Explain why?

Exercise 56 :
If R and S are reflexive, then $R \cup S$ is so. Explain why?

Exercise 57 :
If R and S are symmetric, then $R \cup S$ is so.

Solution to Exercise 35:
Let $(x, y) \in R \cup S$. Then either $(x, y) \in R$ and then $(y, x) \in R$, or $(x, y) \in S$ and then $(y, x) \in S$. Thus, $(y, x) \in R \cup S$.

Exercise 58 :
If R and S are transitive, then $R \cup S$ is not necessarily so.
Counter-example: $R = \{(a, b)\}$ and $S = \{(b, c)\}$.

Exercise 59 :
Let $R = \{(0,1), (0,2), (1,1), (1,3), (2,2), (3,0)\}$. Find its transitive closure R^t, after drawing the directed graph of R.

Exercise 60 :
A relation R induced by a partition is an equivalence relation
reflexive, symmetric, transitive.

1. Let $X = \{0, 1, 2, 3, 4\}$ and let a partition be $P = \{\{0, 2\}, \{1\}, \{3, 4\}\}$. Find the ordered pairs in R.

2. Let $Y = \{0, 1, 2, 3, 4\}$ and let a partition be $P = \{\{0\}, \{1, 3, 4\}, \{2\}\}$.

3. Let $Z = \{0, 1, 2, 3, 4\}$ and let a partition be $P = \{\{0\}, \{1, 2, 3, 4\}\}$.

Solution to Exercise 36:

1. Then equivalence classes are: $\{0, 2\} = [0] = [2]$, $\{1\} = [1]$, $\{3, 4\} = [3] = [4]$ and hence $R = \{(0, 0), (2, 2), (0, 2), (2, 0), (1, 1), (3, 3), (4, 4), (3, 4), (4, 3)\}$.

2. Reasoning as above, $R = \{(0, 0), (1, 1), (3, 3), (4, 4), (1, 3), (3, 1), (1, 4), (4, 1), (3, 4), (4, 3), (2, 2)\}$.

3. Reasoning as above, $R = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 3), (3, 2), (2, 4$

Exercise 61 :
Consider the powerset of $X = \{a, b, c\}$ and define R on the powerset as follows: URV if and only if U and V have the same cardinality. Find the equivalence classes of R.

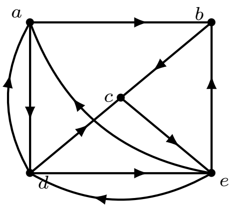
Solution to Exercise 37:
The equivalence classes are: $[\{\emptyset\}] = \{\emptyset\}$, $[\{a\}] = \{\{a\}, \{b\}, \{c\}\}$, $[\{a, b\}] = \{\{a, b\}, \{a, c\}, \{b, c\}\}$, $[\{a, b, c\}] = \{\{a, b, c\}\}$.

Exercise 62 :
Consider the following relation R over reals: $xRy \iff (x - y) \in \mathbb{Z}$.
Prove that it is an equivalence and characterize its equivalence classes.

Exercise 63 :
Let R be a relation on a set A and suppose R is symmetric and transitive. Prove the following: If for every x in X there exists a y in X such that xRy, then R is an equivalence relation.

Solution to Exercise 38:
For every x in X there is a y in X such that xRy, then, by symmetry, yRx, and by transitivity, xRx. Thus R is also reflexive and so it is an equivalence relation.

Exercise 64 :
A relation R on the set $X = \{a, b, c, d, e\}$ has the directed graph shown in the diagram below.



1. List the elements of R.

2. Write down the binary matrix of R.

3. List the elements of $R \circ R$.

4. Draw the directed graph of the relation $R \circ R$.

Solution to Exercise 39:

1. $R = \{(a, b), (a, d), (b, c), (c, e), (d, a), (d, c), (d, e), (e, a), (e, b), (e, d)\}$.

2. $M_R = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$.

3. $R \circ R = \{(a, c), (a, e), (a, a), (b, e), (c, a), (c, b), (c, d), (d, b), (d, d), (d, e), (d, a), (d, b), \; (d, d), (e, b), (e, d), (e, c), (e$

4. $M_{R\circ R} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

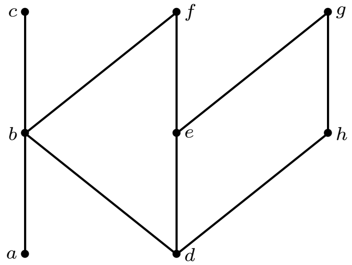
Exercise 65 :
A relation R between the sets $X = \{1,2,3\}$ and $Y = \mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$, has the following binary matrix. $M_R = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$.
List the elements of R.

Solution to Exercise 40:
$R = \{(1,\{1\}), (1,\{1,2\}), (1,\{1,3\}), (1,\{1,2,3\}), (2,\{2\}),\ (2,\{1,2\}), (2,\{2,3\}), (2,\{1,2,3\}), (3,\{3\}), (3,\{1,3\}), (3,\{1$

Exercise 66 :
Let R be the relation on $\mathbb{Z}^+$ defined by nRm if and only if $m = n^2$.
Describe the relation $R^2 = R \circ R$ on $\mathbb{Z}^+$.

Solution to Exercise 41:

Exercise 67 :
A partial order R on a set X with the following Hasse diagram. List the elements of R.



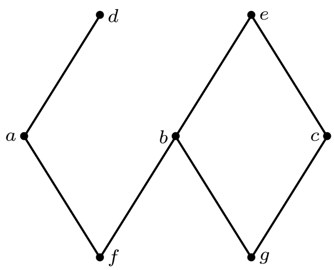
Solution to Exercise 42:

$$\begin{aligned} R \quad = \quad & \{(a,a), (a,b), (a,c), (a,f), (d,d), (d,b), (d,e), (d,h), (d,c), \\ & (d,f), (d,g), (b,b), (b,c), (b,f), (e,e), (e,f), \\ & (e,g), (h,h), (h,g), (c,c), (f,f), (g,g)\} \end{aligned}$$

Exercise 68 :
Draw a Hasse diagram of the partial order relation R on $X = \{a,b,c,d,e,f,g\}$ given by

$$\begin{aligned} R \quad = \quad & \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f), (g,g), (a,d), (b,e), \\ & (c,e), (f,a), (f,b), (f,d), (f,e), (g,b), (g,c), (g,e)\}. \end{aligned}$$

Solution to Exercise 43:



5.1 Order Relation

Exercise 69 :
Among the following relations which are order relations and which are total order relations

1. $X = \mathbb{Z}$, $xRy \iff \exists k \in \mathbb{N} : x = y^k$.
2. $X = \mathbb{N}$, $xRy \iff x < y$.
3. $X = \mathbb{R}$, $xRy \iff x^2 = y^2$.
4. $X = \mathbb{R}^2$, $(x_1, x_2)R(y_1, y_2) \iff (x_1 \leq y_2) \wedge (x_2 \leq y_1)$.

Solution to Exercise 44:

1. R is an order relation.
2. R is not an order relation. (R is not reflexive).
3. R is not an order relation. (R is not antisymmetric, $2R(-2)$ and $(-2)R2$).
4. R is not an order relation. (R is not reflexive $((1,2),(1,2)) \notin R$).

Exercise 70 :
Let $X = \{a, b, c, d\}$ and let R be the relation defined as follows:

$$\begin{aligned} R &= \{(a,a),(b,b),(c,c),(d,d),(c,a),(a,d),(c,d), \\ &\qquad (b,c),(b,d),(b,a)\}. \end{aligned}$$

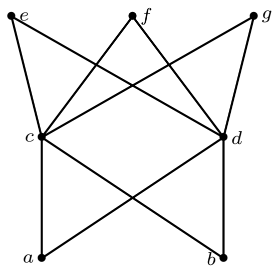
Is R a total order on X?

Solution to Exercise 45:
R is reflexive. R is antisymmetric. R is transitive. Therefore R is a partial order. Since all elements are comparable, R is a total order.

Exercise 71 :
Let $X = \{1, 2, 4, 6, 8\}$ and, for $a, b \in X$, define $a \leq b$ if and only if $\dfrac{b}{a}$ is an integer.

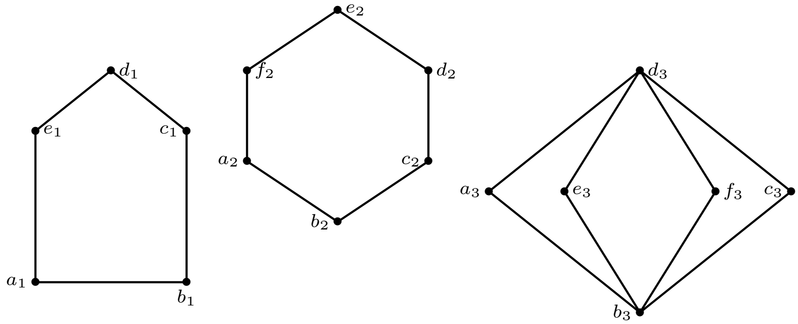
1. Prove that $\leq$ defines a partial order on X.
2. Draw the Hasse diagram for $\leq$.
3. Is $(X, \leq)$ totally ordered? Explain.

Exercise 72 :
Give an example of a poset $(X, \leq)$ so that the following Hasse diagram.

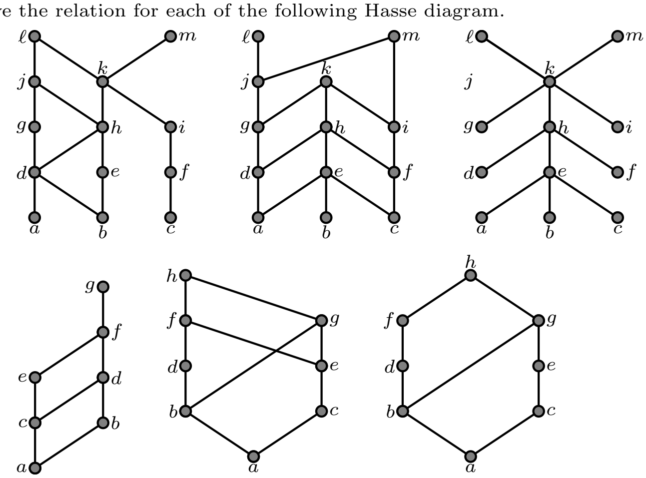


Solution to Exercise 46:
We take the relation R (divides), $a = 2, b = 3, c = 12, d = 18, e = 180, f = 252, g = 396$.

Exercise 73 :
Show that none of the relations whose Hasse diagrams are given below are distributive.



Exercise 74 :
Give the relation for each of the following Hasse diagram.



Exercise 75 :
Determine whether the relation $R \subset \mathbb{Z}^2 \times \mathbb{Z}^2$ defined by:

$$(m, n)R(p, q) \iff m \leq porn \leq q$$

is an order relation.

Solution to Exercise 47:
The relation R is not antisymmetric since $(2,3)R(3,1)$ and $(3,1)R(2,3)$, but $(2,3) \neq (3,1)$.

Exercise 76 :
Determine whether the relation $R = \{(x, y) \in \mathbb{R}^2;\ x^2 \leq y^2\}$ is an order relation.

Solution to Exercise 48:
The relation R is not antisymmetric since $2R-2$ and $-2R2$, but $2 \neq -2$.

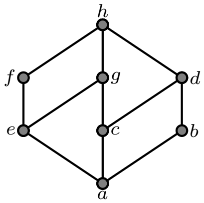
Exercise 77 :
Let $X = \{0,1\}^3$ and the relation R defined on X by:

$$(m_1, n_1, p_1)R(m_2, n_2, p_2) \iff (m_1 \leq m_2) \wedge (n_1 \leq n_2) \wedge (p_1 \leq p_2).$$

Prove that R is an order relation on X and give its Hasse diagram.

Solution to Exercise 49:

Let $a = (0,0,0)$, $b = (0,0,1)$, $c = (0,1,0)$, $d = (0,1,1)$, $e = (1,0,0)$, $f = (1,0,1)$, $g = (1,1,0)$, $h = (1,1,1)$.

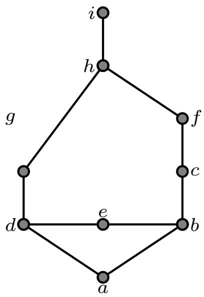


Exercise 78 :
Let $E = \{0,1,2\}$. We define the relation $R \subset E^2 \times E^2$ by:

$$(m,n)R(p,q) \iff m \leq p \text{and} n \leq q.$$

Prove that R is an order relation.

Solution to Exercise 50:
Let $a = (0,0)$, $b = (0,1)$, $c = (0,2)$, $d = (1,0)$, $e = (1,1)$, $f = (1,2)$, $g = (2,0)$, $h = (2,1)$, $i = (2,2)$.
$R = \{(a,a),(a,b),(a,c),(a,d),(a,e),(a,f),(a,g),(a,h),(a,i),$
$(b,b),(b,c),(b,e),(b,f),(b,h),(b,i),(c,c),(c,f),(c,i),(d,d),(d,e),$
$(d,f),(d,g),(d,h),(d,i),(e,e),(e,f),(e,i),(f,f),$
$(f,i),(g,g),(g,h),(g,i),(h,h),(h,i),(i,i)\}$



CHAPTER III

GRAPH THEORY

1 Graphs and Graph Models

Definition 1.1

A graph $G = (V, E)$ is a structure consisting of a set V of vertices (also called nodes), and a set E of edges, which are lines joining vertices.
Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints. If the edge e links the vertex a to the vertex b, we write $e = \{a, b\}$.
The order of a graph $G = (V, E)$ is the cardinality of its vertex set, and the size of a graph is the cardinality of its edge set.

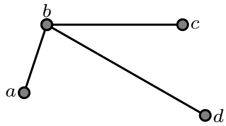
There is several type of graphs, (undirected, directed, simple, multigraph,...) have different formal definitions, depending on what kinds of edges are allowed.

Definition 1.2

1. A simple graph G is a graph that has no loops, (that is no edge $\{a, b\}$ with $a = b$) and no parallel edges between any pair of vertices.

2. A multigraph G is a graph that has no loop and at least two parallel edges between some pair of vertices.

1.1 Simple Undirected Graph

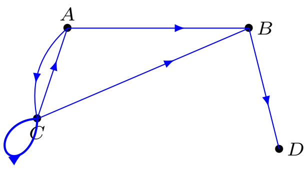


Only undirected edges, at most one edge between any pair of distinct nodes, and no loops.
Directed Graph (Digraph) (with loops)

Definition 1.3

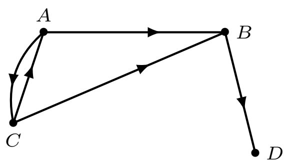
A directed graph (digraph) , $G = (V, E)$, consists of a non-empty set, V, of vertices (or nodes), and a set $E \subset V \times V$ of directed edges (or ordered pairs). Each directed edge $(a, b) \in E$ has a start (tail) vertex a, and a end (head) vertex b.
a is called the initial vertex and b is the terminal vertex.
Note: a directed graph $G = (V, E)$ is simply a set V together with a binary relation E on V.

Example 53 :



Only directed edges, at most one directed edge from any node to any node, and loops are allowed.

1.2 Simple Directed Graph

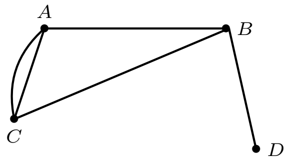


Only directed edges, at most one directed edge from any node to any other node, and no loops allowed.

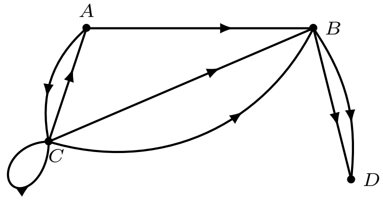
1.3 Undirected Multigraph

Definition 1.4

A (simple,undirected) multigraph, $G = (V, E)$, consists of a non-empty set V of vertices (or nodes), and a set $E \subset [V]^2$ of (undirected) edges, but no loops.



Only undirected edges, may contain multiple edges between a pair of nodes, but no loops.
Directed Multigraph:



Only directed edges, may contain multiple edges from one node to another, the loops are allowed.
Graph Terminology

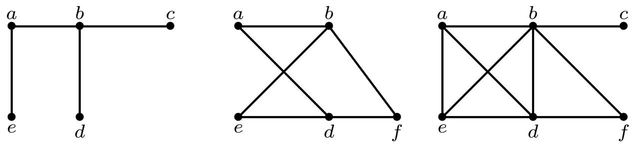
Graph Terminology

	Type	Edges	Multi-Edges	Loops
1	(Simple undirected) graph	Undirected	No	No
2	(Undirected) multigraph	Undirected	Yes	No
3	(Undirected) pseudograph	Undirected	Yes	Yes
4	Directed graph	Directed	No	Yes
5	Simple directed graph	Directed	No	No
6	Directed multigraph	Directed	Yes	Yes
8	Mixed graph	Both	Yes	Yes

Definition 1.5

The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example 54 :



G_1 G_2 $G_1 \cup G_2$

2 Degree and neighborhood of a vertex

Remark 3 :
The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set or an infinite number of edges is called an infinite graph, and in comparison, a graph with a finite vertex

set and a finite edge set is called a finite graph.
In this course we will consider only finite graphs.

Definition 2.1

Two vertices a, b in a graph G are called adjacent in G if $\{a, b\}$ is an edge of G. If $e = \{a, b\}$ is an edge of G, then e is called incident with the vertices a and b or e connects a and b.

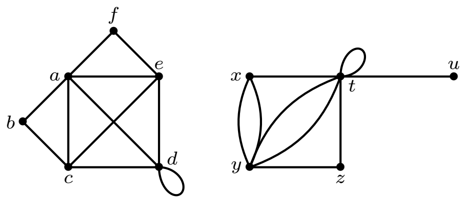
Definition 2.2

The degree of a vertex a in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex a is denoted by $\deg(a)$.

Definition 2.3

The neighborhood (neighbor set) of a vertex a in an undirected graph, denoted $N(a)$ is the set of vertices adjacent to a.

Example 55 :
Let F and G be the following graphs:



The degrees of the vertices in the graphs F and G are respectively: $\deg(a) = 5$, $\deg(b) = 2$, $\deg(c) = 4$, $\deg(d) = 5$, $\deg(e) = 4$, $\deg(f) = 2$.
$\deg(x) = 3$, $\deg(y) = 5$, $\deg(z) = 2$, $\deg(t) = 7$, $\deg(u) = 1$.

$N(a) = \{b, c, d, e, f\}$, $N(b) = \{a, c\}$, $N(c) = \{a, b, d, e\}$. $N(d) = \{a, c, e\}$, $N(e) = \{a, c, d, f\}$, $N(f) = \{a, e\}$.
$N(x) = \{y, z, t\}$, $N(y) = \{x, z, t\}$, $N(z) = \{x, y, t\}$, $N(t) = \{x, y, z, t, u\}$, $N(u) = \{t\}$.

Definition 2.4

For any graph G,we define

$$\delta(G) = \min\{\deg v;\ v \in V(G)\}$$

and

$$\Delta(G) = \max\{\deg v;\ v \in V(G)\}.$$

If all the vertices of G have the same degree r, then $\delta(G) = \Delta(G) = r$ and in this case G is called a regular graph of degree r.
A regular graph of degree 3 is called a cubic graph.

2.1 Handshaking Theorem

Theorem 2.5

If $G = (V, E)$ is a undirected graph with m edges, then:

$$2m = \sum_{a \in V} \deg(a).$$

Proof
Each edge contributes twice to the degree count of all vertices. Hence, both the left-hand and right-hand sides of this equation equal twice the number of edges.

Corollary 2.6

Every cubic graph has an even number of vertices.

Proof
Let G be a cubic graph with p vertices, then $\sum_{v \in V} \deg(v) = 3p$ which is even by Handshaking Theorem. Hence p is even.

Corollary 2.7

An undirected graph has an even number of vertices of odd degree.

Proof
Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{a \in V_1} \deg(a) + \sum_{a \in V_2} \deg(a).$$

$\sum_{a \in V_1} \deg(a)$ must be even since $\deg(a)$ is even for each $a \in V_1$.

$\sum_{a \in V_2} \deg(a)$ must be even because $2m$ and $\sum_{a \in V_1} \deg(a)$ are even.

Example 56 :
Every graph has with at least two vertices contains two vertices of equal degree.

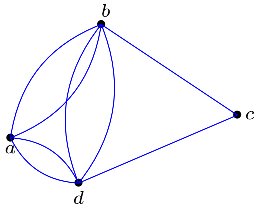
Suppose that the all n vertices have different degrees, and look at the set of degrees. Since the degree of a vertex is at most $n-1$, the set of degrees must be $\{0, 1, 2, \ldots, n-2, n-1\}$.

But that's not possible, because the vertex with degree $n-1$ would have to be adjacent to all other vertices, whereas the one with degree 0 is not adjacent to any vertex.

Example 57 :
If a graph has 7 vertices and each vertices have degree 6. The nombre of edges in the graph is 21. $(6 \times 7 = 42 = 2m = 2 \times 21)$.

Example 58 :
There is a graph with four vertices a, b, c, and d with $\deg(a) = 4$, $\deg(b) = 5 = \deg(d)$, and $\deg(c) = 2$.

The sum of the degrees is $4 + 5 + 2 + 5 = 16$. Since the sum is even, there might be such a graph with $\frac{16}{2} = 8$ edges.

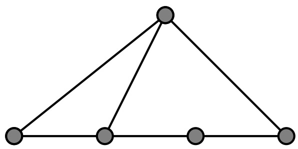


Example 59 :
A graph with 4 vertices of degrees $1, 2, 3,$ and 3 does not exist because $1 + 2 + 3 + 3 = 9$ (The Handshake Theorem.)
Also there is not a such graph because, there is an odd number of vertices of odd degree.

Example 60 :
For each of the following sequences, find out if there is any simple graph of order 5 such that the degrees of its vertices are given by that sequence. If so, give an example.

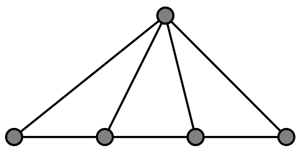
1. $3, 3, 2, 2, 2$
2. $4, 4, 3, 2, 1$.
3. $4, 3, 3, 2, 2$.
4. $3, 3, 3, 2, 2$.
5. $3, 3, 3, 3, 2$.
6. $5, 3, 2, 2, 2$.

1. $3, 3, 2, 2, 2$



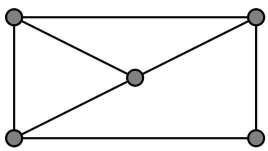
2. $4, 4, 3, 2, 1$. It does not exist. (One vertice v_1 which has degree 4, then there is one edge between v_1 and the others vertices. Also there is an other vertice v_2 which has degree 4, then there is one edge between v_2 and the others vertices. Then the minimum of degree is 2 and not 1).

3. $4, 3, 3, 2, 2$.



4. $3, 3, 3, 2, 2$. It does not exist. (The number of vertives with odd edges is odd).

5. $3, 3, 3, 3, 2$.



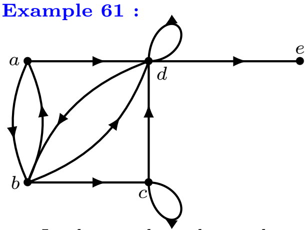
6. $5, 3, 2, 2, 2$. It does not exist. (The order is 5 and one vertex has degree 5).

2.2 Directed Graphs

Definition 2.8

The in-degree of a vertex a, denoted $\deg^-(a)$, is the number of edges directed into a. The out-degree of a, denoted $\deg^+(a)$, is the number of edges directed out of a. Note that a loop at a vertex contributes 1 to both in-degree and out-degree.

Example 61 :



In the graph we have: $\deg^-(a) = 1$, $\deg^+(a) = 2$, $\deg^-(b) = 2$, $\deg^+(b) = 3$, $\deg^-(c) = 2$, $\deg^+(c) = 2$, $\deg^-(d) = 4$, $\deg^+(d) = 3$, $\deg^-(e) = 1$, $\deg^+(e) = 0$.

Theorem 2.9

Let $G = (V, E)$ be a directed graph. Then:

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

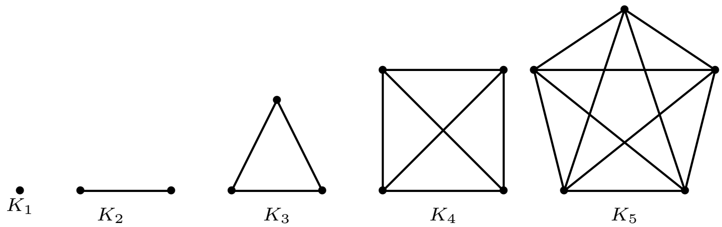
Proof The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. Both sums must be $|E|$.

3 Special Types of Graphs

Definition 3.1

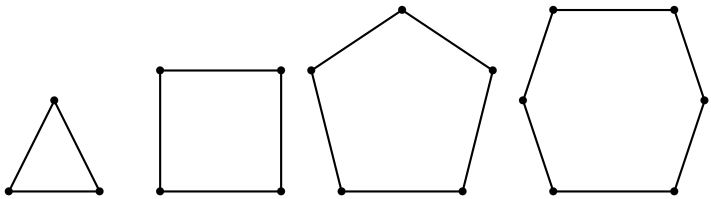
A null graph (or totally disconnected graph) is one whose edge set is empty. (A null graph is just a collection of vertices.)

Complete Graphs A complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



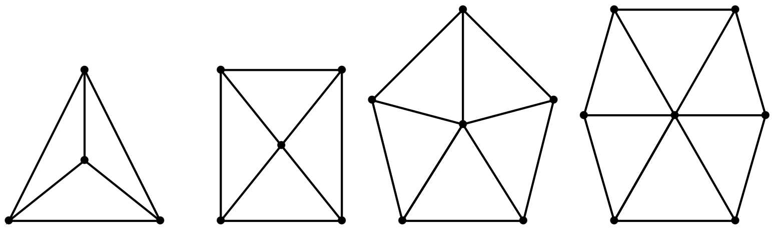
3.1 Cycles

A cycle for $n \geq 3$ consists of n vertices v_1, v_2,..., v_n, and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$,..., $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$.



C_3 C_4 C_5 C_6

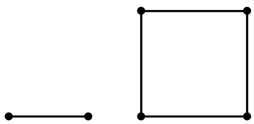
The Wheel Graph The wheel graph W_n ($n \geq 3$) is obtained from C_n by adding a vertex a inside C_n and connecting it to every vertex in C_n.



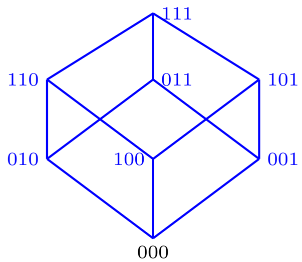
W_3 W_4 W_5 W_6

3.2 $n-$Cubes

An $n-$dimensional hypercube, or $n-$cube, is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices if and only if they differ in exactly one bit position.



Q_1 Q_2



4 Bipartite graphs

Bipartite Graphs

Definition 4.1

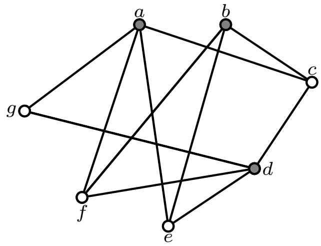
A bipartite graph is an (undirected) graph $G = (V, E)$ whose vertices can be partitioned into two disjoint sets (V_1, V_2), with $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, such that for every edge $e \in E$, $e = \{a, b\}$ such that $a \in V_1$ and $b \in V_2$.

In other words, every edge connects a vertex in V_1 with a vertex in V_2. Equivalently, a graph is bipartite if and only if it is possible to color each vertex red or blue such that no two adjacent vertices have the same color.

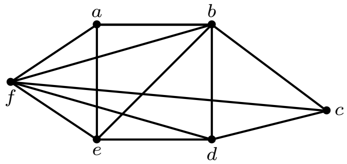
Bipartite Graphs

Definition 4.2

An equivalent definition of a bipartite graph is one where it is possible to color the vertices either red or blue so that no two adjacent vertices are the same color.



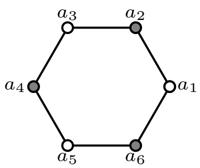
F



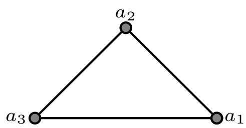
G

F is bipartite. $V_1 = \{a, b, d\}$, $V_2 = \{c, e, f, g\}$.
In G if we color a red, then its neighbors f and b must be blue. But f and b are adjacent. G is not bipartite

Example 62 :



C_6



C_3

C_6 is bipartite. Partition the vertex set of C_6 into $V_1 = \{a_1, a_3, a_5\}$ and $V_2 = \{a_2, a_4, a_6\}$.
If we partition vertices of C_3 into two nonempty sets, one set must contains two vertices. But every vertex is connected to every other. So, the two vertices in the same partition are connected. Hence, C_3 is not bipartite.

Theorem 4.3

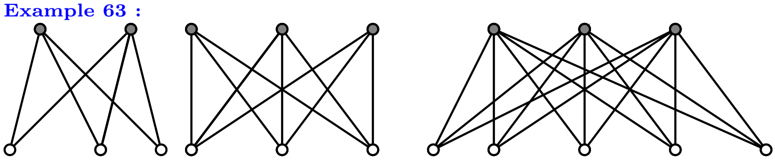
Let G be a graph of n vertices. Then G is bipartite if and only if it contains no cycles of odd length.

4.1 Complete Bipartite Graphs

Definition 4.4

A complete bipartite graph is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

Example 63 :



$K_{2,3}$ $K_{3,3}$ $K_{3,5}$

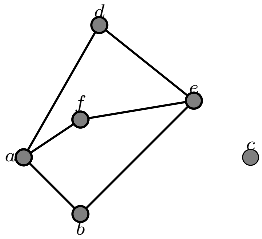
5 Subgraphs

Subgraphs

Definition 5.1

A subgraph of a graph $G = (V, E)$ is a graph (W, F), where $W \subset V$ and $F \subset E$. A subgraph F of G is a proper subgraph of G if $F \neq G$.

Example 64 :
Let $V = \{a, b, c, d, e, f\}$, $E = \{ab, af, ad, be, de, ef\}$ and $G = (V, E)$. Determine all the subgraphs of G of order 4 and size 4.

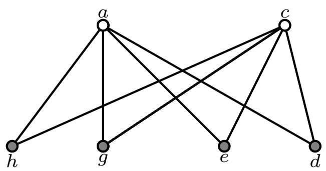


Solution
$(\{a, b, e, d\}, \{ab, be, ed, da\})$, $(\{a, b, e, f\}, \{ab, be, ef, fa\})$, $(\{a, d, e, f\}, \{ad, de, ef, fa\})$

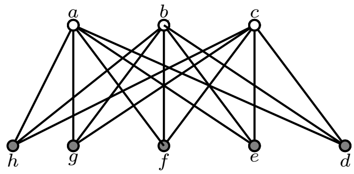
5.1 Induced Subgraphs

Definition 5.2

Let $G = (V, E)$ be a graph. The subgraph induced by a subset W of the vertex set V is the graph $H = (W, F)$, whose edge set F contains an edge in E if and only if both endpoints are in W.



$F = K_{2,4}$



$K_{3,5}$

$K_{2,4}$ is the subgraph of $K_{3,5}$ induced by $W = \{a, c, e, g, h\}$.

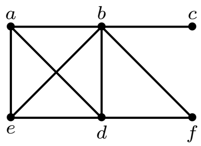
6 Representing Graphs and Graph Isomorphism

Representing Graphs: Adjacency Lists

Definition 6.1

An adjacency list represents a graph (with no multiple edges) by specifying the vertices that are adjacent to each vertex.

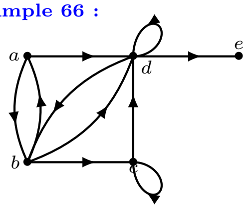
Example 65 :



G

An adjacency list for a simply graph	
Vertex	Adjacent vertices
a	b, d, e
b	a, c, e, d, f
c	b
d	a, b, e, f
e	a, b, d
f	b, d

Example 66 :



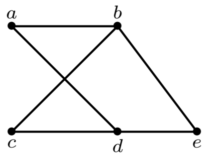
An adjacency list for a directed graph	
Initial vertex	Terminal vertices
a	b, d
b	a, c, d
c	c, d
d	b, d, e
e	

6.1 Representation of Graphs: Adjacency Matrices

Definition 6.2

Let $G = (V, E)$ be a simple graph where $|V| = n$. If $a_1, a_2, \ldots, a_n$ are the vertices of G. The adjacency matrix, A, of G, with respect to this listing of vertices, is the $n \times n$ matrix with its $(i, j)^{\text{th}}$ entry is 1 if a_i and a_j are adjacent, and 0 if they are not adjacent. ($A = (a_{i,j})$, with $a_{i,j} = 1$ if $\{a_i, a_j\} \in E$ and $a_{i,j} = 0$ if $\{a_i, a_j\} \notin E$.)

Example 67 :

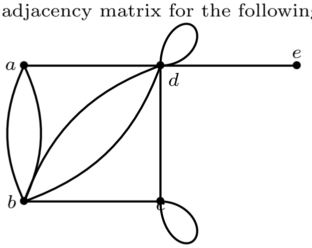


G

The adjacency matrix is $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

The adjacency matrix of an undirected graph is symmetric: Also, since there are no loops, each diagonal entry is zero:

Example 68 :
The adjacency matrix for the following pseudograph is:



$$\begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

6.2 Isomorphism of Graphs

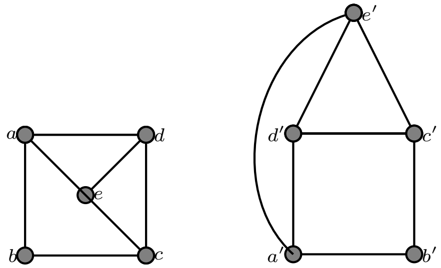
Definition 6.3

Two (undirected) graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called isomorphic if there is a bijection, $f : V_1 \longrightarrow V_2$, with the property that for all vertices $a, b \in V_1$

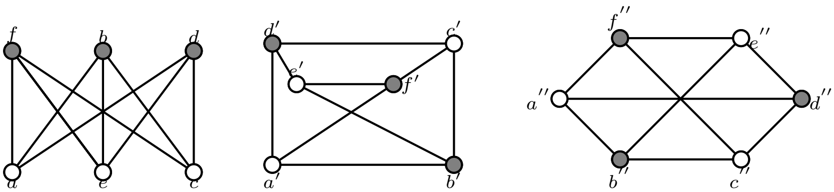
$$\{a, b\} \in E_1 \iff \{f(a), f(b)\} \in E_2.$$

Such function f is called an isomorphism.

The following graphs are isomorphic.



The following graphs are isomorphic.



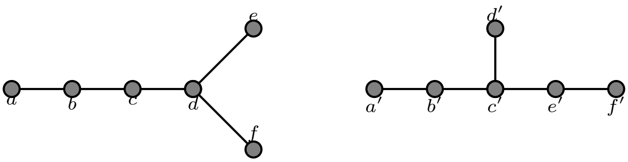
> **Theorem 6.4**
>
> Let f be an isomorphism of the graph $G_1 = (V_1, E_1)$ to the graph $G_2 = (V_2, E_2)$. Let $v \in V_1$. Then $\deg(v) = \deg(f(v)$. i.e., isomorphism preserves the degree of vertices.

Proof A vertex $u \in V_1$ is adjacent to a vertex v in G_1 if and only if $f(u)$ is adjacent to $f(v)$ in G_2. Also f is bijection. Hence the number of vertices in V_1 which are adjacent to v is equal to the number of vertices in V_2 which are adjacent to $f(v)$. Hence $\deg(v) = \deg(f(v))$.

Remarks 4 :

1. Two isomorphic graphs have the same number of vertices and the same number of edges.
2. Two isomorphic graphs have equal number of vertices with a given degree.
   However these conditions are not sufficient to ensure that two graphs are isomorphic.

Example 69 :
Consider the two graphs given in figure below. Under any isomorphism d must correspond to c', a, e, f must correspond to a', d', f' in some order. The remaining two vertices b, c are adjacent whereas b', e' are not adjacent. Hence there does not exist an isomorphism.



7 Connectedness in undirected graphs

7.1 Paths (in undirected graphs)

Informally, a path is a sequence of edges connecting vertices.

Definition 7.1

1. For an undirected graph $G = (V, E)$, an integer $n \geq 0$, and vertices $a, b \in V$, a path of length n from a to b in G is a sequence: $x_0, e_1, x_1, e_2, \ldots, x_{n-1}, e_n, x_n$ of interleaved vertices $x_j \in V$ and edges $e_i \in E$, such that $x_0 = a$ and $x_n = b$, and such that $e_i = \{x_{i-1}, x_i\} \in E$ for all $i \in \{1, \ldots, n\}$.
   Such path starts at a and ends at b.
   The trivial path from v to v consists of the single vertex v.

Definition 7.2

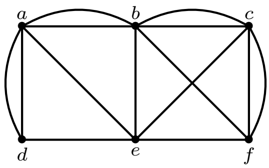
1. A path of length $n \geq 1$ is called a circuit (or cycle) if $n \geq 1$ and the path starts and ends at the same vertex, i.e., $a = b$.

2. A path or circuit is called simple if it does not contain the same edge more than once.

Remarks 5 :

1. When $G = (V, E)$ is a simple undirected graph a path $x_0, e_1, \ldots, e_n, x_n$ is determined uniquely by the sequence of vertices $x_0, x_1, \ldots, x_n$. So, for simple undirected graphs we can denote a path by its sequence of vertices $x_0, x_1, \ldots, x_n$.

2. Don't confuse a simple undirected graph with a simple path. There can be a simple path in a non-simple graph, and a non-simple path in a simple graph.



1. d, a, b, c, f is a simple path of length 4.

2. d, e, c, b, a, d is a simple circuit of length 5.

3. d, a, b, c, f, b, a, e is a path, but it is not a simple path, because the edge $\{a, b\}$ occurs twice in it.

4. c, e, a, d, e, f is a simple path, but it is not a tidy path, because vertex e occurs twice in it.

7.2 Paths in directed graphs

Definition 7.3

1. For a directed graph $G = (V, E)$, an integer $n \geq 0$, and vertices $a, b \in V$, a path of length n from a to b in G is a sequence of vertices and edges $x_0, e_1, x_1, e_2, \ldots, x_n, e_n$, such that $x_0 = a$ and $x_n = b$, and such that $e_i = (x_{i-1}, x_i) \in E$ for all $i \in \{1, \ldots, n\}$.

2. When there are no multi-edges in the directed graph G, the path can be denoted (uniquely) by its vertex sequence $x_0, x_1, \ldots, x_n$.

3. A path of length $n \geq 1$ is called a circuit (or cycle) if the path starts and ends at the same vertex, i.e., $a = b$.

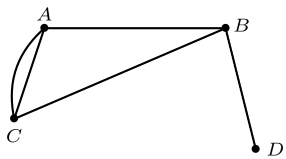
Definition 7.4

4. A path or circuit is called **simple** if it does not contain the same edge more than once. We call it **tidy** if it does not contain the same vertex more than once, except possibly the first and last in case $a = b$ and the path is a circuit (cycle).

7.3 Connectedness in undirected graphs

Definition 7.5

An undirected graph $G = (V, E)$ is called connected, if there is a path between every pair of distinct vertices. It is called disconnected otherwise.



This graph is connected

Theorem 7.6

A graph G is connected if and only if for any partition of V into subsets V_1 and V_2 there is an edge joining a vertex of V_1 to a vertex of V_2.

Theorem 7.7

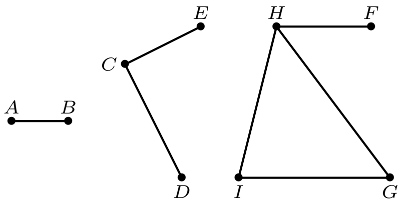
There is always a simple, and tidy, path between any pair of vertices a, b of a connected undirected graph G.

Proof By definition of connectedness, for every pair of vertices a, b, there must exist a shortest path $x_0, e_1, x_1, \ldots, e_n, x_n$ in G such that $x_0 = a$ and $x_n = b$. Suppose this path is not tidy, and $n \geq 1$. (If $n = 0$, the Proposition is trivial.) Then $x_j = x_k$ for some $0 \leq j < k \leq n$. But then $x_0, e_1, x_1, \ldots, x_j, e_{k+1}, x_{k+1}, \ldots, e_n, x_n$ is a shorter path from a to b, contradicting the assumption that the original path was shortest.

7.4 Connected Components of Undirected Graphs

Definition 7.8

A connected component $H = (V', E')$ of a graph $G = (V, E)$ is a maximal connected subgraph of G, meaning H is connected and $V' \subset V$ and $E' \subset E$, but H is not a proper subgraph of a larger connected subgraph R of G.



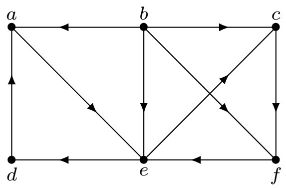
This graph, $G = (V, E)$, has 3 connected components. (It is thus a disconnected graph.)

7.5 Connectedness in Directed Graphs

Definition 7.9

1. A directed graph $G = (V, E)$ is called strongly connected, if for every pair of vertices a and b in V, there is a (directed) path from a to b, and a directed path from b to a.

2. ($G = (V, E)$ is weakly connected if there is a path between every pair of vertices in V in the underlying undirected graph (meaning when we ignore the direction of edges in E.) A strongly connected component of a directed graph G, is a maximal strongly connected subgraph H of G which is not contained in a larger strongly connected subgraph of G.

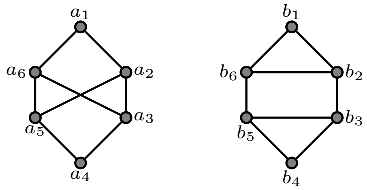


This digraph, G, is not strongly connected, because, for example, there is no directed path from e to b.
One strongly connected component of G is $H = (V_1, E_1)$, where $V_1 = \{a, c, d, e, f\}$ and $E_1 = \{(a, e), (e, c), (c, f), (f, e), (e, d), (d, a)\}$.

8 Paths and Isomorphism

Paths and Isomorphism There are several ways that paths and circuits can help determine whether two graphs are isomorphic. For example, the existence of a simple circuit of a particular length is a useful invariant that can be used to show that two graphs are not isomorphic. In addition, paths can be used to construct mappings that may be isomorphisms. As we mentioned, a useful isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a positive integer greater than 2.

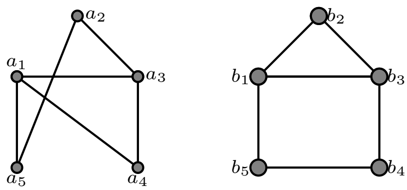
Let G and H be the following graphs.



G H

Both G and H have six vertices and eight edges. Each has 4 vertices of degree 3, and two vertices of degree 2. So, the three invariants number of vertices, number of edges, and degrees of vertices all agree for the two graphs. However, H has a simple circuit of length 3, namely, b_1, b_2, b_6, b_1, whereas G has no simple circuit of length 3. Then G and H are not isomorphic.

Example 70 :
Let G and H be the following graphs.



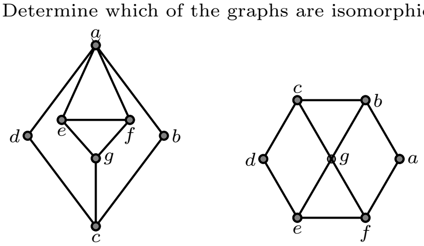
G H

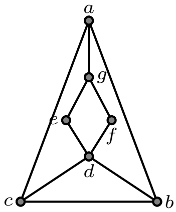
Both G and H have 5 vertices and 6 edges, both have 2 vertices of degree 3 and 3 vertices of degree 2, and both have a simple circuit of length 3, a simple circuit of length 4, and a simple circuit of length 5.

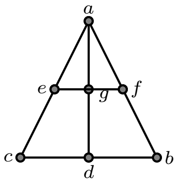
Because all these isomorphic invariants agree, G and H may be isomorphic.
To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree. For example, the paths a_1, a_4, a_3, a_2, a_5 in G and b_3, b_2, b_1, b_5, b_4 in H both go through every vertex in the graph, start at a vertex of degree 3, go through vertices of degrees 2, three, and two, respectively, and end at a vertex of degree 2. By following these paths through the graphs, we define the mapping f with $f(a_1) = b_3$, $f(a_4) = b_2$, $f(a_3) = b_1$, $f(a_2) = b_5$, and $f(a_5) = b_4$.

Exercise 79 :
Construct two graphs that have the same degree sequence but are not isomorphic.
solution Let F be of a cycle on 6 vertices, and let G be the union of two disjoint cycles on 3 vertices each. In both graphs each vertex has degree 2, but the graphs are not isomorphic, since one is connected and the other is not.

Determine which of the graphs are isomorphic.







9 Counting Paths Between Vertices

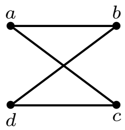
9.1 Counting Paths Between Vertices

The number of paths between two vertices in a graph can be determined using its adjacency matrix.

Theorem 9.1

Let G be a graph with adjacency matrix A with respect to the ordering $b_1, b_2, \ldots, b_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from b_i to b_j, where r is a positive integer, equals the $(i, j)^{th}$ entry of A^r.

Example 71 :
How many paths of length four are there from a to d in the simple graph G



The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Hence, the number of paths of length 4 from a to d is the $(1, 4)^{th}$ entry of A^4. Because

$$A = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}.$$

There are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths of length four from a to d.

CHAPTER IV

TREES

1 General Properties of Trees

1.1 General Properties of Trees

Definition 1.1

A tree is a connected simple undirected graph with no simple circuits.
A forest is a (not necessarily connected) simple undirected graph with no simple circuits.

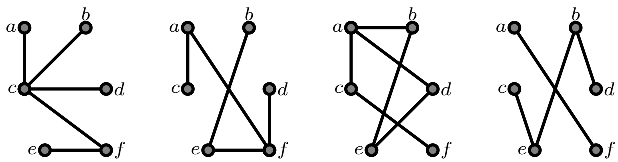
Remark 6 :
A graph G is a tree if and only if it contains no cycles, but adding any new edge creates a cycle.

Properties 1.2:

f T is a tree:

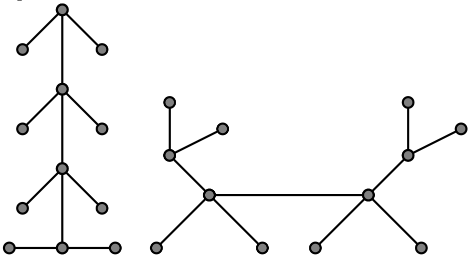
1. There is a unique simple path between any 2 of its vertices.
2. No loops.
3. No multiple edges.

Examples 72 :

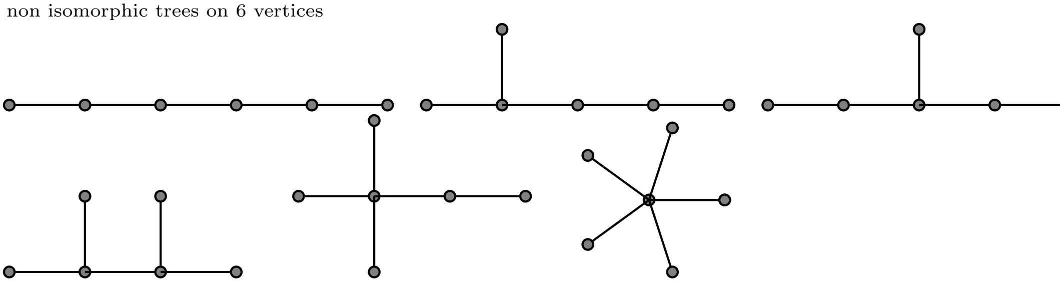


The first and the second are trees. The third contains a circuit a, b, e, d. The forth is not connected.

Examples 73 :
Example of forest:



Examples 74 :
The non isomorphic trees on 6 vertices



Theorem 1.3

Any undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Theorem 1.4

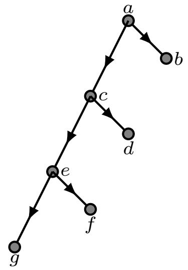
Every tree is a bipartite graph.

1.2 Rooted (Directed) Trees

Definition 1.5

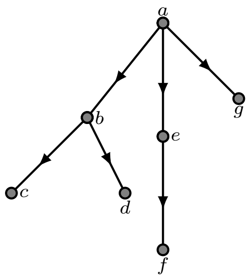
A rooted tree is a tree in which one vertex has been designed as the root and every edge is directed away from the root.

Examples 75 :



The root is a vertex with in-degree 0. [Node a is the root]

Example 76 :

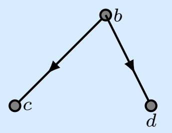


Definition 1.6

1. Parent: a vertex u is a parent of v if there is a directed edge from u to v.
2. Child: If u is parent of v, then v is child of u. [c and d are children of b]
3. Siblings: Vertices with the same parents. [c and d]
4. Ancestors: Vertices in path from the root to vertex v, excluding v itself, including the root. [Ancestors of d : b, a]
5. Descendents: All vertices that have v as ancestors. [Descendents of b : c, d]

Definition 1.7

6. Leaf: Vertex with no children. [c, d, f, g]
7. Internal vertices: Vertices that have children. [a, b, e].
8. Subtree: Subgraphs consisting of v and its descendents and their incident edges. (Subtree rooted at b)



Definition 1.8

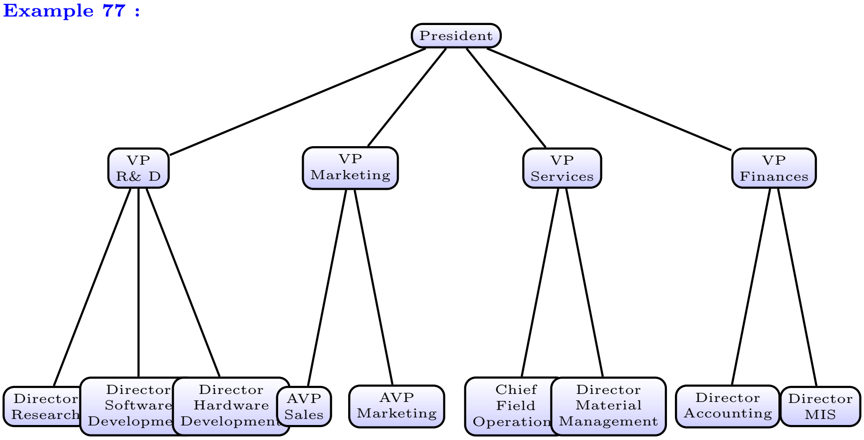
9. Level (of a vertex v) is the length of the unique path from root to v. [level of root $= 0$, level of $b = 1$, level of $c = 2$]

10. Height is maximum of vertices levels. [Height $= 2$]

1.3 Trees as Models

Trees are used as models in such diverse areas as computer science, chemistry, geology, botany, and psychology.We will describe a variety of such models based on trees. For example the representation of organizations. Each vertex in this tree represents a position in the organization. An edge from one vertex to another indicates that the person represented by the initial vertex is the (direct) boss of the person represented by the terminal vertex.

Example 77 :



2 Properties of Trees

Theorem 2.1

Every tree, $T = (V, E)$ with $|V| \geq 2$, has at least two vertices that have degree equal 1.

Proof
Take any longest simple path $x_1, \dots, x_m$ in T. Both x_1 and x_m must have degree 1: otherwise there is a longer path in T.

Theorem 2.2

Every tree with n vertices has exactly $n-1$ edges.

The proof is by induction on n. For $n = 1$, there is no edges. Suppose the result for n and take a tree $T = (V, E)$ with $n+1$ vertices. There is $x \in V$ with degree 1. Since T is a tree, $T - x$ is a tree with n vertices. Then $|E| - 1 = |V - \{x\}| - 1$ and $|E| = |V| - 1$.

Theorem 2.3

Let $T = (V, E)$ be a connected graph such that $|V| = n$ and $|E| = n-1$, then T is a tree.

Proof
To prove that T is a tree it suffices to prove that do not contains circuits. If $x_1, \dots, x_m$ is a circuit from a to a. Then x_2 is not a bridge and $T - x_2$ is a connected graph with n vertices and $n-2$ edges, which is impossible. Then T does not contain circuits.

Definition 2.4

If a graph G is connected and e is an edge such that $G - e$ is not connected, then e is said to be a bridge or a cut edge.

Theorem 2.5

Any edge in a tree is a bridge.

Remark 7 :
If $T = (V, E)$ is a tree such that $|V| = n$ and $|E| = n-1$, then the sum of degrees is $2(n-1)$.

3 $m-$ary Trees

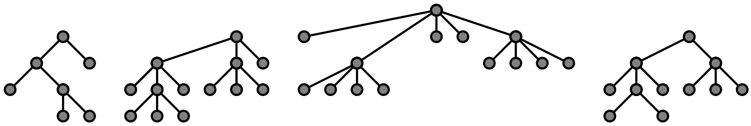
$m-$ary Trees

Definition 3.1

Let $m \geq 1$

1. A rooted tree is called a $m-$ary tree if every internal node has at most m children.
2. A rooted tree is called a full $m-$ary tree if every internal node has exactly m children.

3. An $m-$ary tree with $m = 2$ is called a binary tree.



F $\quad$ G $\quad$ H $\quad$ I

F is a binary tree, G is a full $3-$ary, H is a $5-$ary, I is $3-$ary.
Counting vertices in a full $m-$ary trees

Theorem 3.2

For all $m \geq 1$, every full $m-$ary tree with i internal vertices has exactly $m.i + 1$ vertices.

Proof Every vertex other than the root is a child of an internal vertex. There are thus $m.i$ such children, plus 1 root.

Theorem 3.3

For all $m \geq 1$, a full $m-$ary tree with:

1. n vertices has $i = \dfrac{n-1}{m}$ internal vertices and $\ell = \dfrac{(m-1)n+1}{m}$ leaves.

2. i internal vertices has $n = m.i + 1$ vertices and $\ell = (m-1)i + 1$ leaves.

3. if $m \geq 2$, then if the $m-$ary tree has ℓ leaves then it has $n = m\ell - 1m - 1$ vertices and $i = \dfrac{\ell - 1}{m-1}$ internal vertices.

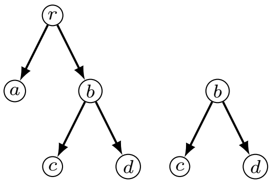
4 Ordered Rooted Trees

1. A rooted ordered tree is a rooted tree $(T, \leq)$ where in addition the children of each internal vertex v are linearly ordered according to some ordering $\leq$.

2. When drawing the tree, we usually write ordered children (from least to greatest) from left to right.

3. If the rooted ordered tree is a binary tree, then the first child is called left child and the second child is called right child.

4. The tree rooted at the left child of a vertex is called the left subtree of this vertex, and the tree rooted at the right child of a vertex is called the right subtree of the vertex.



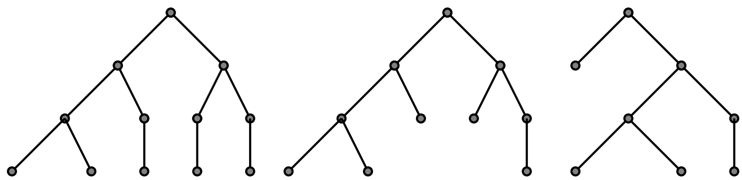
G H

In the tree G, a is the left child of r, b is the right child of r and the tree H is the right sub-tree of r.

Definition 4.1

1. The height of the rooted tree is the maximum of the levels of vertices (length of the longest path from the root to any vertex)

2. **Balanced Tree** A rooted $m-$ary tree of height h is balanced if all leaves are at levels h or $h-1$.



Balanced Balanced not Balanced

5 Applications of Trees

5.1 Applications of Trees

We will discuss three problems that can be studied using trees.

1. The first problem is: How should items in a list be stored so that an item can be easily located?

2. The second problem is: What series of decisions should be made to find an object with a certain property in a collection of objects of a certain type?

3. The third problem is: How should a set of characters be efficiently coded by bit strings?

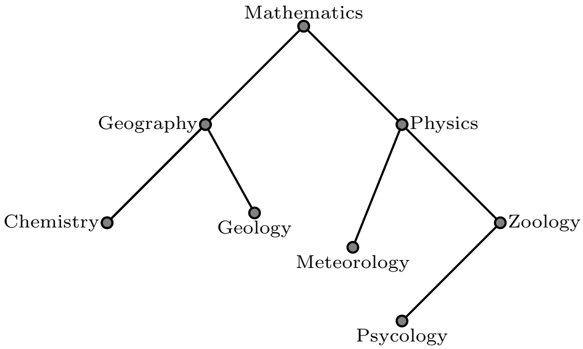
5.2 Binary Search Trees

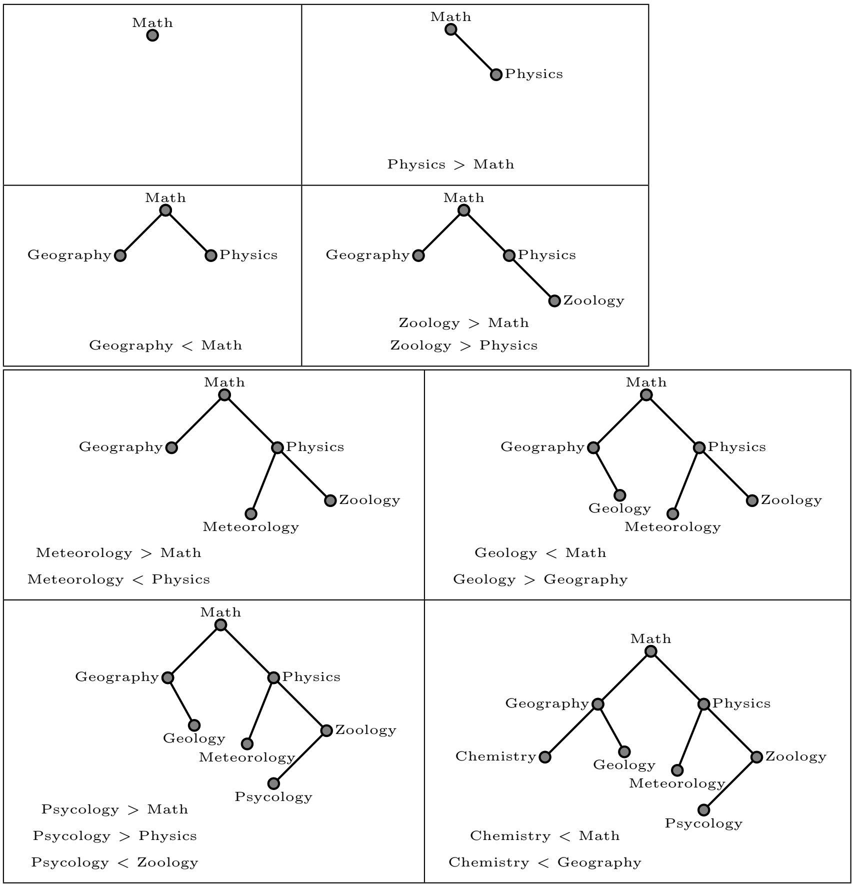
1. Searching for items in a list is one of the most important tasks that arises in computer science. Our primary goal is to implement a searching algorithm that finds items efficiently when the items are totally ordered. This can be accomplished through the use of a binary search tree.

2. A binary search tree is a binary tree in which each child of a vertex is designated as a right or left child, no vertex has more than one right child or left child, and each vertex is labeled

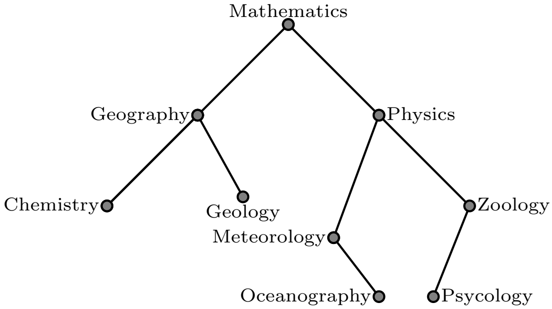
with a key, which is one of the items. Furthermore, vertices are assigned keys so that the key of a vertex is both larger than the keys of all vertices in its left subtree and smaller than the keys of all vertices in its right subtree.

Example 78 :
Form a binary search tree for the words: mathematics, Physics, Geography, Zoology, Meteorology, Geology, Psychology, and Chemistry (using alphabetical order).

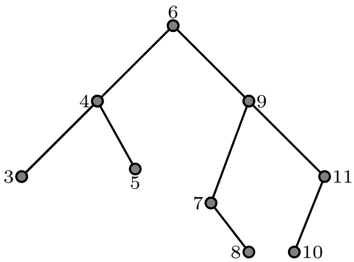




Example 79 :
Insert the word Oceanography into the binary search tree in the previous example.
mathematics $<$ oceanography, physics $>$ oceanography, meteorology $<$ oceanography.



Example 80 :
Form a binary search tree for the numbers: 6, 9, 4, 11, 7, 5, 10, 3 and 8 (using the order on $\mathbb{N}$).



5.3 Decision Trees

1. Rooted trees can be used to model problems in which a series of decisions leads to a solution.

2. For instance, a binary search tree can be used to locate items based on a series of comparisons, where each comparison tells us whether we have located the item, or whether we should go right or left in a subtree.

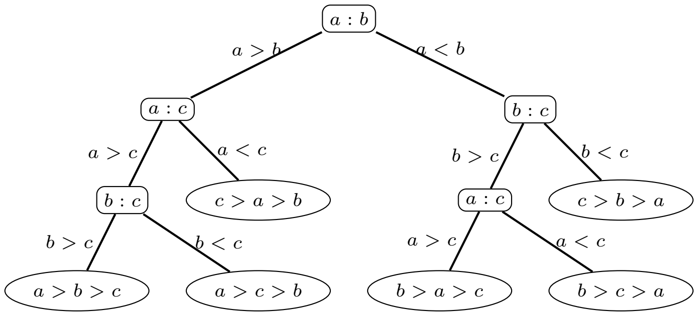
3. A rooted tree in which each internal vertex corresponds to a decision, with a subtree at these vertices for each possible outcome of the decision, is called a **decision tree**.

4. The possible solutions of the problem correspond to the paths to the leaves of this rooted tree.

Example 81 :
Suppose there are seven coins, all with the same weight, and a counterfeit coin that weights less than the others. How many weighings are necessary using a balance scale to determine which of the eight coins is the counterfeit one?

We do a weighing of the coins $(1, 2, 3)$ and $(3, 4, 5)$. If they have the same weight, we do an other weighing of the coins $(7, 8)$ and we have done. If $(1, 2, 3)$ is lighter than $(3, 4, 5)$, we do an other weighing of the coins $(1, 2)$ and we have done.

Example 82 :
A decision tree that orders the elements of the list a, b, c.



5.4 Prefix Codes

1. Consider using bit strings of different lengths to encode letters.

2. When letters are encoded using varying numbers of bits, some method must be used to determine where the bits for each character start and end.

3. One way to ensure that no bit string corresponds to more than one sequence of letters is to encode letters so that the bit string for a letter never occurs as the first part of the bit string for another letter. Codes with this property are called **prefix codes**.

4. A prefix code can be represented using a binary tree, where the characters are the labels of the leaves in the tree. The edges of the tree are labeled so that an edge leading to a left child is assigned a 0 and an edge leading to a right child is assigned a 1.

1. The bit string used to encode a character is the sequence of labels of the edges in the unique path from the root to the leaf that has this character as its label.

2. For instance, the tree in Figure (??) represents the encoding of e by 0, a by 10, t by 110, n by 1110, and s by 1111.

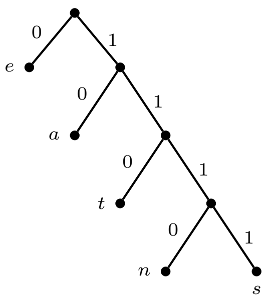


figure IV.1: A Binary Tree with a Prefix Code.

1. The tree representing a code can be used to decode a bit string. For instance, consider the word encoded by 11111011100 using the code in Figure (??). This bit string can be decoded by starting at the root, using the sequence of bits to form a path that stops when a leaf is reached.

5.5 Tree Traversal

1. Ordered rooted trees are often used to store information.
2. We need procedures for visiting each vertex of an ordered rooted tree to access data.
3. We will describe several important algorithms for visiting all the vertices of an ordered rooted tree.
4. Ordered rooted trees can also be used to represent various types of expressions, such as arithmetic expressions involving numbers, variables, and operations.

5.6 Traversal Algorithms

Procedures for systematically visiting every vertex of an ordered rooted tree are called traversal algorithms. We will describe three of the most commonly used such algorithms.

1. **preorder traversal**,**R**oot, **L**eft, **R**ight.
2. **inorder traversal**, **L**eft, **R**oot, **R**ight.
3. **postorder traversal**, **L**eft, **R**ight, **R**oot.

5.7 Preorder Traversal

Definition 5.1

Let T be an ordered rooted tree with root r. If T consists only of r, then r is the preorder traversal of T . Otherwise, suppose that $T_1, T_2, \cdots, T_n$ are the subtrees at r from left to right in T . The preorder traversal begins by visiting r. It continues by traversing T_1 in preorder, then T_2 in preorder, and so on, until T_n is traversed in preorder.

Example 83 :
In which order does a preorder traversal visit the vertices in the ordered rooted tree T shown in Figure (??) below

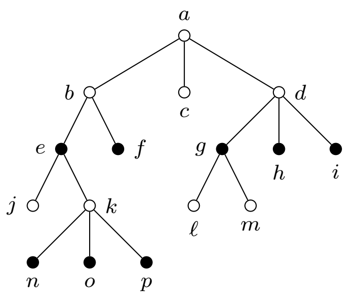


figure IV.2: The Ordered Rooted Tree T.

The steps of the preorder traversal of T are shown in Figure (??) below.

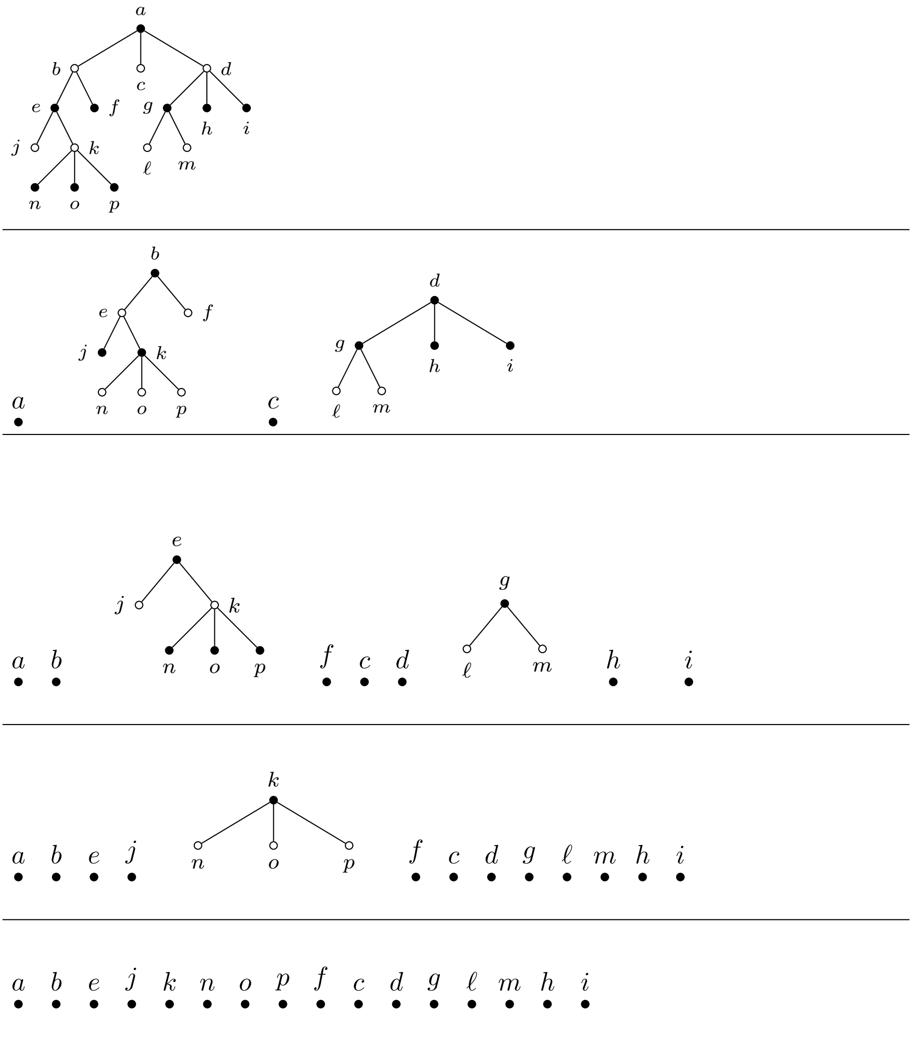


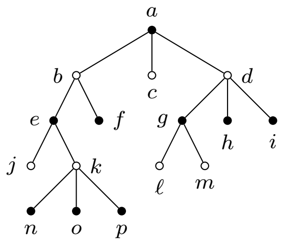
figure IV.3: The Preorder Traversal of T.

Inorder Traversal

Definition 5.2

Let T be an ordered rooted tree with root r. If T consists only of r, then r is the inorder traversal of T . Otherwise, suppose that $T_1, T_2, \cdots, T_n$ are the subtrees at r from left to right. The inorder traversal begins by traversing T_1 in inorder, then visiting r. It continues by traversing T_2 in inorder, then T_3 in inorder, ... , and finally T_n in inorder.

Example 84 :
In which order does an inorder traversal visit the vertices of the ordered rooted tree T below?



The steps of the inorder traversal of T are as fololows:

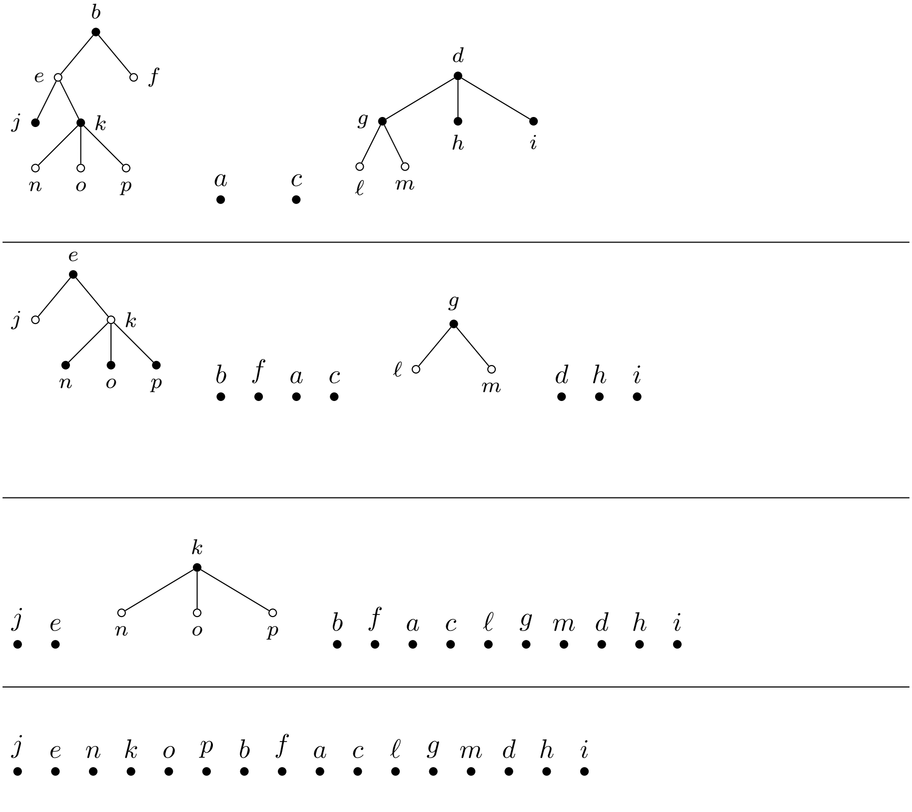


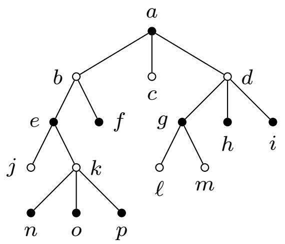
figure IV.4: The Inorder Traversal of T.

5.8 Postorder Traversal

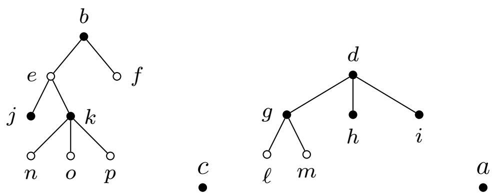
Definition 5.3

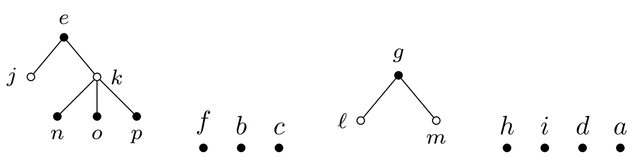
Let T be an ordered rooted tree with root r. If T consists only of r, then r is the postorder traversal of T . Otherwise, suppose that $T_1, T_2, \cdots, T_n$ are the subtrees at r from left to right. The postorder traversal begins by traversing T_1 in postorder, then T_2 in postorder, $\cdots$, then T_n in postorder, and ends by visiting r.

Example 85 :
In which order does a postorder traversal visit the vertices of the ordered rooted tree T shown below?



The steps of the postorder traversal of the ordered rooted tree T is as follows:





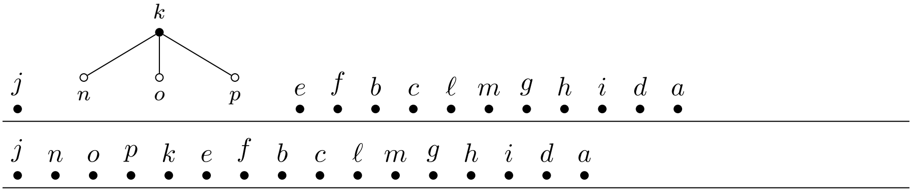


figure IV.5: The Postorder Traversal of T.

5.9 Infix, Prefix, and Postfix Notation

We can represent complicated expressions, such as compound propositions, combinations of sets, and arithmetic expressions using ordered rooted trees. For instance, consider the representation of an arithmetic expression involving the operators $+$ (addition), $-$ (subtraction), $*$ (multiplication), $/$ (division), and $\uparrow$ (exponentiation). We will use parentheses to indicate the order of the operations.An ordered rooted tree can be used to represent such expressions, where the internal vertices represent operations, and the leaves represent the variables or numbers. Each operation operates on its left and right subtrees (in that order).

Example 86 :
What is the ordered rooted tree that represents the expression $((x + y) \uparrow 2) + ((x - 4)/3)$?

The binary tree for this expression can be built from the bottom up. First, a subtree for the expression $x+y$ is constructed. Then this is incorporated as part of the larger subtree representing $(x + y) \uparrow 2$. Also, a subtree for $x - 4$ is constructed, and then this is incorporated into a subtree representing $(x-4)/3$. Finally the subtrees representing $((x+y) \uparrow 2)$ and $((x-4)/3)$ are combined to form the ordered rooted tree representing $((x + y) \uparrow 2) + ((x - 4)/3)$. These steps are shown as follows (Figure **??**).

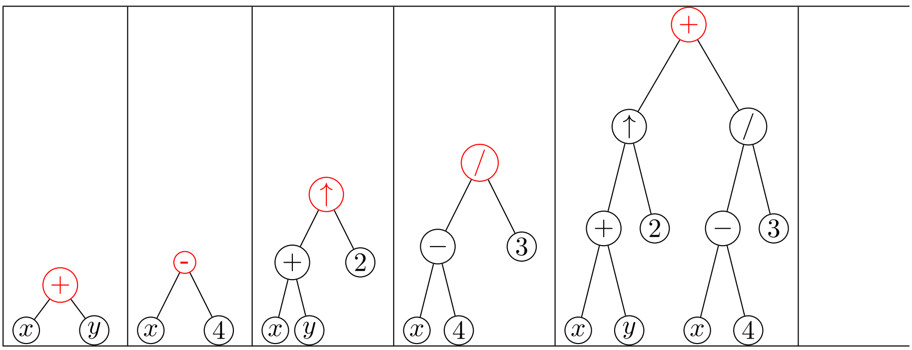


figure IV.6: A Binary Tree Representing $((x + y)^2) + (\frac{x-4}{3})$.

An inorder traversal of the binary tree representing an expression produces the original expression with the elements and operations in the same order as they originally occurred, except for unary operations, which instead immediately follow their operands. For instance, inorder traversals of the binary trees in Figure ??, which represent the expressions $\dfrac{x+y}{x+3}$, $(x + (\dfrac{y}{x})) + 3$, and $x + (\dfrac{y}{x+3})$, all lead to the infix expression $\dfrac{x+y}{x+3}$.
To make such expressions unambiguous it is necessary to include parentheses in the inorder traversal whenever we encounter an operation.

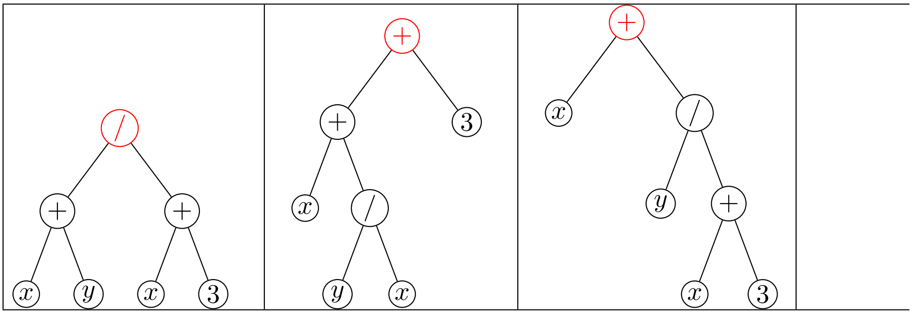
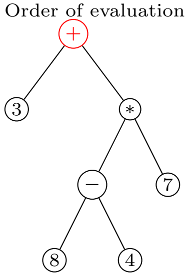


figure IV.7: Rooted Trees Representing $\dfrac{x+y}{x+3}$, $(x+\dfrac{y}{x})+3$, and $x+(\dfrac{y}{x+3})$.

- The fully parenthesized expression obtained in this way is said to be in **infix form**.
- We obtain the **prefix form** of an expression when we traverse its rooted tree in preorder.
- We obtain the **postfix form** of an expression by traversing its binary tree in postorder.

Order of evaluation



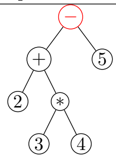
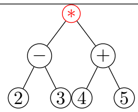
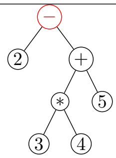
1. Prefix form: $+3 * -8\ 4\ 7$ (lecture) $+3*(8-4)7 \to +3(4*7) \to +3\ 28 = 31$
2. Infix form: $3+8-4*7$ (lecture) $3+((8-4)*7) = 31$.
3. Postfix form: $3\ 8\ 4{-}7{*}{+}$ (lecture) $3(8{-}4)7{*}{+} \to 347{*}{+} \to 3(4{*}7){+} \to 3\ 28{+} = 28{+}3 = 31$.

5.10 Evaluations of Arithmetic Expressions

Expression	Prefix forms	Infix forms	Postfix forms
(a + b)	+ a b	a + b	a b +
a - (b * c)	- a * b c	a - b * c	a b c * -

Remarks 8 :

1. A prefix or postfix form corresponds to exactly one expression tree.
2. An infix form may correspond to more than one expression tree. It is therefore not suitable for expression evaluation.

Expression	Expression Tree	Infix form
$2-3*4+5$		$2-3*4+5$
$(2-3)*(4+5)$		$2-3*4+5$
$2-(3*4+5)$		$2-3*4+5$

6 Spanning Trees of undirected graphs

6.1 Spanning Trees of undirected graphs

Definition 6.1

For a simple undirected graph G, a spanning tree of G is a subgraph T of G such that T is a tree and T contains every vertex of G.

Theorem 6.2

A simple graph is connected if and only if it has a spanning tree.

6.2 Depth-First Search

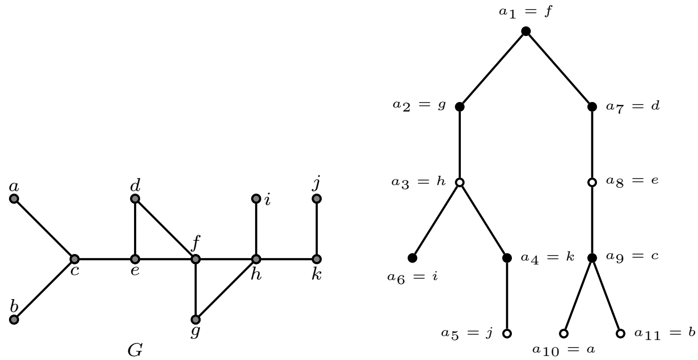
We can build a spanning tree for a connected simple graph using **depth-first search**. We will form a rooted tree, and the spanning tree will be the underlying undirected graph of this rooted tree.
Arbitrarily choose a vertex of the graph as the root. Form a path starting at this vertex by successively adding vertices and edges, where each new edge is incident with the last vertex in the path and a vertex not already in the path. Continue adding vertices and edges to this path as long as possible. If the path goes through all vertices of the graph, the tree consisting of this path is a spanning tree. However, if the path does not go through all vertices, more vertices and edges must be added. Move back to the next to last vertex in the path, and, if possible, form a new path starting at this vertex passing through vertices that were not already visited.

If this cannot be done, move back another vertex in the path, that is, two vertices back in the path, and try again.
Repeat this procedure, beginning at the last vertex visited, moving back up the path one vertex at

a time, forming new paths that are as long as possible until no more edges can be added. Because the graph has a finite number of edges and is connected, this process ends with the production of a spanning tree. Each vertex that ends a path at a stage of the algorithm will be a leaf in the rooted tree, and each vertex where a path is constructed starting at this vertex will be an internal vertex.

Depth-first search is also called backtracking, because the algorithm returns to vertices previously visited to add paths.

Example 87 :

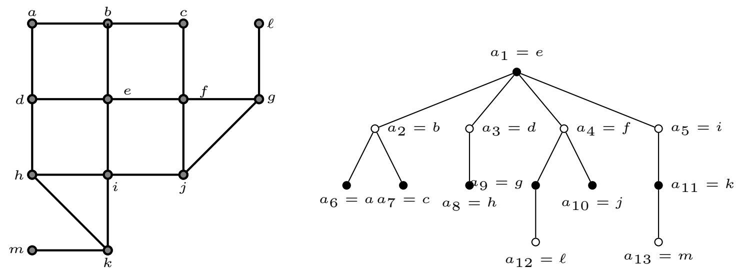


6.3 Breadth-First Search

We can also produce a spanning tree of a simple graph by the use of **breadth-first search.** Arbitrarily choose a root from the vertices of the graph. Then add all edges incident to this vertex. The new vertices added at this stage become the vertices at level 1 in the spanning tree. Arbitrarily order them. Next, for each vertex at level 1, visited in order, add each edge incident to this vertex to the tree as long as it does not produce a simple circuit. Arbitrarily order the children of each vertex at level 1.

This produces the vertices at level 2 in the tree. Follow the same procedure until all the vertices in the tree have been added. The procedure ends because there are only a finite number of edges in the graph. A spanning tree is produced because we have produced a tree containing every vertex of the graph.

Example 88 :



CHAPTER V

BOOLEAN ALGEBRAS

1 The Abstract Definition of a Boolean Algebra

Definition 1.1

A Boolean algebra is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation $^{-}$ such that these properties hold for all x, y, and z in B:

Identity laws $\begin{cases} x \vee 0 = x \\ x \wedge 1 = x \end{cases}$

Complement laws $\begin{cases} x \vee \bar{x} = 1 \\ x \wedge \bar{x} = 0 \end{cases}$

Associative laws $\begin{cases} (x \vee y) \vee z = x \vee (y \vee z) \\ (x \wedge y) \wedge z = x \wedge (y \wedge z) \end{cases}$

Commutative laws $\begin{cases} x \vee y = y \vee x \\ x \wedge y = y \wedge x \end{cases}$

Distributive laws $\begin{cases} x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \end{cases}$

If we denote $\vee$ by $+$ and $\wedge$ by ., we have the following:

Identity laws $\begin{cases} x + 0 = x \\ x.1 = x \end{cases}$

Complement laws $\begin{cases} x + \bar{x} = 1 \\ x.\bar{x} = 0 \end{cases}$

Associative laws $\begin{cases} (x + y) + z = x + (y + z) \\ (x.y).z = x.(y.z) \end{cases}$

Commutative laws $\begin{cases} x + y = y + x \\ x.y = y.x \end{cases}$

Distributive laws $\begin{cases} x + (y.z) = (x + y).(x + z) \\ x.(y + z) = (x.y) + (x.z) \end{cases}$

From our previous discussion, $B = \{0, 1\}$. The three operations in this Boolean algebra are

- The complement of an element, denoted with a bar, is defined by

$$\overline{0} = 1 \text{ and } \overline{1} = 0$$

- The Boolean sum, denoted by + or by OR, has the following values:

$$1+1=1, 1+0=1, 0+1=1, 0+0=0.$$

- The Boolean product, denoted by . or by AND, has the following values:

$$1.1=1, 1.0=0, 0.1=0, 0.0=0.$$

Example 89 :
Using the definitions of complementation, the Boolean sum, and the Boolean product, it follows that
$1.0+\overline{(0+1)}=0+\overline{1}=0+0=0$

The complement, Boolean sum, and Boolean product correspond to the logical operators, $\neg$, $\vee$ and $\wedge$, respectively, where 0 corresponds to **F** (false) and 1 corresponds to **T** (true). Equalities in Boolean algebra can be directly translated into equivalences of compound propositions. Conversely, equivalences of compound propositions can be translated into equalities in Boolean algebra.We will see later in this section why these translations yield valid logical equivalences and identities in Boolean algebra. Example 2 illustrates the translation from Boolean algebra to propositional logic.

Example 90 :
Translate $1.0+\overline{(0+1)}$, the equality found in Example 1, into a logical equivalence.
We obtain a logical equivalence when we translate each 1 into a T, each 0 into an F, each Boolean sum into a disjunction, each Boolean product into a conjunction, and each complementation into a negation.We obtain $(T \wedge F) \vee \neg(T \vee F) \equiv F$.

Example 91 :
The following example illustrates the translation from propositional logic to Boolean algebra.
Translation of the logical equivalence $(T \wedge T) \vee \neg F \equiv T$ into an identity in Boolean algebra.
We obtain an identity in Boolean algebra when we translate each T into a 1, each F into a 0, each disjunction into a Boolean sum, each conjunction into a Boolean product, and each negation into a complementation. We obtain $(1.1)+\overline{0}=1$.

Identities of Boolean Algebra There are many identities in Boolean algebra. The most important of these are displayed in Table 5. These identities are particularly useful in simplifying the design of circuits.

Boolean Identities	
Identity	Name
$\bar{\bar{x}}=x$	Low of double complement
$x+x=x$ $x.x=x$	Idempotent lows
$x+0=x$ $x.1=x$	Identity lows
$x+1=1$ $x.0=0$	Domination lows

Boolean Identities	
Identity	Name
$x+y=y+x$ $x.y=y.x$	Commutative lows
$x+(y+z)=(x+y)+z$ $x.(y.z)=(x.y).z$	Associative lows
$x+(y.z)=(x+y).(x+z)$ $x.(y+z)=x.y+x.z$	Distributive lows
$\overline{(x+y)}=\bar{x}.\bar{y}$ $\overline{(x.y)}=\bar{x}+\bar{y}$	De Morgan's lows
$x+x.y=x$ $x.(x+y)=x$	Absorption lows
$x+\bar{x}=1$	Unit property
$x.\bar{x}=0$	Zero property

Example 92 :
Show that the distributive law $x(y+z)=xy+xz$ is valid.

The verification of this identity is shown in Table ??. The identity holds because the last two columns of the table agree.

Verification of one of the Distributive Laws							
x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

2 Boolean Functions

Boolean Expressions and Boolean Functions Let B be a boolean algebra and $B_n = \{(x_1, \ldots, x_n);\ x_1, \ldots, x_n \in B\}$. The variable x is called a **Boolean variable** if it assumes values only from B. A function from B_n to B is called a **Boolean function of degree** n.

Example 93 :
The function $F(x, y) = x\bar{y}$ from the set of ordered pairs of Boolean variables to the set $\{0, 1\}$ is a Boolean function of degree 2 with $F(1, 1) = 0, F(1, 0) = 1, F(0, 1) = 0$, and $F(0, 0) = 0$.
We display these values of F in the following Table.

x	y	F(x,y)
1	1	0
1	0	1
0	1	0
0	0	0

1. Two different Boolean expressions that represent the same function are called **equivalent**.

2. For instance, the Boolean expressions xy, $xy + 0$, and $xy.1$ are equivalent.

3. The **complement** of the Boolean function F is the function $\overline{F}$, where $\overline{F}(x_1, \cdots, x_n) = \overline{F(x_1, \cdots, x_n)}$.

4. Let F and G be Boolean functions of degree n. The **Boolean sum** $F+G$ and the **Boolean product** FG are defined by $(F+G)(x_1, \cdots, x_n) = F(x_1, \cdots, x_n)+G(x_1, \cdots, x_n)$, $(FG)(x_1, \cdots, x_n) = F(x_1, \cdots, x_n)G(x_1, \cdots, x_n)$.

3 Representation of Boolean Functions

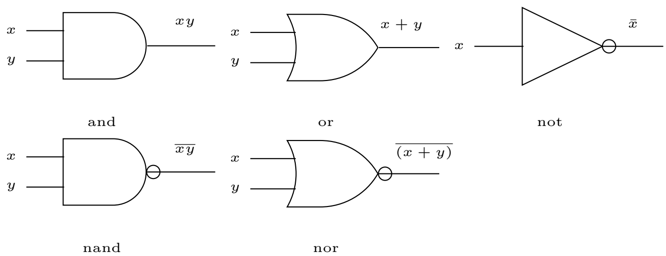
Representation of Boolean Functions Two important problems of Boolean algebra will be studied in this section. The first problem is: Given the values of a Boolean function, how can a Boolean expression that represents this function be found? This problem will be solved by showing that any Boolean function can be represented by a **Boolean sum of Boolean products** of the variables and their complements. The solution of this problem shows that every Boolean function can be represented using the three Boolean operators ., +, and –. The second problem is: Is there a smaller set of operators that can be used to represent all Boolean functions? We will answer this question by showing that all Boolean functions can be represented using only one operator. Both of these problems have practical importance in circuit design.

Sum-of-Products Expansions Let $F(x, y, z)$ and $G(x, y, z)$ be the Boolean functions defined by the following table.

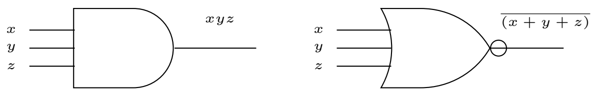
x	y	z	F(x,y,z)	G(x,y,z)
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	1
0	1	1	1	0
0	1	0	0	1
0	0	1	1	0
0	0	0	0	1

The expressions of these functions are:
$F(x,y,z) = x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}z$, $G(x,y,z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$.

Logic Gates A logic gate is an electronic circuit or device which makes the logical decisions. To arrive at this decisions, the most common logic gates used are OR, AND, NOT, NAND, and NOR gates. The NAND and NOR gates are called universal gates. The exclusive-OR gate is another logic gate which can be constructed using AND, OR and NOT gate. Logic gates are also called switches.



Remark 9 :
The associative laws show that there is no ambiguity about a term such as $x+y+z$ or xyz, so we can introduce multiple-input primitive gates:



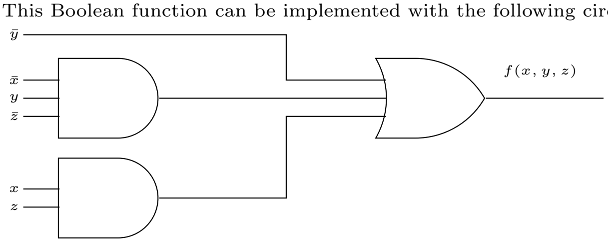
Any logic function can be implemented using NAND gates. To achieve this, first the logic function has to be written in Sum of Product (SOP) form. Once logic function is converted to SOP, then is very easy to implement using NAND gate. In other words any logic circuit with AND gates in first level and OR gates in second level can be converted into a NAND-NAND gate circuit.

Any logic function can be implemented using NOR gates. To achieve this, first the logic function has to be written in Product of Sum (POS) form. Once it is converted to POS, then it's very easy to implement using NOR gate. In other words any logic circuit with OR gates in first level and AND gates in second level can be converted into a NOR-NOR gate circuit.

Example 94 :
For example, let f be the following Boolean function:

$$f(x,y,z) = \bar{y} + \bar{x}y\bar{z} + xz.$$

This Boolean function can be implemented with the following circuit.



Remarks 10 :

1. We can write expressions in many ways, but some ways are more useful than others.
2. A sum of products (SOP) expression is characterized by:
   - There are only OR (sum) operations at the "outermost" level.
   - Each term in the sum must be a product of literals.
3. The advantage is that a sum of products expression can be implemented using a fairly simple two-level circuit:
   Literals are at the "0th" level.
   AND gates are at the first level.
   A single OR gate is at the second level.

Definition 3.1

A **literal** is a Boolean variable or its complement.
A **minterm** of the Boolean variables $x_1, x_2, \cdots, x_n$ is a Boolean product $y_1 y_2 \cdots y_n$, where $y_i = x_i$ or $y_i = \overline{x_i}$. Hence, a minterm is a product of n literals, with one literal for each variable.

For example, a three-variable function like $f(x, y, z)$ has up to 8 minterms:

$$xyz,\ xy\bar{z},\ x\bar{y}z,\ x\bar{y}\bar{z},\ \bar{x}yz,\ \bar{x}y\bar{z},\ \bar{x}\bar{y}z,\ \bar{x}\bar{y}\bar{z}$$

Every function can be written as a sum of minterms, which is a special kind of sum of products form.
The sum of minterms form for any function is unique. This sum is called the complete sum of product (CSP) of the function.
If you have a truth table for a function, you can write a sum of minterms expression just by picking out the rows of the table where the function output is 1.

Definition 3.2

- We denote by **CSP** form of Boolean function its (**Complete sum-of-product**).
- We denote by **CPS** form of Boolean function its (**Complete product-of-sum**). This form is obtained by giving the CSP for the complement of the function, and we take the complement of the CSP give the CPS.

• Every Boolean function can be written in many different ways, so it's sometimes useful to use standard representations like sums of products or sums of minterms.
• Every Boolean expression can be converted to a circuit.
Now we look at a graphical technique for simplifying an expression into a minimal sum of products (MSP) form:

•There are a minimal number of product terms in the expression.
• Each term has a minimal number of literals.

4 Minimization of Circuits - Karnaugh Maps

The Karnaugh map

- To reduce the number of terms in a Boolean expression representing a circuit, it is necessary to find terms to combine. There is a graphical method, called a **Karnaugh map** or **K-map**, for finding terms to combine for Boolean functions involving a relatively small number of variables.

- We will first illustrate how K-maps are used to simplify expansions of Boolean functions in two variables.

- We will continue by showing how K-maps can be used to minimize Boolean functions in three variables and then in four variables.

There are four possible minterms in the sum-of-products expansion of a Boolean function in the two variables x and y. A K-map for a Boolean function in these two variables consists of four cells, where a 1 is placed in the cell representing a minterm if this minterm is present in the expansion. Cells are said to be **adjacent** if the minterms that they represent differ in exactly one literal. For instance, the cell representing $\overline{x}y$ is adjacent to the cells representing xy and $\overline{xy}$. The four cells and the terms that they represent are shown in Figure ??.

	y	$\bar{y}$
x	xy	$x\bar{y}$
$\bar{x}$	$\bar{x}y$	$\bar{x}\bar{y}$

K-maps in Two Variables.

A K-map in three variables is a rectangle divided into eight cells. The cells represent the eight possible minterms in three variables. Two cells are said to be adjacent if the minterms that they represent differ in exactly one literal. One of the ways to form a K-map in three variables is shown in Figure ??.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

K-maps in Three Variables.

A K-map in four variables is a rectangle divided into sixteen cells. The cells represent the sixteen possible minterms in three variables. Two cells are said to be adjacent if the minterms that they represent differ in exactly one literal. One of the ways to form a K-map in three variables is shown in Figure ??.

	zw	$z\bar{w}$	$\bar{z}\bar{w}$	$\bar{z}w$
xy	$xyzw$	$xyz\bar{w}$	$xy\bar{z}\bar{w}$	$xy\bar{z}w$
$x\bar{y}$	$x\bar{y}zw$	$x\bar{y}z\bar{w}$	$x\bar{y}\bar{z}\bar{w}$	$x\bar{y}\bar{z}w$
$\bar{x}\bar{y}$	$\bar{x}\bar{y}zw$	$\bar{x}\bar{y}z\bar{w}$	$\bar{x}\bar{y}\bar{z}\bar{w}$	$\bar{x}\bar{y}\bar{z}w$
$\bar{x}y$	$\bar{x}yzw$	$\bar{x}yz\bar{w}$	$\bar{x}y\bar{z}\bar{w}$	$\bar{x}y\bar{z}w$

K-maps in Four Variables.

Definition 4.1

- We denote by **MSP** form (**Minimal sum-of-product**), is obtained using K-maps method.

- We denote by **MPS** form (**Minimal product-of-sum**), is obtained by giving the CSP for the complement of the function, and we take the complement of the CSP give the CPS.

Grouping terms in the K-map: The power of K-maps is in minimizing the terms, K-maps can be minimized with the help of grouping the terms to form single terms.
When forming groups of squares, consider the following:

1. Every square containing 1 must be considered at least once.

2. A square containing 1 can be included in as many groups as desired.

3. A group must be as large as possible.
4. If a square containing 1 cannot be placed in a group, then leave it out to include in final expression.
5. The number of squares in a group must be equal to 2 .i.e. $2, 4, 8, .$
6. The map is considered to be folded or spherical, therefore squares at the end of a row or column are treated as adjacent squares.
7. The simplified logic expression obtained from a K-map is not always unique. Groupings can be made in different ways.
8. Before drawing a K-map the logic expression must be in canonical form.

Example 95 :
Find the K-maps for $f(x,y) = xy + \overline{x}y$, $g(x,y) = x\overline{y} + \overline{x}y$, and $h(x,y) = x\overline{y} + \overline{x}y + \overline{x}\,\overline{y}$.
The three K-maps of these functions is as follows respectively:

	y	$\bar{y}$
x	1	0
$\bar{x}$	1	0

	y	$\bar{y}$
x	0	1
$\bar{x}$	1	0

	y	$\bar{y}$
x	0	1
$\bar{x}$	1	1

figure V.1: K-maps for the Sum-of-Products Expansions

Using the K-maps, the minimal expansions for these Boolean functions are $f(x,y) = y$, $g(x,y) = x\overline{y} + \overline{x}y$, and $h(x,y) = \overline{x} + \overline{y}$.

Example 96 :
Let $f(x,y,z) = xy + \overline{y}z + xz + x\overline{y}\overline{z}$.
• First, we look for the (CSP) form of f.
Here is the truth table and sum of minterms for f

x	y	z	f(x,y,z)
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

The K-map of f is:

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$	0	0	0	1

The (MSP) of f is $f(x,y,z) = x + \bar{y}z$.
Example Let f be the following Boolean function: $f(x,y,z) = x + \overline{y}(\overline{x} + z)$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$	0	0	1	1

The (CSP) of f is $xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$.
The (CSP) of $\bar{f}$ is $\bar{x}yz + \bar{x}y\bar{z}$.
The (MSP) of f is $\bar{x}\bar{y} + x$.
The (MPS) of f is $(x + \bar{y} + \bar{z}).(x + \bar{y} + z)$.

Example 97 :
Let g be the following Boolean function: $g(x,y,z) = \overline{x}(x + \overline{y} + y\overline{z})$.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	0	0	0	0
$\bar{x}$	0	1	1	1

The (CSP) of g is $\bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z$.
The (CSP) of $\bar{g}$ is $\bar{x}yz + xyz + xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}z$.
The (MSP) of g is $\bar{x}y\bar{z} + \bar{x}\bar{y}$.
The (MPS) of g is $(x + \bar{y} + z).(\bar{x} + \bar{y} + \bar{z}).(\bar{x} + \bar{y} + z).(\bar{x} + y + z).(\bar{x} + y + \bar{z})$.

Example 98 :
We consider the Boolean function of degree 4: $f(x,y,z,w) = x\bar{y}zw + xy\bar{z}w + \bar{x}yz\bar{w}$.
The (CSP) of f is $x\bar{y}zw + xy\bar{z}w + \bar{x}yz\bar{w}$.
The (CSP) of $\bar{f}$ is $\bar{x}yzw + \bar{x}\bar{y}zw + \bar{x}yzw + xyz\bar{w} + x\bar{y}z\bar{w} + \bar{x}\bar{y}z\bar{w} + xy\bar{z}\bar{w} + x\bar{y}\bar{z}\bar{w} + \bar{x}\bar{y}\bar{z}\bar{w} + \bar{x}y\bar{z}\bar{w} + x\bar{y}\bar{z}w + \bar{x}\bar{y}\bar{z}w + \bar{x}y\bar{z}w$.
The (MSP) of f is $x\bar{y}zw + \bar{x}yz\bar{w} + xy\bar{z}w$.
The (MPS) of f is $(\bar{x} + \bar{y} + \bar{z}).(z + w).(x + \bar{w}).(y + w).(y + z)$.
Example Consider the following Boolean function:

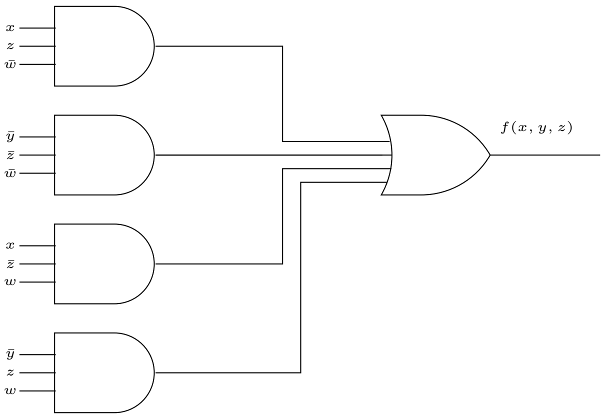
$$f(x,y,z) = xyz\bar{w} + xy\bar{z}w + x\bar{y} + \bar{x}\bar{y}zw + \bar{x}\bar{y}\bar{z}\bar{w}.$$

The Karnaugh -map for $f(x,y,z,w)$ is:

	zw	$z\bar{w}$	$\bar{z}\bar{w}$	$\bar{z}w$
xy	0	1	0	1
$x\bar{y}$	1	1	1	1
$\bar{x}\bar{y}$	1	0	1	0
$\bar{x}y$	0	0	0	0

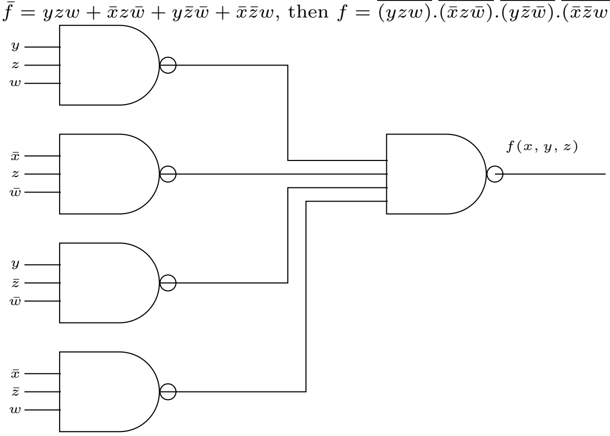
MSP(f) $= xz\bar{w} + \bar{y}\bar{z}\bar{w} + x\bar{z}w + \bar{y}zw$.
$\bar{f} = yzw + \bar{x}z\bar{w} + y\bar{z}\bar{w} + \bar{x}\bar{z}w$.
MPS(f) $= (\bar{y} + \bar{z} + \bar{w})(x + \bar{z} + w)(\bar{y} + z + w)(x + z + \bar{w})$

Construction of a minimal circuit using (AND-OR) gates, with $f(x, y, z, w)$ output.



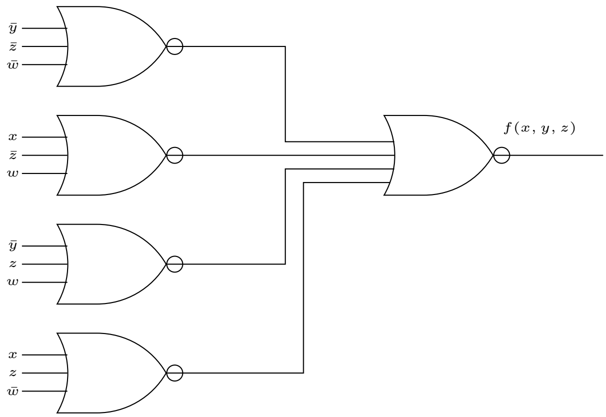
A minimal circuit using (AND-OR) gates, with $f(x, y, z, w)$ output.

Construction of circuits with $f(x, y, z, w)$ output using **NAND** gates.
$\bar{f} = yzw + \bar{x}z\bar{w} + y\bar{z}\bar{w} + \bar{x}\bar{z}w$, then $f = \overline{(yzw)}.\overline{(\bar{x}z\bar{w})}.\overline{(y\bar{z}\bar{w})}.\overline{(\bar{x}\bar{z}w)}$.



A circuit using (NAND) gates, with $f(x, y, z, w)$ output.

Construction of circuits with $f(x, y, z, w)$ output using **NOR** gates.
$f = (\bar{y} + \bar{z} + \bar{w})(x + \bar{z} + w)(\bar{y} + z + w)(x + z + \bar{w})$, then $\bar{f} = \overline{(\bar{y} + \bar{z} + \bar{w})} + \overline{(x + \bar{z} + w)} + \overline{(\bar{y} + z + w)} + \overline{(x + z + \bar{w})}$

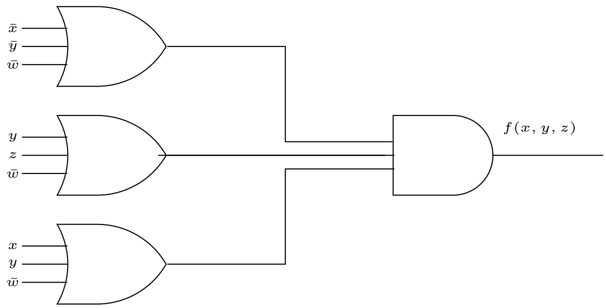


A circuit using (NOR) gates, with $f(x,y,z,w)$ output.

Example 99 :
Let $f(x,y,z,w) = z\bar{w} + \bar{z}\bar{w} + x\bar{y}z + \bar{x}yzw + \bar{x}y\bar{z}w$. The Karnaugh -map for f is

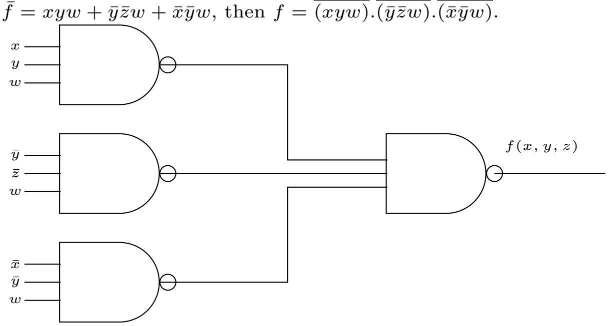
	zw	$z\bar{w}$	$\bar{z}\bar{w}$	$\bar{z}w$
xy	0	1	1	0
$x\bar{y}$	1	1	1	0
$\bar{x}\bar{y}$	0	1	1	0
$\bar{x}y$	1	1	1	1

$\mathbf{MSP(f)} = \bar{w} + \bar{x}yw + x\bar{y}z$,
$\bar{f} = xyw + \bar{x}\bar{y}w + \bar{y}\bar{z}w$. $\mathbf{MPS(f)} = (\bar{x} + \bar{y} + \bar{w})(x + y + \bar{w})(y + z + \bar{w})$.
Construction of a minimal circuit using (AND-OR) gates, with $f(x,y,z,w)$ output.



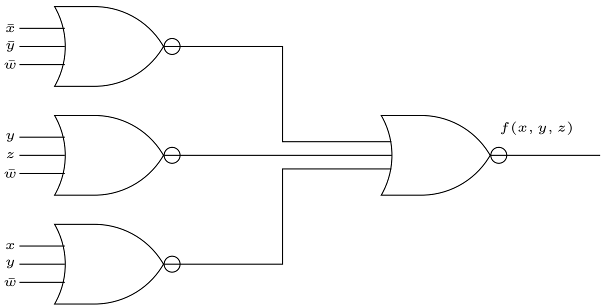
A minimal circuit using (AND-OR) gates, with $f(x, y, z, w)$ output.

Construction of circuits with $f(x, y, z, w)$ output using **NAND** gates.
$\bar{f} = xyw + \bar{y}\bar{z}w + \bar{x}\bar{y}w$, then $f = \overline{(xyw)}.\overline{(\bar{y}\bar{z}w)}.\overline{(\bar{x}\bar{y}w)}$.



A circuit using (NAND) gates, with $f(x, y, z, w)$ output.

Construction of circuits with $f(x, y, z, w)$ output using **NOR** gates.
$f = (\bar{x} + \bar{y} + \bar{w})(x + y + \bar{w})(y + z + \bar{w})$, then $\bar{f} = \overline{(\bar{x} + \bar{y} + \bar{w})} + \overline{(x + y + \bar{w})} + \overline{(y + z + \bar{w})}$.



A circuit using (NOR) gates, with $f(x, y, z, w)$ output.

Example 100 :
Use K-maps to minimize these sum-of-products expansions.

1. $xy\overline{z} + x\overline{y}\ \overline{z} + \overline{x}\ \overline{y}\ \overline{z}$
2. $x\overline{y}z + x\overline{y}\ \overline{z} + \overline{x}\ \overline{y}z + \overline{x}yz + \overline{x}\ \overline{y}\ \overline{z}$
3. $xyz + xy\overline{z} + x\overline{y}z + x\overline{y}\ \overline{z} + \overline{x}\ \overline{y}z + \overline{x}\ \overline{y}\ \overline{z}$
4. $xy\overline{z} + x\overline{y}\ \overline{z} + \overline{x}\ \overline{y}z + \overline{x}\ \overline{y}\ \overline{z}$

1. $xy\overline{z} + x\overline{y}\ \overline{z} + \overline{x}\ \overline{y}\ \overline{z} = y\bar{z} + x\bar{z}$.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	0	1	1	0
$\bar{x}$	0	1	0	0

2. $x\overline{y}z + x\overline{y}\ \overline{z} + \overline{x}\ \overline{y}z + \overline{x}yz + \overline{x}\ \overline{y}\ \overline{z} = \bar{y} + \bar{x}\bar{y}$.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	0	0	1	1
$\bar{x}$	1	0	1	1

3. $xyz + xy\overline{z} + x\overline{y}z + x\overline{y}\ \overline{z} + \overline{x}\ \overline{y}z + \overline{x}\ \overline{y}\ \overline{z} = x + \bar{y}$.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$	0	0	1	1

4. $xy\overline{z} + x\overline{y}\ \overline{z} + \overline{x}\ \overline{y}z + \overline{x}\ \overline{y}\ \overline{z} = x\bar{z} + \bar{x}\bar{y}$.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	0	1	1	0
$\bar{x}$	0	0	1	1