INTEGRAL CALCULUS

CLASS NOTES

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Preface

This book represents the standard content of an Integral Calculus course (Calculus II), more precisely it is objected to engineering and computer science students at King Saud University to cover the material of the course MATH 106 (Integral Calculus). This book can be regarded as a general reference to any Calculus II course.

The book consists of five chapters besides an appendix. Each section has various examples to make sure that students understand and absorb mathematical concepts and theories represented in this course.

Chapter One focuses on Riemann Integral, anti-derivative, indefinite integral and the fundamental theorem of Calculus.

Chapter Two represents the logarithmic function, the exponential function, the hyperbolic functions and their inverse functions.

Chapter Three focuses on the most famous techniques of integration such as integration by parts, trigonometric substitutions and the method of partial fractions.

Chapter Four deals with the applications of definite integral, especially evaluating the area of a plane region, the volume of a solid of revolution, the arc length and the area of a surface of revolution.

Chapter Five represents parametric equations of plane curves, polar curves and focuses on the area between two polar curves.

Chapter Six is an appendix and it aims to present the mathematical tools for students to plot parametric curves in the plane in a rigorous way.

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Dr Mongi Blel and Dr Tariq Alfadhel

CONTENTS

CHAPTER 1

THE RIEMANN INTEGRAL

1 Anti-Derivative, Indefinite Integral

1.1 Anti-Derivative

In the classical calculus course, we defined the derivative of a function if it exists. In this section we are interested in the inverse problem. If f is a function defined on an interval I, we look for (if possible) a function F such that $F' = f$ on I.

Definition 1.1

Let $f\colon I \longrightarrow \mathbb{R}$ be a function defined on an interval I. A function $F\colon I \longrightarrow \mathbb{R}$ is called an anti-derivative of f on I, if F is differentiable on I and

$$F'(x) = f(x), \quad \forall x \in I.$$

Example 1 :

1. The function $F(x) = x^2 + 1$ is an anti-derivative of the function $f(x) = 2x$ on $\mathbb{R}$.

2. The function $2\sqrt{x}$ is an anti-derivative of the function $\dfrac{1}{\sqrt{x}}$ on $(0, +\infty)$.

Theorem 1.2

Let F and G be two anti-derivatives of a function f on an interval I, then there is a constant $c \in \mathbb{R}$ such that

$$F(x) = G(x) + c, \quad \forall x \in I.$$

Proof .
$(F - G)'(x) = F'(x) - G'(x) = 0$, then $F - G$ is the constant function on the interval I.

□

1.2 The Indefinite Integral

Definition 1.3

If a function $f\colon I \longrightarrow \mathbb{R}$ has an anti-derivative on I, $\displaystyle\int f(x)dx$ denotes any anti-derivative of f. The function $\displaystyle\int f(x)dx$ is called an indefinite integral of f on I. Therefore,

$$\frac{d}{dx}\int f(x)dx = f(x), \quad \forall x \in I \tag{1.1}$$

In (1.1), x is called the variable of integration and f the integrand.

Example 2 :

1. $\displaystyle\int x^r dx = \frac{x^{r+1}}{r+1} + c, r \in \mathbb{Q} \setminus \{-1\},$

2. $\displaystyle\int \cos(x)dx = \sin(x) + c,$

3. $\displaystyle\int \sin(x)dx = -\cos(x) + c,$

4. $\displaystyle\int \sec^2(x)dx = \tan(x) + c,$

5. $\displaystyle\int \csc^2(x)dx = -\cot(x) + c,$

6. $\displaystyle\int \sec(x)\tan(x)dx = \sec(x) + c,$

7. $\displaystyle\int \csc(x)\cot(x)dx = -\csc(x) + c,$

Theorem 1.4: Important formulas

Let $f, g \colon I \longrightarrow \mathbb{R}$ be two functions.

1. If f is differentiable and $\frac{d}{dx}f(x)$ has an anti-derivative, then
$$\int \frac{d}{dx}f(x)dx = f(x) + c.$$

2. If f has an anti-derivative, then
$$\frac{d}{dx}\int f(x)dx = f(x).$$

3. If f has an anti-derivative on I, then for all $\lambda \in \mathbb{R}$,
$$\int \lambda f(x)dx = \lambda \int f(x)dx.$$

4. If f and g have anti-derivatives, then the functions $f \pm g$ have anti-derivatives and
$$\int (f(x) \pm g(x))\, dx = \int f(x)dx \pm \int g(x)dx.$$

Example 3 :

1.

$$\int \left(\frac{3}{x^4} - 5x\right) dx = \int (3x^{-4} - 5x)\, dx = \int 3x^{-4} dx - \int 5x dx$$
$$= -x^{-3} - \frac{5}{2}\, x^2 + c.$$

2.

$$\int \frac{2x^2+3}{\sqrt{x}} dx = \int x^{\frac{-1}{2}}\, (2x^2+3)\, dx = 2\int x^{\frac{3}{2}} dx + 3\int x^{\frac{-1}{2}} dx$$
$$= \frac{4}{5}\, x^{\frac{5}{2}} + 6\, x^{\frac{1}{2}} + c.$$

1.3 Exercises

1-1-1 Evaluate the following indefinite integrals:

1) $\int \sec^2(x) dx$,

2) $\int \csc^2(x) dx$,

3) $\int \tan^2(x) dx$,

4) $\int \cot^2(x) dx$,

5) $\int \sec(x)\tan(x) dx$,

6) $\int \csc(x)\cot(x) dx$,

7) $\int \left(x - \frac{1}{x^{\frac{2}{3}}} + \frac{1}{x^2}\right) dx$,

8) $\int \left(x + 2 + \frac{4}{(x+1)^2}\right) dx$,

9) $\int \left(\frac{1}{\sec(x)} - \frac{1}{\csc(x)}\right) dx$.

2 Substitution Method or Change of Variables

Theorem 2.1: Integration by Substitution

Let $g\colon I \longrightarrow J$ be a continuously differentiable function and $f\colon J \longrightarrow \mathbb{R}$ be a function which has an anti-derivative F on J, then $F(g(x))$ is an anti-derivative of $f(g(x))g'(x)$ and

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

This identity is called also "substitution" since it can be obtained by substituting $u = g(x)$ and $du = g'(x)dx$ into the integral $\int f(u)du = F(u) + c$.
This formula is obtained by the chain rule formula.
The substitution method is also called the changing variable method.

Example 1 :

1. $\displaystyle\int \cos(2x)dx \overset{(u=2x)}{=} \int \cos(u)\frac{1}{2}du = \frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin(2x) + c.$

2. $\displaystyle\int (x^2+1)^n 2xdx \overset{(u=x^2+1)}{=} \int u^n du = \frac{u^{n+1}}{n+1} + c = \frac{(x^2+1)^{n+1}}{n+1} + c,$ for $n \neq -1$.

3.
$$\begin{aligned}\int \sin(2x+3)dx &\overset{(u=2x+3)}{=} \frac{1}{2}\int \sin(u)du = -\frac{1}{2}\cos(u) + c\\ &= -\frac{1}{2}\cos(2x+3) + c.\end{aligned}$$

4. $\displaystyle\int \sec^2(\pi x)dx \overset{(u=\pi x)}{=} \frac{1}{\pi}\int \sec^2(u)du = \frac{1}{\pi}\tan(\pi x) + c.$

Theorem 2.2

Let I be an interval, $r \in \mathbb{Q} \setminus \{-1\}$ and $f\colon I \longrightarrow \mathbb{R}$ a continuously differentiable function. Assume also that the function f^r is continuous on I. Then

$$\int f^r(x) f'(x) dx = \frac{1}{r+1} f^{r+1}(x) + c.$$

The proof is given by substituting $u = f(x)$ in the integral $\displaystyle\int f^r(x) f'(x) dx$.

Example 2 :

1. $\displaystyle\int \sin(f(x))\, f'(x) dx = -\cos(f(x)) + c,$

2. $\displaystyle\int \sec^2(f(x))\, f'(x) dx = \tan(f(x)) + c,$

3. $\displaystyle\int \sec(f(x)) \tan(f(x))\, f'(x) dx = \sec(f(x)) + c.$

4. $\displaystyle\int (2x^3+1)^7 6x^2 dx \overset{(u=2x^3+1)}{=} \int u^7 du = \frac{1}{8}(2x^3+1)^8 + c,$

5. $\displaystyle\int (7-6x^2)^{\frac{1}{2}} x dx \overset{(u=7-6x^2)}{=} -\frac{1}{12}\int u^{\frac{1}{2}} du = -\frac{1}{18}(7-6x^2)^{\frac{3}{2}} + c,$

6. $\displaystyle\int \frac{x^2-1}{(x^3-3x+1)^6} dx \overset{(u=x^3-3x+1)}{=} \frac{1}{3}\int \frac{du}{u^6} = -\frac{1}{15}(x^3-3x+1)^{-5} + c,$

7. $\displaystyle\int \cos(3x+4) dx \overset{u=3x+4}{=} \frac{1}{3}\int \cos(u) du = \frac{1}{3}\sin(3x+4) + c,$

8. $\displaystyle\int \left(1+\frac{5}{x}\right)^3 \frac{1}{x^2} dx \overset{u=1+\frac{5}{x}}{=} \frac{-1}{5}\int u^3 du = \frac{-1}{20}\left(1+\frac{5}{x}\right)^4 + c,$

9. $\displaystyle\int \sqrt{9-x^2}\; x dx \overset{u=9-x^2}{=} \frac{-1}{2}\int u^{\frac{1}{2}} du = \frac{-1}{3}\;(9-x^2)^{\frac{3}{2}} + c,$

10. $\displaystyle\int \frac{1}{\sqrt{x}\;(1+\sqrt{x})^3} dx \overset{u=1+\sqrt{x}}{=} 2\int u^{-3} du = -\frac{1}{(1+\sqrt{x})^2} + c,$

11. $$\int \tan^2(x)\ \sec^2(x)\ dx \overset{u=\tan(x)}{=} \int u^2 du = \frac{1}{3}\tan^3(x) + c$$

12. $$\int \frac{\sin\left(1+\sqrt{x}\right)}{\sqrt{x}} dx \overset{u=1+\sqrt{x}}{=} 2 \int \sin(u)du = -2\ \cos\left(1+\sqrt{x}\right) + c,$$

13. $$\int \frac{\cos\left(\sqrt[3]{x}\right)}{\sqrt[3]{x^2}} dx \overset{u=\sqrt[3]{x}}{=} 3 \int \cos(u)du = 3\sin\left(x^{\frac{1}{3}}\right) + c,$$

14. $$\int \frac{\cos(\sqrt{x})}{\sqrt{x}\ \sin^2(\sqrt{x})} dx \overset{u=\sqrt{x}}{=} 2 \int \frac{\cos(u)}{\sin^2(u)} du = -\frac{2}{\sin(\sqrt{x})} + c.$$

2.1 Exercises

1-2-1 Evaluate the following integrals

1) $\int \sin(2x+3)dx,$

2) $\int \frac{1}{\cos^2(\pi x)} dx,$

3) $\int x\sqrt{x+1}dx,$

4) $\int \frac{x}{\sqrt{3-4x^2}} dx,$

5) $\int \frac{1}{\sqrt{x}\cos^2\left(\sqrt{x}\right)} dx,$

6) $\int \frac{x^2+3x+6}{\sqrt{x+1}} dx,$

7) $\int x^3\sqrt{x^4+1}\ dx$

8) $\int \frac{3x}{(1+x^2)^{\frac{2}{3}}}\ dx$

9) $\int \frac{3x^5}{\sqrt[3]{x^6+1}}\ dx$

10) $\int x^2 \sec^2(x^3-2)\ dx$

11) $\int \frac{\csc^2\left(\sqrt{x}\right)}{\sqrt{x}}\ dx$

12) $\int \frac{\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)}{x^2}\ dx$

13) $\int \frac{(2+\sin x)^4}{\sec x}\ dx$

1-2-2 Evaluate the following integrals with the indicated change of variable:

1) $\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^3(x)} dx, \quad (t = \cos(x)),$

2) $\int_0^1 \frac{dx}{(1+x^2)^2}, \quad (x = \tan(\theta)),$

3) $\int_0^1 x\sqrt{x^2+1}dx, \quad (t = x^2+1),$

3 Riemann Sums, Area and Definite Integral

3.1 Summation Notation

Definition 3.1

Given a set of real numbers $\{a_1, a_2, ..., a_n\}$, the symbol $\sum_{k=1}^{n} a_k$ represents their sum as follows

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n.$$

Theorem 3.2

For $m, n \in \mathbb{N}$, the following summation properties hold:

1. $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$
2. $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k.$
3. $\sum_{k=1}^{n} Ca_k = C\sum_{k=1}^{n} a_k$, where $C \in \mathbb{R}$
4. For $1 \leq m \leq n$, $\sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=1}^{n} a_k.$

Theorem 3.3

For $n \in \mathbb{N}$ and $C \in \mathbb{R}$, the following properties hold:

1. $\displaystyle\sum_{k=1}^{n} C = \underbrace{C + \cdots + C}_{n \text{ times}} = nC,$
2. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$
3. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$
4. $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$

Example 1 :
Evaluation of the following sums

1. $\displaystyle\sum_{k=1}^{100} k = \frac{100 \times (100+1)}{2} = 5050,$
2. $\displaystyle\sum_{k=1}^{20} k^2 = \frac{20.(20+1).(2.20+1)}{6} = 2870,$
3. $\displaystyle\sum_{k=1}^{10} k^3 = \left[\frac{10.(10+1)}{2}\right]^2 = 55^2 = 3025,$
4.

$$\begin{aligned}\sum_{k=1}^{n} (k+1)^2 k &= \sum_{k=1}^{n} k^3 + 2k^2 + k \\ &= \left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{12}\left(3n^2 + 11n + 10\right),\end{aligned}$$

5.

$$\begin{aligned}\sum_{k=1}^{n}(3k^2-2k+1) &= 3\sum_{k=1}^{n}k^2-2\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\\ &= \frac{n(n+1)(2n+1)}{2}-n(n+1)+n\\ &= \frac{n}{2}(2n^2+n+1).\end{aligned}$$

Example 2 :
Find the following limits:

1. $\lim\limits_{n\to\infty}\frac{1}{n^2}\sum\limits_{k=1}^{n}5k,$

2. $\lim\limits_{n\to\infty}\frac{1}{n^3}\sum\limits_{k=1}^{n}(k-1)^2.$

Solution

1. $\frac{1}{n^2}\sum\limits_{k=1}^{n}5k=\frac{5}{2}(1+\frac{1}{n})$, then $\lim\limits_{n\to\infty}\frac{1}{n^2}\sum\limits_{k=1}^{n}5k=\frac{5}{2}$,

2.

$$\begin{aligned}\frac{1}{n^3}\sum_{k=1}^{n}(k-1)^2 &= \frac{1}{n^3}\sum_{k=0}^{n-1}k^2\\ &= \frac{1}{n^3}\left(\frac{n(n-1)(2n-1)}{6}\right)=\frac{1}{6}(1-\frac{1}{n})(2-\frac{1}{n}).\end{aligned}$$

Then $\lim\limits_{n\to\infty}\frac{1}{n^3}\sum\limits_{k=1}^{n}(k-1)^2=\frac{1}{3}$.

3.2 The Riemann Integral

Let $f\colon[a,b]\longrightarrow\mathbb{R}$ be a **bounded** function on a **closed and bounded** interval. The aim of the section is to define the Riemann integral of the function f on $[a,b]$ if it is possible.

The integral of f on $[a,b]$ is a real number whose geometrical interpretation is the signed area under the graph of the function f on $[a,b]$.

This number is also called the definite integral of f.
By integrating the function f over the interval $[a, x]$ with varying x in $[a, b]$, we get a function F of x. The most important result about integration is the fundamental Theorem of calculus, which states that if the function f is continuous, the function F is an anti derivative of f.

Definition 3.4

1. A partition P of the closed interval $[a, b]$ is a finite set of points $P = \{a_0, a_1, \dots, a_n\}$ such that $a = a_0 < \dots < a_n = b$. Each $[a_{j-1}, a_j]$ is called a sub-interval of the partition and the number $h_j = a_j - a_{j-1}$ is called the amplitude of this interval.

2. The norm of a partition $P = \{a_0, a_1, \dots, a_n\}$ is the length of the longest sub-interval $[a_j, a_{j+1}]$, that is:
$||P|| = \max\{h_j, j = 1, \dots, n\}$.

3. A partition $P = \{a_0, a_1, \dots, a_n\}$ of the closed interval $[a, b]$ is a called uniform if $a_{k+1} - a_k = \dfrac{b-a}{n}$. In this case

$$a_k = a + k\frac{b-a}{n}, \; 0 \leq k \leq n. \tag{3.2}$$

4. A mark on the partition $P = \{a_0, a_1, \dots, a_n\}$ is a set of points $w = \{x_1, \dots, x_n\}$ such that $x_j \in [a_{j-1}, a_j]$ for all $1 \leq j \leq n$.

5. A pointed partition of the interval $[a, b]$ is a partition of the interval together with a mark $w = \{x_1, \dots, x_n\}$ on this partition. This pointed partition will be denoted by:

$$(P, w) = \{([a_{j-1}, a_j], x_j)\}_{1 \leq j \leq n}.$$

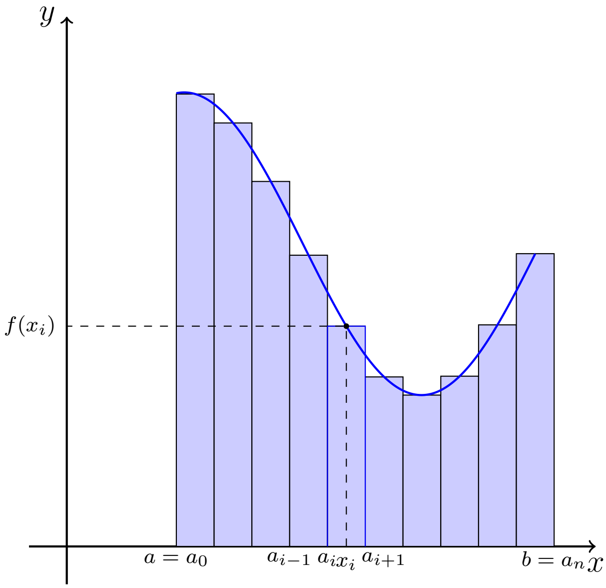
Definition 3.5

Let $(P, w) = \{([a_{j-1}, a_j], x_j)\}_{1 \le j \le n}$ be a pointed partition of the interval $[a, b]$. The Riemann sum of f with respect to the pointed partition P is the number

$$R(f, P, w) = \sum_{j=1}^{n} f(x_j)(a_j - a_{j-1}) = \sum_{j=1}^{n} f(x_j) h_j \tag{3.3}$$

Each term in the sum is the product of the value of the function at a given point by the length of an interval. Consequently, each term represents the area of a rectangle with height $f(x_j)$ and length $a_j - a_{j-1}$.

The Riemann sum $R(f, P, w)$ is the algebraic area of the union of the rectangles of width h_j and height $f(x_j)$. This is an algebraic area since $f(x_j)h_j$ is counted positively if $f(x_j) > 0$ and negatively if $f(x_j) < 0$.



Example 1 :

Let $f\colon [0,1] \longrightarrow \mathbb{R}$ the function defined by $f(x) = 2x - 2x^2$.

If $P = \{a_k = \dfrac{k}{10},\ 0 \le k \le 10\}$ is the uniform partition of the interval

$[0,1]$ and the mark $w = \{x_k = a_k,\ 1 \le k \le 10\}$, we get the Riemann sum

$$\begin{aligned} R(f,P,w) &= \frac{1}{10}\sum_{k=1}^{10} f(x_k) = \frac{1}{10}\sum_{k=1}^{10}(2x_k - 2x_k^2) \\ &= \frac{1}{10}[0.18+0.32+0.42+0.48+0.5+0.48+0.42+0.32+0.18+0] \\ &= 0.33. \end{aligned}$$

Example 2 :

Consider the function $f(x) = 4x+1$ on the interval $[-1,6]$ and the partition $P = \{-1,\ 0,\ 2,\ 4,\ 6\}$. Looking for the Riemann sums for the function by choosing in each sub-interval of P

1. the left hand end point,
2. the right hand end point,
3. the middle point.

In this case: $h_1 = 1$, $h_2 = 2$, $h_3 = 2$, $h_4 = 2$.

1. The left hand endpoints are $w_1 = -1,\ w_2 = 0,\ w_3 = 2,\ w_4 = 4$, and
   $f(w_1) = -3,\ f(w_2) = 1,\ f(w_3) = 9,\ f(w_4) = 17$ and
   $$R(f,P,w) = \sum_{k=1}^{4} f(w_k)h_k = 51.$$
2. The right hand endpoint are $w_1 = 0,\ w_2 = 2,\ w_3 = 4,\ w_4 = 6$. Then
   $f(w_1) = 1,\ f(w_2) = 9,\ f(w_3) = 17,\ f(w_4) = 25.$
   Therefore $R(f,P,w) = \sum_{k=1}^{4} f(w_k)h_k = 103.$
3. The middle points are $w_1 = -\frac{1}{2},\ w_2 = 1,\ w_3 = 3,\ w_4 = 5$. Then $f(w_1) = -1,\ f(w_2) = 5,\ f(w_3) = 13,\ f(w_4) = 21.$
   Therefore $R(f,P,w) = \sum_{k=1}^{4} f(w_k)h_k = 77.$

Example 3 :
Consider the function $f(x) = x$ on the interval $[0,1]$ and the uniform partition $P = \{\frac{k}{n},\ 0 \le k \le n\}$, for $n \ge 1$. Presenting three principal cases of Riemann sums, as we put the x_k at the left, the middle or the right end point of the intervals $[a_{k-1}, a_k]$, where $a_k = \dfrac{k}{n}$, for $1 \le k \le n$.

1. $x_k = a_{k-1}$:
$$R(f,P,w) = \frac{1}{n}\sum_{k=1}^{n}\frac{k-1}{n} = \frac{1}{n^2}\sum_{k=0}^{n-1} k = \frac{n-1}{2n}.$$

2. $x_j = \dfrac{a_{k-1}+a_k}{2}$
$$R(f,P,w) = \frac{1}{n}\sum_{k=1}^{n}\frac{2k-1}{2n} = \frac{1}{2n^2}\sum_{k=1}^{n} 2k-1 = \frac{1}{2}.$$

3. $x_k = a_k$:
$$R(f,P,w) = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n} = \frac{n+1}{2n}.$$

The second sum is equal to $\dfrac{1}{2}$ for every n, the other sums tend to $\dfrac{1}{2}$ when n tends to infinity.

Example 4 :
Let $f\colon [1,3] \longrightarrow \mathbb{R}$ be the function defined by: $f(x) = 3x+1$, the uniform partition $P = \{a_k,\ 0 \le k \le n\}$ of the interval $[1,3]$ and the mark $w = \{x_k,\ 1 \le k \le n\}$, where x_k is the middle point of the sub-interval $[a_{k-1}, a_k]$, $x_k = 1 + \dfrac{2k-1}{n}$. The Riemann sum is

$$\begin{aligned} R(f,P,w) &= \frac{2}{n}\sum_{k=1}^{n} f(x_k) = \frac{2}{n}\sum_{k=1}^{n}\left(3\left(1+\frac{2k-1}{n}\right)+1\right) \\ &= \frac{2}{n}\sum_{k=1}^{n}\left(4+\frac{6k}{n}-\frac{3}{n}\right) \\ &= 8 + 6(1+\frac{1}{n}) - \frac{6}{n}. \end{aligned}$$

Example 5 :

Referring to the last example with x_k the right end point of the sub-interval $[a_{k-1}, a_k]$, $x_k = 1 + \dfrac{2k}{n}$.

$$\begin{aligned} R(f,P,w) &= \frac{2}{n}\sum_{k=1}^{n} f(x_k) = \frac{2}{n}\sum_{k=1}^{n}\left(3\left(1+\frac{2k}{n}\right)+1\right) \\ &= \frac{2}{n}\sum_{k=1}^{n}\left(4+\frac{6k}{n}\right) = 8 + 6(1+\frac{1}{n}). \end{aligned}$$

3.3 Fundamental Properties

Theorem 3.6

Let $f, g\colon [a,b] \to \mathbb{R}$ be two functions and $\alpha, \beta \in \mathbb{R}$ and (P, w) a pointed partition of the interval $[a,b]$.

1. Linearity: $R(\alpha f + \beta g, P, w) = \alpha R(f, P, w) + \beta R(g, P, w)$.
2. Monotony: If $f \leq g$, then $R(f, P, w) \leq R(g, P)$. In particular, if $f \geq 0$, then $R(f, P, w) \geq 0$.
3. Chasles's Formula: Let $c \in (a,b)$, (P_1, w_1) a pointed partition of $[a,c]$ and (P_2, w_2) a pointed partition of $[b,c]$, then $(P_1 \cup P_2, w = w_1 \cup w_2)$ is a pointed partition of $[a,b]$ and
$$R(f, P_1 \cup P_2, w) = R(f, P_1, w_1) + R(f, P_2, w_2).$$

Definition 3.7

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function, the Riemann integral of f on the interval $[a,b]$ is

$$\lim_{\|P\|\to 0} R(f, P, w) \tag{3.4}$$

whenever the limit exists. (The limit is over all pointed partitions $P = \{([x_{j-1}, x_j], w_j)\}_{1\leq j\leq n}$).
If the limit exists, it is said that f is Riemann integrable (or integrable) on $[a, b]$. This limit if it exists, is denoted by: $\displaystyle\int_a^b f(x)dx$ and called the definite integral of f on the interval $[a, b]$.

Theorem 3.8

If $f\colon [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable, then

$$\int_a^b f(x)dx = \lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(a + k\frac{b-a}{n}). \tag{3.5}$$

Theorem 3.9

Let $f, g\colon [a, b] \longrightarrow \mathbb{R}$ be two Riemann integrable functions and $\alpha, \beta \in \mathbb{R}$. Then

1. $\displaystyle\int_a^b \alpha dx = \alpha(b-a).$
2. The function αf is Riemann integrable on $[a, b]$ and
$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx.$$
3. The functions $f \pm g$ are Riemann integrable on $[a, b]$ and
$$\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

4. For all $c \in (a, b)$ the function f is Riemann integrable on $[a, c]$, on $[c, b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

5. If $f \geq 0$, then $\int_a^b f(x)dx \geq 0$.

6. If $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Theorem 3.10

If $f\colon [a, b] \longrightarrow \mathbb{R}$ is Riemann integrable, the function $F(x) = \int_a^x f(t)dt$ is continuous.

Proof .
If $m \leq f \leq M$ and $a \leq x \leq y \leq b$, $m(y-x) \leq F(y) - F(x) \leq M(y-x)$. Then F is continuous.

□

Definition 3.11

A function $f\colon [a, b] \longrightarrow \mathbb{R}$ is called piecewise continuous if there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that f is continuous on every interval $]x_k, x_{k+1}[$, $\lim_{x \to x_k^+} f(x)$ and $\lim_{x \to x_{k+1}^-} f(x)$ exist in $\mathbb{R}$, for all $k = 0, \ldots, n-1$.

Theorem 3.12

Any piecewise continuous function $f\colon [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable.

Example 6 :
Evaluation of the following definite integrals

1. $$\int_{-1}^{3} 4dx = \lim_{n\to+\infty} \frac{4}{n}\sum_{k=1}^{n} 4 = \lim_{n\to+\infty} \frac{16n}{n} = 16,$$

2. $$\int_{0}^{4} xdx = \lim_{n\to+\infty} \frac{4}{n}\sum_{k=1}^{n} \frac{4k}{n} = \lim_{n\to+\infty} \frac{8(n+1)}{n} = 8,$$

3. $$\begin{aligned}\int_{0}^{1} (3x+7)dx &= \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{3k}{n} + 7\\ &= \lim_{n\to+\infty} \frac{3(n+1)}{2n} + 7 = \frac{3}{2} + 7 = \frac{17}{2}\end{aligned}$$

4. $$\int_{-1}^{0} (1-x)dx = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}(1-\frac{k}{n}) = \frac{1}{n}(n - \frac{n+1}{2}) = \frac{1}{2},$$

5. $$\int_{-1}^{4} |x|dx = -\int_{-1}^{0} xdx + \int_{0}^{4} xdx = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n} + 8 = \frac{1}{2} + 8 = \frac{17}{2},$$

6. $$\begin{aligned}\int_{1}^{4} (x^2+x+2)dx &= \lim_{n\to+\infty} \frac{3}{n}\sum_{k=1}^{n}\left(1+3\frac{k}{n}\right)^2 + (1+3\frac{k}{n}) + 2\\ &= \lim_{n\to+\infty} \frac{3}{n}\sum_{k=1}^{n}\left(1+6\frac{k}{n}+9\frac{k^2}{n^2}+1+3\frac{k}{n}+2\right)\\ &= \lim_{n\to+\infty} \frac{3}{n}\left(4n + \frac{9(n+1)}{2} + \frac{3(n+1)(2n+1)}{2n}\right)\\ &= \frac{69}{2}.\end{aligned}$$

7.

$$\begin{aligned}\int_0^2 (6x^3+1)dx &= \lim_{n\to+\infty} \frac{2}{n}\sum_{k=1}^{n} 6\left(2\frac{k}{n}\right)^3 + 1 \\ &= \lim_{n\to+\infty} \frac{2}{n}\left(12\frac{(n+1)^2}{n} + n\right) = 26.\end{aligned}$$

Example 7 :

Using the definition of the Riemann integral, the following limits can be expressed as definite integrals

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}((2+\frac{k}{n})^2-4), \text{ and } \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}((-4+\frac{k}{n})^{\frac{1}{3}}+4(-4+\frac{k}{n})),$$

If $f(x) = x^2 - 4$ on the interval $[2,3]$,

$$\int_2^3 (x^2-4)dx = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}((2+\frac{k}{n})^2-4).$$

If $f(x) = x^{\frac{1}{3}} + 4x$ on the interval $[-4,-3]$,

$$\int_{-4}^{-3} (x^{\frac{1}{3}}+4x)dx = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}((-4+\frac{k}{n})^{\frac{1}{3}}+4(-4+\frac{k}{n})).$$

Conventions and Notations:

1. If $a > b$, we denote $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$.

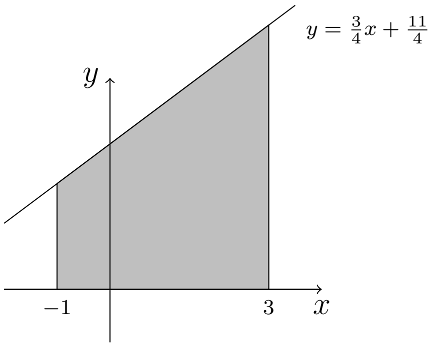
2. If $f(a)$ exists, $\displaystyle\int_a^a f(x)dx = 0$.

Theorem 3.13

If $f : [a,b] \to \mathbb{R}$ is Riemann integrable and $f(x) \geq 0, \quad \forall x \in [a,b]$, then the area A of the region under the graph of f from a to b is

$$A = \int_a^b f(x)dx.$$

Example 8 :
Consider the following shaded region



$$3x - 4y = 11 \iff y = \frac{3}{4}x + \frac{11}{4} = f(x).$$

The area A of the shaded region is:

$$A = \int_{-1}^{3} f(x)dx = \int_{-1}^{3} (\frac{3}{4}x + \frac{11}{4})dx = 19.$$

Example 9 :
Consider the functions $f(x) = 3x + 1$ and $g(x) = 2x + 2$ on the interval $[1, 4]$. $f(x) - g(x) = (3x + 1) - (2x + 2) = x - 1 \geq 0, \ \forall x \in [1, 4]$. Then $f(x) \geq g(x), \ \forall x \in [1, 4]$ and $\int_{1}^{4} (2x + 2)dx \leq \int_{1}^{4} (3x + 1)dx$.

3.4 Symmetry and Definite Integrals

Definition 3.14

Let $f\colon [-a, a] \longrightarrow \mathbb{R}$ be a function.

1. The function f is called odd if $f(-x) = -f(x)$ for all $x \in [-a, a]$.

2. The function f is called even if $f(-x) = f(x)$ for all $x \in [-a, a]$.

3. A function $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is called $T-$periodic if $g(x+T) = g(x)$ for all
$x \in \mathbb{R}$.

Theorem 3.15

Let $f\colon [-a,a] \longrightarrow \mathbb{R}$ be a Riemann integrable function.

1. If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.
2. If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
3. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a $T-$periodic function. If f is Riemann integrable on $[0,T]$, then f is Riemann integrable on any closed interval $[a,b]$ and

$$\int_{a}^{a+T} f(x)\,dx = \int_{0}^{T} f(x)\,dx, \quad \forall a \in \mathbb{R}.$$

Proof .

1. If f is odd, then $\displaystyle\int_{-a}^{0} f(x)\,dx \overset{t=-x}{=} -\int_{a}^{0} f(-t)\,dt = -\int_{0}^{a} f(t)\,dt$ and $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.
2. If f is even, then $\displaystyle\int_{-a}^{0} f(x)\,dx \overset{t=-x}{=} -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(t)\,dt$ and $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
3. If f is $T-$periodic, then

$$\int_T^{a+T} f(x)\,dx \overset{t=x-T}{=} \int_0^a f(t+T)\,dt = \int_0^a f(t)\,dt \text{ and}$$

$$\begin{aligned}\int_a^{a+T} f(x)\,dx &= \int_a^0 f(x)\,dx + \int_0^T f(x)\,dx + \int_T^{a+T} f(x)\,dx \\ &= -\int_0^a f(x)\,dx + \int_0^T f(x)\,dx + \int_0^a f(x)\,dx \\ &= \int_0^T f(x)\,dx.\end{aligned}$$

□

Example 1 :

$$\int_{-1}^{1} x^2\,dx = 2\int_0^1 x^2\,dx = 2\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3}.$$

$$\int_{-1}^{1} x^3\,dx = \left[\frac{1}{4}x^4\right]_{-1}^1 = 0.$$

$$\int_{5-\pi}^{5+\pi} \sin(x)\,dx = \int_{-\pi}^{\pi} \sin(x)\,dx = 0.$$

3.5 Exercises

Recall that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

1-3-1 1) Find the value of n such that $\sum_{k=1}^{n}(2k^2 - k + 1) = 147$.

2) Find the value of α such that $\sum_{k=1}^{6}\left(k^2 + 3k + 2\alpha\right) = 130$,

1-3-2 Express the sum $\sum_{k=1}^{n} k(k+1)$ in terms of n.

1-3-3 Find the value of a satisfying the following identities

1) $\sum_{k=1}^{10}(ak-10)=120$

3) $\sum_{k=5}^{15}(ak+5)=275$

2) $\sum_{k=1}^{5}(ak^2+2)=120$

1-3-4 Find the following limits.

1) $\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}(3k-2)$

2) $\lim_{n\to+\infty}\sum_{k=1}^{n}(2\frac{k}{n^2}-\frac{3}{n})$

3) $\lim_{n\to+\infty}\frac{1}{n^3}\sum_{k=1}^{n}(3k^2-2k+1)$

4) $\lim_{n\to+\infty}\frac{1}{n^4}\sum_{k=1}^{n}(k^3-3k^2+2)$

5) $\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}(\frac{k^2}{n^2}-3\frac{k}{n}+1)$

6) $\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}(k^2-k+1)$

7) $\lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n}(2k^3+4)$

1-3-5 Find the Riemann sum $R(f,P,w)$ for the function f defined by $f(x)=3x-2$ on the interval $[-2,2]$ with respect to the partition $P=\{-2,0,1,1.5,2\}$ by choosing on each sub-interval of the partition

1) The left-hand end point $w_k=x_{k-1}$

2) The right-hand end point $w_k=x_k$

3) The mid-point $w_k=\frac{x_{k-1}+x_k}{2}$

1-3-6 Use the Riemann sums to find the following integrals:

1) $\int_0^1 (3x+7)dx,$

2) $\int_1^4 (x^2+x+2)dx,$

3) $\int_0^2 (6x^3+1)dx,$

4) $\int_0^2 (3x-2)\ dx$

5) $\int_1^3 (5x-6)\ dx$

6) $\int_{-1}^4 (2x+1)\ dx$

7) $\int_0^4 (x^2+1)\ dx$

8) $\int_2^4 (x^2-x)\ dx$

9) $\int_0^3 (x^3-1)\ dx$

10) $\int_1^4 (x^3+x)\ dx$

1-3-7 Find the following limits:

1) $\lim\limits_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \sec^2\left(\frac{k}{n}\right).$

2) $\lim\limits_{n\to+\infty} \frac{1}{n^3}\sum_{k=1}^{n} (k-1)(k+2).$

1-3-8 Evaluate the following integrals:

1) $\int_{-\frac{\pi}{2}}^{\pi} f(t)dt$, where $f(t) = \begin{cases} \cos(t), & \text{for } t\in[-\frac{\pi}{2},\frac{\pi}{2}] \\ \sin(t), & \text{for } t\in[\frac{\pi}{2},\pi]. \end{cases},$

2) $\int_0^2 |x-1|\, dx.$

1-3-9 Express the following limits as an indefinite integrals:

1) $\lim\limits_{n\longrightarrow+\infty} \sum_{k=1}^{n} \frac{1}{n+k},$

2) $\lim\limits_{n\to+\infty} \sum_{k=1}^{n} \frac{n}{(2n+k)^2},$

3) $\lim\limits_{n\longrightarrow+\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k^2}{n^2},$

4) $\lim\limits_{n\longrightarrow+\infty} \frac{1}{n}\sum_{k=1}^{n} \sin(\frac{kx}{n}),\ x \in \mathbb{R},$

5) $\lim\limits_{n\to+\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2},$

6) $\lim\limits_{n\longrightarrow+\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2},$

7) $\lim\limits_{n\longrightarrow+\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^2}},$

8) $\lim\limits_{n\longrightarrow+\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2\sin(\frac{k\pi}{n})$,

9) $\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n}\cos\left(\frac{k\pi}{n}\right)$,

10) $\lim_{n\to+\infty}\sum_{k=1}^{2^n}\frac{k^3}{2^{4n}}$.

4 The Fundamental Theorem of Calculus

Theorem 4.1: The Mean Value Theorem

Let $f\colon[a,b]\longrightarrow\mathbb{R}$ be a continuous function. There exists $c\in[a,b]$ such that

$$\int_a^b f(x)dx=(b-a)f(c).$$

Proof .
Let $m=\inf\limits_{x\in[a,b]}f(x)$ and $M=\sup\limits_{x\in[a,b]}f(x)$. Since $m\le f\le M$ on the interval $[a,b]$, then $m\le\frac{1}{b-a}\int_a^b f(x)dx\le M$. By the Intermediate Value Theorem, there exists $c\in[a,b]$ such that $\frac{1}{b-a}\int_a^b f(x)dx=f(c)$.
□

Remark 1 :

If f is a non negative continuous function on $[a,b]$. The integral $\int_a^b f(x)dx$ represents the area under the graph of f and $(b-a)f(c)$ represents the area of the rectangle with side measurements $f(c)$ and $b-a$.

Definition 4.2

Let f be a continuous function on $[a,b]$. The average value of f is defined by:

$$f_{av}=\frac{1}{b-a}\int_a^b f(x)dx.$$

Example 1 :

1. The average value of the function $f(x) = 3x + 7$ on the interval $[0, 1]$ is $\int_0^1 (3x + 7)dx = \frac{17}{2}$. The number c where f reaches its average value satisfies the equation $3c + 7 = \frac{17}{2}$, then $c = \frac{1}{2}$.
2. The average value of the function $f(x) = x^2 + x + 2$ on the interval $[1, 4]$ is $\int_0^1 (x^2 + x + 2)dx = \frac{17}{6}$. The number c where f reaches its average value verifies $c^2 + c + 2 = \frac{17}{6}$, then $c = \frac{\sqrt{13} - \sqrt{3}}{2\sqrt{3}}$.
3. The average value of the function $f(x) = 6x^3 + 1$ on the interval $[0, 2]$ is $\int_0^1 (6x^3 + 1)dx = \frac{5}{2}$. The number c where f reaches its average value satisfies the equation $6c^3 + 1 = \frac{5}{2}$, then $c = 2^{-\frac{2}{3}}$.

Example 2 :

Let f be a continuous function on $[a, b]$ such that $\int_a^b f(x)dx = 0$, then the equation $f(x) = 0$ has a solution in $[a, b]$. The average value of f on $[a, b]$ is 0. Then by the Mean Value Theorem, f reaches this value at some point $c \in [a, b]$.

Theorem 4.3: (First Fundamental Theorem of Calculus)

Let $f\colon [a, b] \longrightarrow \mathbb{R}$ be a continuous function, then the function F defined by $F(x) = \int_a^x f(t)dt$ is differentiable on $[a, b]$ and $F'(x) = f(x)$.

Proof .
Let $x \in [a, b]$ and $h \neq 0$ such that $x + h \in [a, b]$. Using the Mean Value Theorem there exists c between x and $x + h$ such that

$$f(c) = \frac{F(x + h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt.$$

Since f is continuous, $F'(x) = \lim_{h \to 0} f(c) = f(x)$. □

Remark 2 :

1. The continuity of the function f is important. It is possible that a discontinuous function never equals its average value. We can take the function $f(x) = 0$ on the interval $[0, 1]$ and $f(x) = 1$ on the interval $[1, 2]$. The average value of f on the interval $[0, 2]$ is $\frac{1}{2}\int_0^2 f(x)dx = \frac{1}{2}\int_1^2 dx = \frac{1}{2}$. But $f(x) \neq \frac{1}{2}$, for all $x \in [0, 2]$.
2. Let f be a continuous function on a closed interval $[a, b]$. For any $c \in [a, b]$, the function $G(x) = \int_c^x f(t)dt; \quad x \in [a, b]$ is an anti derivative of f i.e. $G'(x) = f(x); \quad \forall x \in [a, b]$ because
$G(x) = \int_a^x f(t)dt - \int_a^c f(t)dt$.

Theorem 4.4: (Second Fundamental Theorem of Calculus)

Let $f\colon [a, b] \longrightarrow \mathbb{R}$ be a continuous function, and F an anti-derivative of f on $[a, b]$, then $\int_a^b f(t)dt = F(b) - F(a)$

Proof .
Let $G(x) = \int_a^x f(t)dt$. We know that $G'(x) = f(x)$, then there exists $c \in \mathbb{R}$ such that $F(x) = G(x) + c$ for some constant c and all $a \leq x \leq b$. Since $G(a) = 0$, then $c = F(a)$, and $G(x) = F(x) - F(a)$, for all $x \in [a, b]$.
□

Notations: $[F(x)]_a^b = F(b) - F(a)$.

Theorem 4.5

Let f be a continuous function on an interval I. If u and v are two differentiable functions on an interval J such that $v(J) \subset I$ and

$u(J) \subset I$, then the function $x \longmapsto \int_{u(x)}^{v(x)} f(t)dt$ is differentiable on the interval J and

$$\frac{d}{dx}\Big(\int_{u(x)}^{v(x)} f(t)dt\Big) = v'(x)f(v(x)) - u'(x)f(u(x)); \quad \forall x \in J.$$

Proof .
Let $F(x) = \int_a^x f(t)dt$, where $a \in I$. $\int_{u(x)}^{v(x)} f(t)dt = F(u(x)) - F(v(x))$.
Since $F'(x) = f(x)$, the Chain Rule Formula yields

$$\frac{d}{dx}\Big(\int_{u(x)}^{v(x)} f(t)dt\Big) = v'(x)f(v(x)) - u'(x)f(u(x)); \quad \forall x \in J.$$

□

Example 1 :

1. $\frac{d}{dx}\Big(\int_{3x}^{x^2} (t^3+1)^7 dt\Big) = 2x(x^6+1)^7 - 3(27x^3+1)^7$

2. If $G(x) = \int_{1-x}^{x^2} \frac{1}{4+3t^2}\, dt$,

$$\begin{aligned} G'(x) &= \frac{1}{4+3\left(x^2\right)^2}(2x) - \frac{1}{4+3\left(1-x\right)^2}(-1) \\ &= \frac{2x}{4+3x^4} + \frac{1}{4+3\left(1-x\right)^2}. \end{aligned}$$

3. $\frac{d}{dx}\int_0^5 \sqrt{t^2+3}\, dt = 0$ since $\int_0^5 \sqrt{t^2+3}\, dt$ is constant,

4. $\frac{d}{dx}\int_x^1 u^2\cos(u)\, du = -x^2\cos(x)$,

5. $\frac{d}{dx}\int_x^{x^2} \frac{1}{t-1}\, dt = \frac{2x}{x^2-1} - \frac{1}{x-1} = \frac{1}{x+1}$.

4.1 Exercises

1-4-1 Let I and J be the integrals defined by:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sin(x)+\cos(x)}dx \text{ and } J = \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)+\cos(x)}dx.$$

1) Prove that $I = J$, (Hint: use the substitution $t = \frac{\pi}{2} - x$).
2) Evaluate $I + J$.
3) Deduce the values of I and J.

1-4-2 Differentiate the following functions:

1) $f(x) = x\int_{\sqrt{\pi}}^{x} \cos(t^2)dt,$
2) $f(x) = \int_1^x \sin^3(t)dt,$
3) $f(x) = \int_x^{x^2} \cos^5(t)dt,$
4) $f(x) = \int_{\sin(x)}^{\cos(x)} (1-t^2)^{\frac{3}{2}}dt,$
5) $f(x) = \int_{\tan(x)}^{\sec(x)} (1+t^3)^{\frac{1}{3}}dt,$
6) $f(x) = \int_{\frac{1}{x}}^{2} (4+t^2)^{\frac{5}{2}}dt,$
7) $f(x) = \int_{2x}^{x^2} t\ln t dt,\ x > 0,$
8) $f(x) = \int_{-1}^{x} \sqrt{2+\sin t}\ dt$
9) $f(x) = \int_2^x \frac{1}{\sqrt{1+t^2}}\ dt$
10) $f(x) = \int_0^{\sqrt{x}} \frac{1}{1+t^2}\ dt$
11) $f(x) = \int_{-3}^{\sin x} \frac{t}{1+t^4}\ dt$
12) $f(x) = \int_{-1}^{x^2} \sqrt{3+\cos t}\ dt$
13) $f(x) = \int_{x^2}^{3} \sqrt{3+\cos t}\ dt$
14) $f(x) = \int_{\sin x}^{x^3} \frac{t}{2+t^2}\ dt$
15) $f(x) = \int_{\sqrt{x}}^{x^2+1} \frac{t^2}{1+t^2}\ dt$

1-4-3 Find $F'(0)$ if $F(x) = \int_{3x}^{3x^2+1} \frac{t}{4+t^2}\ dt$

1-4-4 Find $F'\left(\frac{\pi}{2}\right)$ if $F(x) = \int_{\cos x}^{\sin x} \frac{1}{\sqrt{t^2+1}}\ dt$

1-4-5 Find the number c that satisfies the conclusion of the Mean Value Theorem for the following functions

1) $f(x) = 3x + 7$ on $[0, 1]$.
2) $f(x) = x^2 + x + 2$ on $[1, 4]$.
3) $f(x) = 6x^3 + 1$ on $[0, 2]$.
4) $f(x) = ax + b, \quad a \neq 0$, on $[\alpha, \beta]$.

5 Numerical Integration

Very often definite integration cannot be done in closed form. When this happens some simple and useful techniques are needed to approximate the definite integrals. This section discuss two such simple and useful methods.

5.1 Trapezoidal Rule

Let $f\colon [a, b] \longrightarrow \mathbb{R}$ be a non negative continuous function. To approximate the area under the graph of f, the function f on $[x_j, x_{j+1}]$ is replaced by the polynomial P of degree 1 such that $P(x_j) = f(x_j)$ and $P(x_{j+1}) = f(x_{j+1})$. It is said that the polynomial P interpolates the function f on the points x_j and x_{j+1}. Then

$$P(x) = f(x_j)\frac{x_{j+1} - x}{x_{j+1} - x_j} + f(x_{j+1})\frac{x - x_j}{x_{j+1} - x_j}.$$

The area under the graph of P on the interval $[x_j, x_{j+1}]$ is the area of a trapezoid equal to

$$\frac{1}{2}(x_{j+1} - x_j)(f(x_{j+1}) + f(x_{j+1})).$$

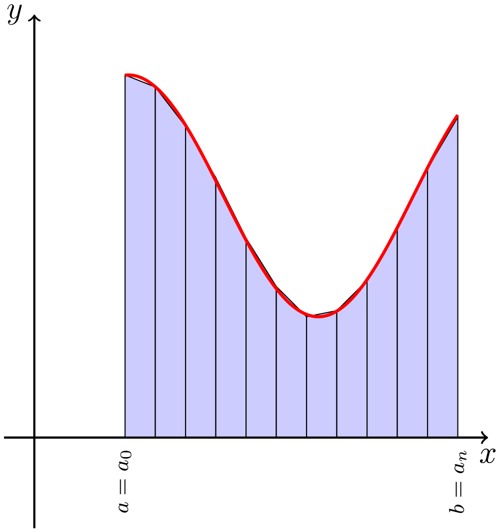
The area under the graph of f is approximated by:

$$\sum_{j=1}^{n} \frac{1}{2}(x_{j+1} - x_j)(f(x_{j+1}) + f(x_j)). \tag{5.6}$$

In the case where $x_{j+1} - x_j = \dfrac{b-a}{n}$, this area is approximated by

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) \quad + f(b)\right). \tag{5.7}$$

This formula is called the **trapezoidal rule**.
This formula is exact for polynomials of degree at most 1.



Theorem 5.1

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a twice continuously differentiable function. The remainder for this method is approximated as follows

$$|R_n| \leq \frac{(b-a)^3 M_2}{12n^2}, \quad \text{where } M_2 = \sup_{x\in[a,b]} |f^{(2)}(x)|.$$

Example 1 :

Let $f(x) = 2x - 1$ and $g(x) = x^2 + 3x - 1$ defined on the interval $[1,3]$. Using trapezoidal method for $n = 5$. An approximation of the integrals $\int_1^3 f(x)dx$ and $\int_1^3 g(x)dx$ is given by:

$$x_k = 1 + \frac{2k}{5},\ f(x_k) = 1 + \frac{4k}{5} \text{ and } g(x_k) = 3 + 2k + \frac{4k^2}{25}.$$

$$\int_1^3 (2x-1)dx \approx \frac{1}{5}\left(1+5+2\sum_{k=1}^{4}(1+\frac{4k}{5})\right) = 6.$$

$$\int_1^3 (2x-1)dx = \left[x^2-x\right]_1^3 = 6.$$

The reminder $R = 0$.

$$\begin{aligned}\int_1^3 (x^2+3x-1)dx &\approx \frac{1}{5}\left(3+17+2\sum_{k=1}^{4}(1+\frac{4k}{5})^2+3(1+\frac{4k}{5})-1\right)\\ &= \frac{1}{5}\left(20+2\sum_{k=1}^{4}(\frac{4k^2}{25}+2k+3)\right) = \frac{1}{5}(93+\frac{3}{5})\\ &= 18.72.\end{aligned}$$

$$\int_1^3 (x^2+3x-1)dx = \left[\frac{x^3}{3}+\frac{3x^2}{2}-x\right]_1^3 = \frac{37}{2}.$$

The reminder $|R| \leq 0.06$.

Example 2 :

Approximation of the integral $\int_0^1 \sqrt{1+x+x^2}\ dx$ using trapezoidal rule with $n = 4$.

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	0	1	1	1
1	0.25	1.1456	2	2.2913
2	0.5	1.3228	2	2.6457
3	0.75	1.5207	2	3.0414
4	1	1.73205	1	3.4641
				12.44248

$$\int_0^1 \sqrt{1+x+x^2}\ dx \approx \frac{1-0}{2(4)}\left[f(0)+2f(0.25)+2f(0.5)+2f(0.75)+f(1)\right],$$

$$\int_0^1 \sqrt{1+x+x^2}\ dx \approx \frac{1}{8}\left[12.44248\right] \approx 1.5553.$$

The reminder R fulfills $|R| \leq \frac{1}{4^4} = \frac{1}{256}$.

Example 3 :

Approximation of the integral $\int_2^4 \frac{1}{x-1}\,dx$ using trapezoidal rule with $n = 4$.

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	2	1	1	1
1	2.5	0.6666	2	1.333
2	3	0.5	2	1
3	3.5	0.4	2	0.8
4	4	0.3333	1	0.3333
				4.1666

$\int_2^4 \frac{1}{x-1}\,dx \approx 1.0415$. The reminder R fulfills $|R| \leq \frac{1}{12}$.

5.2 The Simpson Method

In this method, the function f on the interval $[x_j, x_{j+1}]$ is replaced by the polynomial P of degree 2 which interpolates the function f at the points x_j, x_{j+1} and the middle point $m_j = \frac{x_j + x_{j+1}}{2}$.

$$\int_{x_j}^{x_{j+1}} f(x)dx \approx \int_{x_j}^{x_{j+1}} P_j(x)dx = \frac{x_{j+1} - x_j}{6}(f(x_j) + f(x_{j+1}) + 4f(m_j)).$$

$$\begin{aligned} P_j(x) &= f(x_j)\frac{(x_{j+1}-x)(x-m_j)}{(x_{j+1}-x_j)(x_j-m_j)} + f(m_j)\frac{(x_{j+1}-x)(x-x_j)}{(x_{j+1}-m_j)(m_j-x_j)} \\ &+ f(x_{j+1})\frac{(x-x_j)(x-m_j)}{(x_{j+1}-x_j)(x_{j+1}-m_j)}. \end{aligned}$$

$$\int_{x_j}^{x_{j+1}} f(x)dx \approx \int_{x_j}^{x_{j+1}} P_2(x)dx = \frac{x_{j+1} - x_j}{6}(f(x_j) + f(x_{j+1}) + 4f(m_j)).$$

If the partition is uniform, $x_{j+1} - x_j = \frac{b-a}{n}$, then

$$\begin{aligned} S_n(f) &= \frac{b-a}{6n}\sum_{j=0}^{n-1}(f(x_j)+f(x_{j+1})+4f(m_j)) \\ &= \frac{b-a}{6n}\Big(f(a)+f(b)+2\sum_{j=1}^{n-1}f(x_j)+4\sum_{j=0}^{n-1}f(m_j)\Big). \quad (5.8) \end{aligned}$$

This formula is called **The Simpson formula** and it is exact for polynomials of degree at most 3.
If the middle point is not used, taking $n = 2m$ and $P = \{x_0, x_1, \ldots, x_{2m-1}\}$ a partition of the interval $[a, b]$. **The Simpson Formula** take the following form

$$S_n(f) = \frac{b-a}{3n}\Big(f(a)+f(b)+4\sum_{j=0}^{m-1}f(x_{2j+1})+2\sum_{j=1}^{m-1}f(x_{2j})\Big). \qquad (5.9)$$

Example 4 :
Let $g(x) = x^2+3x-1$ and $h(x) = x^3$ defined on the interval $[1, 3]$. Using Simpson method for $n = 8$, to approximate the integrals
$\displaystyle\int_1^3 (x^2+3x-1)dx$ and $\displaystyle\int_1^3 x^3 dx$.
$x_k = 1+\frac{k}{4}$, $x_{2k} = 1+\frac{k}{2}$ and $x_{2k+1} = 1+\frac{2k+1}{4}$, $g(x_k) = 3+\frac{5k}{4}+\frac{k^2}{16}$ and $h(x_k) = 1+\frac{3k}{4}+\frac{3k^2}{16}+\frac{k^3}{64}$.

$$\int_1^3 (x^2+3x-1)dx \approx \frac{1}{12}\left(3+17+4\sum_{k=0}^{3}g(x_{2k+1})+2\sum_{k=1}^{3}g(x_{2k})\right) = 18.666$$

k	x_k	m_k	$m_k g(x_k)$
0	1	1	3
1	5/4	4	17.25
2	3/2	2	11.5
3	7/4	4	29.25
4	2	2	18
5	9/4	4	43.25
6	5/2	2	25.25
7	11/4	4	59.25
8	3	1	17
			224

$$\int_1^3 x^3 dx \approx \frac{1}{12}\left(1 + 27 + 4\sum_{k=0}^{3} h(x_{2k+1}) + 2\sum_{k=1}^{3} h(x_{2k})\right) = 20.$$

k	x_k	m_k	$m_k h(x_k)$
0	1	1	1
1	5/4	4	7.8125
2	3/2	2	6.75
3	7/4	4	21.4375
4	2	2	16
5	9/4	4	45.5625
6	$\frac{5}{2}$	2	31.25
7	11/4	4	83.1875
8	3	1	27
			240

Example 5 :

Let $f(x) = \sqrt{1+x^3}$ defined on the interval $[0, 3]$. Use the Simpson method for $n = 6$ to approximate the integral $\int_0^3 \sqrt{1+x^3}dx$.

Solution

$x_k = \frac{k}{2}$,

k	x_k	m_k	$m_k f(x_k)$
0	1	1	1
1	1/2	4	4.24264
2	1	2	2.82842
3	3/2	4	8.3666
4	2	2	3.4641
5	5/2	4	16.3095
6	3	1	5.2915
			41.50276

$$\int_0^3 \sqrt{1+x^3}dx \approx 6.9171.$$

Example 6 :

Approximation of the integral $\int_1^3 \sqrt{1-x+x^2}\ dx$ using Simpson's rule with $n=4$.

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	1	1	1	1
1	1.5	1.32287	4	5.2915.3
2	2	1.73205	2	3.4641
3	2.5	2.179449	4	8.717798
4	3	2.645751	1	2.645751
				21.119181

$$\int_1^3 \sqrt{1-x+x^2}\ dx \approx 3.5198.$$

Theorem 5.2

Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a function of class C^4 on the interval $[a,b]$. If $n=2m$ and $P=\{x_0,x_1,\ldots,x_{2m-1}\}$ a partition of the interval $[a,b]$. Then the remainder of the approximation of f by the following sum S_n

$$S_n(f) = \frac{b-a}{3n}\Big(f(a)+f(b)+4\sum_{j=0}^{m-1} f(x_{2j+1}) + 2\sum_{j=1}^{m-1} f(x_{2j})\Big).$$

is approximated as follows:

$$|R_n| \leq \frac{(b-a)^5 M_4}{180n^4}, \qquad M_4 = \sup_{x\in[a,b]} |f^{(4)}(x)|.$$

Example 7 :
Let $f(x) = \sqrt{2+x^2}$ defined on the interval $[0,2]$. Use the Simpson method for $n=6$ to approximate the integral $\int_0^2 \sqrt{2+x^2}dx$.

Solution
$x_k = \frac{k}{3}$,

k	x_k	m_k	$m_k f(x_k)$
0	0	1	$\sqrt{2}$
1	1/3	4	5.81186
2	2/3	2	3.1269438
3	1	4	6.92820
4	4/3	2	3.8873
5	5/3	4	8.7432513
6	2	1	2.44948974
			32.361254842

$\int_0^2 \sqrt{2+x^2}dx \approx 3.59569498.$
$f(x) = (2+x^2)^{\frac{1}{2}}$, $f'(x) = x(2+x^2)^{-\frac{1}{2}}$, $f''(x) = 2(2+x^2)^{-\frac{3}{2}}$,
$f^{(3)}(x) = -6x(2+x^2)^{-\frac{5}{2}}$, $f^{(4)}(x) = 12(2x^2-1)(2+x^2)^{-\frac{7}{2}}$ and
$f^{(5)}(x) = 60x(3-2x^2)(2+x^2)^{-\frac{9}{2}}$.
Using the variation of the function $f^{(4)}$ on the interval $[0,2]$, the value of M_4 is 1.07 , then $|R| \leq 10^{-5}$.

5.3 Exercises

1-5-1 1) Approximate the integral $\int_0^\pi \sqrt{1+\sin(x)}dx$ using trapezoidal rule with $n=4$ and the regular partition. Give an approximation of the error.

2) Approximate $\int_0^5 \frac{dx}{\sqrt{1+x^4}}$ using trapezoidal rule with $n = 5$.

3) Approximate the integral $\int_0^2 \frac{x}{\sqrt{x+1}} dx$ using Simpson's rule for $n = 4$ and $n = 8$. Give an approximate of the reminder in each case.

1-5-2 Let $f(x) = 2x - 1$ and $g(x) = x^2 + 3x - 1$ defined on the interval $[1, 3]$. Use trapezoidal method for $n = 5$ to approximate the integrals $\int_1^3 f(x)dx$ and $\int_1^3 g(x)dx$.

1-5-3 Let $g(x) = x^2 + 3x - 1$ and $h(x) = x^3$ defined on the interval $[1, 3]$. Use Simpson method for $n = 8$ to approximate the integrals $\int_1^3 (x^2 + 3x - 1)dx$ and $\int_1^3 x^3 dx$.

CHAPTER 2

THE TRANSCENDENTAL FUNCTIONS

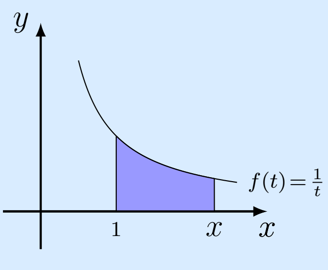
1 The Natural Logarithmic Function

1.1 The Natural Logarithmic Function

For $\alpha \in \mathbb{Q}$, the function $x \longmapsto x^\alpha$ is continuous on $(0, +\infty)$, then it is Riemann integrable on any interval $[a, b] \subset (0, +\infty)$.
For $\alpha \in \mathbb{Q}$, $\alpha \neq -1$, $\displaystyle\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c$.

Definition 1.1

For $x > 0$, the function $\ln(x) = \displaystyle\int_1^x \frac{dt}{t}$ represents the algebraic area of the region between the graph of the function $f(t) = \dfrac{1}{t}$, the $x-$ axis and the straight lines $t = 1$ and $t = x$.
The function $x \longmapsto \ln(x)$ is called **the Natural Logarithmic Function.**



Theorem 1.2

For all x, y in $]0, +\infty[$, we have:

1. $\ln xy = \ln x + \ln y$.
2. $\ln \frac{1}{x} = -\ln x$.
3. $\ln x^n = n \ln x$, for all $n \in \mathbb{N}$.
4. $\ln x^r = r \ln x$, for all $r \in \mathbb{Q}$.

Proof .

1. Let $y \in]0, +\infty[$ and $f(x) = \ln xy$. As $f'(x) = \dfrac{y}{xy} = \dfrac{1}{x}$, there exists a real number c (which depends on y) such that $f(x) = c + \ln x$. For $x = 1$, we shall have: $c = \ln y$.
2. We deduce from 1. that for $y = \frac{1}{x}$,
$0 = \ln(1) = \ln(x.\dfrac{1}{x}) = \ln x + \ln(\dfrac{1}{x})$.
3. We prove $\ln x^n = n \ln x$ by induction.
The result is true for $n = 1$ and $n = 2$. Assume the result is true for n, then $\ln x^{n+1} = \ln x^n.x = \ln x + \ln x^n = (n+1)\ln x$.
4. $\ln x^{\frac{m}{n}} = \dfrac{n}{n} \ln x^{\frac{m}{n}} = \dfrac{1}{n} \ln x^m = \dfrac{m}{n} \ln x$.

□

Example 1 :

Simplification of $\dfrac{1}{5}\left[2\ln|x+1| + \ln|x| - \ln|x^2-2|\right]$.

$$\begin{aligned}
\frac{1}{5}\left[2\ln|x+1| + \ln|x| - \ln|x^2-2|\right] &= \frac{1}{5}\left[\ln|x(x+1)^2| - \ln|x^2-2|\right] \\
&= \ln\left|\left(\frac{x(x+1)^2}{x^2-2}\right)^{\frac{1}{5}}\right|
\end{aligned}$$

Theorem 1.3

1. $\lim\limits_{h\to 0}\dfrac{\ln(1+h)}{h}=1,$
2. $\lim\limits_{x\to+\infty}\ln x=+\infty,$
3. $\lim\limits_{x\to 0^+}\ln x=-\infty,$
4. $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}=0,$
5. $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x^s}=0;\ \forall\ s\in\mathbb{Q}_+^*.$

Proof .

1. The result is obtained by definition of the derivative of the natural logarithmic function at $x=1$. (i.e. we use the formula $\dfrac{d}{dx}\ln x=\dfrac{1}{x}=\lim\limits_{h\to 0}\dfrac{\ln(x+h)-\ln x}{h}$ for $x=1$).
2. $\dfrac{d}{dx}\ln x>0$, thus the Logarithmic function is strictly increasing, moreover $\ln 2^n=n\ln 2$ and $\ln 2>0$ thus the function is not bounded above, this yields that $\lim\limits_{x\to+\infty}\ln x=+\infty$.
3. $\ln(\dfrac{x}{x})=0=\ln x+\ln\dfrac{1}{x}$, thus $\lim\limits_{x\to 0^+}\ln x=-\infty$.
4. The function $g(x)=\ln x-x$ has its absolute maximum at 1 because $g'(x)=\dfrac{1-x}{x}$. Thus $\forall\ x\in]0,+\infty[$, $\ln x-x\leq -1$. Then $\ln x\leq x-1<x$ and $\ln(\sqrt{x})=\frac{\ln x}{2}\leq\sqrt{x}$ for $x>1$.
   Hence $0<\dfrac{1}{2}\dfrac{\ln x}{x}=\dfrac{\ln\sqrt{x}}{x}<\dfrac{1}{\sqrt{x}}$, for $x>1$, and $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}=0$.
5. If $s\in\mathbb{Q}_+^*$, $\ln x^s=s\ln x$.

$$\lim_{x\to+\infty}\frac{\ln x}{x^s}=\lim_{x\to+\infty}\frac{\ln x^s}{sx^s}=0$$

□

Corollary 1.4

The Logarithmic function $\ln\colon]0, +\infty[\longrightarrow \mathbb{R}$ is bijective. There exists a unique real number which will be denoted by e such that $\ln(e) = 1$, $(2 < e < 3)$, e is called the base of the Natural Logarithmic function. $(e \approx 2.71828)$

Remark 3 :

1. $\ln(x) > 0$, $\forall x > 1$,
2. $\ln(x) < 0$, $\forall 0 < x < 1$,
3. $\ln(x) = 0 \iff x = 1$,
4. $\dfrac{d^2}{dx^2}(\ln(x)) = -\dfrac{1}{x^2} > 0$; $\forall x > 0$, (i.e. The function $x \longmapsto \ln(x)$ is concave on $(0, \infty)$).

1.2 The Logarithmic Differentiation

In some cases, the derivative of the function $\ln|f|$ is used to compute the derivative of f.

Theorem 1.5: (The Logarithmic Differentiation)

Let $u\colon I \longrightarrow \mathbb{R} \setminus \{0\}$ be a differentiable function, then

$$\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}.$$

Examples 1 :

1. If $f(x) = \ln(x^2 + 2x + 4)$, then $f'(x) = \dfrac{2x + 2}{x^2 + 2x + 4}$, $\forall x \in \mathbb{R}$.

2. If $f(x) = (x+1)^2(x+2)^3(x-5)^7$, then
$\ln|f(x)| = 2\ln|x+1| + 3\ln|x+2| + 7\ln|x-5|$ and

$$\frac{f'(x)}{f(x)} = \frac{2}{x+1} + \frac{3}{x+2} + \frac{7}{x-5}$$

So

$$f'(x) = \left(\frac{2}{x+1} + \frac{3}{x+2} + \frac{7}{x-5}\right)(x+1)^2(x+2)^3(x-5)^7$$

3. If $f(x) = \ln(|2-3x|^5)$, then $f'(x) = \dfrac{5.(-3).(2-3x)^4}{(2-3x)^5} = \dfrac{15}{3x-2}$.

4. If $f(x) = \ln\left(\sqrt{\dfrac{4+x^2}{4-x^2}}\right)$,

$$f(x) = \ln\left(\sqrt{\frac{4+x^2}{4-x^2}}\right) = \frac{1}{2}\ln(4+x^2) - \frac{1}{2}\ln(4-x^2).$$

Then $f'(x) = \dfrac{1}{2}\dfrac{2x}{4+x^2} - \dfrac{1}{2}\dfrac{(-2x)}{4-x^2} = \dfrac{8x}{(4+x^2)(4-x^2)}$.

5. If $y = \sqrt{\dfrac{(x+1)^4(x+2)^3}{(x-1)^2}}$, $\ln y = \dfrac{1}{2}\left[4\ln|x+1| + 3\ln|x+2| - 2\ln|x-1|\right]$.
Differentiate both sides, we get $\dfrac{y'}{y} = \dfrac{1}{2}\left[\dfrac{4}{x+1} + \dfrac{3}{x+2} - \dfrac{2}{x-1}\right]$.
Hence $y' = \dfrac{1}{2}\sqrt{\dfrac{(x+1)^4(x+2)^3}{(x-1)^2}}\left[\dfrac{4}{x+1} + \dfrac{3}{x+2} - \dfrac{2}{x-1}\right]$.

1.3 Exercises

2-1-1 Solve the following equations:

1) $\ln|x-1| = 7$ 2) $\ln|x^3-1| = 0$

2-1-2 Differentiate the following functions:

1) $f(x) = \ln(x^2 + 2x + 4)$,
2) $f(x) = \ln(|2 - 3x|^5)$,
3) $f(x) = \ln\left(\dfrac{1-x}{1+x}\right)$, $-1 < x < 1$,
4) $f(x) = \ln\left|x^4 + x^3 + 1\right|$
5) $f(x) = \ln\left|x^2 + \cos(2x)\right|$
6) $f(x) = \sin x \;\ln|5x|$
7) $f(x) = \tan(\ln|3x|)$
8) $f(x) = [3x + \ln|\sin x|]^8$
9) $f(x) = \ln\left|\dfrac{\sqrt{x^2+1}\;\sin^5 x}{(x^3+4)^2}\right|$
10) $f(x) = \dfrac{(3x+1)^{\frac{3}{2}}\;(x^2-1)^{\frac{2}{3}}}{\sqrt[3]{x^2+2}}$

2-1-3 Find the derivative of the following functions:

1) $f(x) = \mathrm{Log}(x^2 + 4)$,
2) $f(x) = \ln(x + \sqrt{x^2 - 4})$,
3) $f(x) = \ln(x + \sqrt{4 + x^2})$,

2-1-4 Differentiate the following functions:

1) $f(x) = \dfrac{(x^2+1)^3(x^2+4)^{10}}{(x^2+2)^5(x^2+3)^4}$,
2) $f(x) = \dfrac{(x+1)^3(2x-3)^{\frac{3}{4}}}{(1+7x)^{\frac{1}{3}}(2x+3)^{\frac{3}{2}}}$,
3) $f(x) = \sqrt{(3x^2+2)\sqrt{6x-7}}$,
4) $f(x) = (x+1)^2(x+2)^3(x-5)^7$.

2-1-5 Use implicit differentiation to find y' if

1) $y^2 + \ln(\dfrac{x}{y}) - 4x = -3$,
2) $xe^y + 2x - \ln(y+1) = 3$.

2 The Exponential Function

The natural logarithmic function $\ln\colon]0,+\infty[\longrightarrow \mathbb{R}$ is increasing and bijective, then it has an inverse function.

Definition 2.1

The natural exponential function is the inverse of the natural logarithmic function. It is denoted by e^x.

Properties 2.2

1. The exponential function is bijective and increasing.
2. $\frac{d}{dx}e^x = e^x$,
3. $e^{x+y} = e^x e^y$,
4. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$,
5. $\lim\limits_{x\to -\infty} e^x = 0$,
6. $\lim\limits_{x\to +\infty} e^x = +\infty$,
7. $\lim\limits_{x\to +\infty} \dfrac{e^x}{x} = +\infty$,
8. $\lim\limits_{x\to -\infty} xe^x = 0$.

Proof .

1. Obtained from the properties of the inverse of continuous functions.
2. Since $\ln e^x = x$, then after differentiation both sides, we get, $\dfrac{\frac{d}{dx}e^x}{e^x} = 1$, then $\frac{d}{dx}e^x = e^x$.
3. $\ln e^{x+y} = x + y = \ln e^x + \ln e^y = \ln e^x e^y$, then $e^{x+y} = e^x e^y$.
4. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$, which represents the derivative of the exponential function at 0.
5. $\lim\limits_{x\to 0^+} \ln x = -\infty$, then $\lim\limits_{x\to -\infty} e^x = 0$.

6. $\lim_{x\to+\infty} \ln x = +\infty$, then $\lim_{x\to+\infty} e^x = +\infty$.

7. $\lim_{x\to+\infty} \ln(\frac{e^x}{x}) = \lim_{x\to+\infty} x - \ln x = \lim_{x\to+\infty} x(1 - \frac{\ln x}{x}) = +\infty$, then $\lim_{x\to+\infty} \frac{e^x}{x} = +\infty$.

8. If $x<0$, $\ln(-xe^x)=x+\ln(-x)=-x(\frac{\ln(-x)}{-x}-1)$, then $\lim_{x\to-\infty}(-xe^x)=0$.

□

Corollary 2.3

If $u\colon I \longrightarrow \mathbb{R}$ is a differentiable function, then

$$\frac{d}{dx}(e^{u(x)}) = u'(x)e^{u(x)}.$$

The proof is obtained directly from the Chain Rule Theorem.

Examples 1 :

1. $\frac{d}{dx}e^{1-x^2} = -2xe^{1-x^2}$.

2. $\frac{d}{dx}e^{x\ln(x)} = (\ln(x)+1)e^{x\ln(x)}$.

3. $\frac{d}{dx}\left(e^{5x} + \frac{1}{e^x}\right) = 5e^{5x} - e^{-x}$.

4. If g is a continuous function on $\mathbb{R}$ and $\int e^{3x^2} g(x)dx = -e^{3x^2}$, then $\frac{d}{dx}\left[-e^{3x^2}\right] = e^{3x^2}g(x)$ and $g(x) = -6x$.

5. If $xe^y + 2x - \ln(y+1) = 3$, then using implicit differentiation, we get $e^y + xy'e^y + 2 - \frac{y'}{y+1} = 0$ and $y' = -\frac{2+e^y}{xe^y - \frac{1}{y+1}}$.

6. To know the equation of the tangent line to the graph of the function $f(x) = x - e^{-x}$ that is parallel to the line (D) of equation $6x - 2y = 7$.

   Recall that two straight lines (D_1) and (D_2) with equations $y = ax + c$ and $y = a'x + c'$ respectively are parallel if and only if $a = a'$ and (D_1) and (D_2) are orthogonal if and only if $a.a' = -1$.

   The required tangent line equation is

   $$y - y_1 = m(x - x_1), \quad \text{with } m = f'(x_1).$$

   The equation of D is $y = 3x - \frac{7}{2}$, then the tangent line is parallel to D if and only if $m = 3$. Now, it suffices to find x such that $f'(x) = 3$.
   $f'(x) = 1 + e^{-x} = 3 \iff e^{-x} = 2 \iff x = \ln(\frac{1}{2})$. Then $x_1 = -\ln(2)$ and $f(x_1) = -\ln(2) - e^{\ln(2)} = 2 - \ln(2)$. Therefore the equation of the tangent line is $y - 2 + \ln(2) = 3(x + \ln(2))$.

Exercise 1 :

1. Solve the equation $e^{5x+3} = 4$.
2. Simplify the expression $\ln\left(e^{x}\right)^2$.

Solutions

1. $e^{5x+3} = 4 \iff \ln e^{5x+3} = \ln 4 \iff 5x + 3 = \ln 4$. Then $x = \dfrac{-3 + \ln 4}{5}$.
2. $\ln\left(e^{x}\right)^2 = \ln\left(e^{2x}\right) = 2x$.

2.1 Exercises

2-2-1 Solve the following equations:

1) $e^{2x-1} = 5$ 2) $e^{x^2-4} = 1$

2-2-2 Differentiate the following functions:

1) $f(x) = e^{1-x^2}$,
2) $f(x) = e^{x\ln(x)}$,
3) $f(x) = x^2 e^{-x^3}$,

2-2-3 Find the equation of the tangent line to the graph of the function $f(x)=x-e^{-x}$ that is parallel to the line (D) of equation $6x-2y=7$.

2-2-4 Solve the following equation for x:

$$\frac{e^x}{1+e^x} = \frac{1}{3}.$$

3 Integration Using " ln " and " exp " Functions

Theorem 3.1

Using the last properties of the logarithmic and exponential functions, we have

1. $\displaystyle\int \frac{dx}{x} = \ln|x| + c$,
2. $\displaystyle\int e^x dx = e^x + c$,
3. $\displaystyle\int \frac{u'(x)}{u(x)} dx = \ln|u(x)| + c$,
4. $\displaystyle\int u'(x)e^{u(x)} dx = e^{u(x)} + c$.

Examples 2 :
Evaluation of the following integrals:

1. $$\int \frac{e^{-x}}{(1-e^{-x})^2} dx \overset{u=e^{-x}}{=} -\int \frac{du}{(1-u)^2} = \frac{-1}{(1-e^{-x})} + c,$$

2. $$\int \frac{e^{\frac{3}{x}}}{x^2} dx \overset{u=e^{\frac{3}{x}}}{=} -\frac{1}{3}\int du = -\frac{1}{3}e^{\frac{3}{x}} + c,$$

3. $$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \overset{u=\sqrt{x}}{=} 2\int e^u du = 2e^{\sqrt{x}} + c,$$

4. $\displaystyle\int \frac{e^{\sin(x)}}{\sec(x)}dx = \int e^{\sin(x)}\cos(x)dx = e^{\sin(x)} + c,$

5. $\displaystyle\int_1^e \frac{\sqrt[3]{\ln x}}{x}\,dx \overset{u=\ln x}{=} \int_0^1 u^{\frac{1}{3}}du = \frac{3}{4}\,(\ln e)^{\frac{4}{3}} = \frac{3}{4}.$

6. $\displaystyle\int e^{(x^2+\ln x)}dx = \int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c.$

7. $\displaystyle\int x^2e^{3x^3}dx = \frac{1}{9}\int (9x^2)e^{3x^3}dx = \frac{1}{9}e^{3x^3} + c,$

8. $\displaystyle\int \frac{dx}{x(\ln(x))^2} = -\frac{1}{\ln(x)} + c,$

9. $\displaystyle\int \frac{e^x}{(e^x+1)^2}dx \overset{u=e^x+1}{=} \int \frac{du}{u^2} = -\frac{1}{e^x+1} + c,$

10. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx \overset{u=e^x+e^{-x}}{=} \int \frac{du}{u} = \ln(e^x + e^{-x}) + c,$

11. $\displaystyle\int \tan(x)dx \overset{u=\cos x}{=} -\int \frac{du}{u} = -\ln|\cos(x)| + c = \ln|\sec(x)| + c,$

12. $\displaystyle\int \cot(x)dx \overset{u=\sin x}{=} \int \frac{du}{u} = \ln|\sin(x)| + c,$

13. $\displaystyle\int \frac{dx}{x\sqrt{\ln x}} \overset{u=\ln x}{=} \int u^{-\frac{1}{2}}du = 2\,(\ln x)^{\frac{1}{2}} + c.$

14. $\displaystyle\int \frac{dx}{x\ln\sqrt{x}} = 2\int \frac{dx}{x\ln x} \overset{u=\ln x}{=} 2\int \frac{du}{u} = \ln|\ln x| + c.$

Theorem 3.2

$$\begin{aligned}\int \sec(x)dx &= \ln|\sec(x)+\tan(x)|+c\\ \int \csc(x)dx &= \ln|\csc(x)-\cot(x)|+c\\ &= -\ln|\csc(x)+\cot(x)|+c.\end{aligned}$$

Proof .
$\sec'(x) = \sec(x)\tan(x), \quad \tan'(x) = \sec^2(x)$, then
$\dfrac{\frac{d}{dx}(\sec(x)+\tan(x))}{\sec(x)+\tan(x)} = \sec(x)$. Therefore

$$\int \sec(x)dx = \ln|\sec(x)+\tan(x)|+c.$$

$\csc'(x) = -\csc(x)\cot(x), \quad \cot'(x) = -\csc^2(x)$, then
$\dfrac{\frac{d}{dx}(\csc(x)-\cot(x))}{\csc(x)-\cot(x)} = \csc(x)$ and $\dfrac{\frac{d}{dx}(\csc(x)+\cot(x))}{\csc(x)+\cot(x)} = -\csc(x)$.
Therefore

$$\int \csc(x)dx = \ln|\csc(x)-\cot(x)|+c.$$

□

3.1 Exercises

2-3-1 Evaluate the following integrals with the indicate change of variable:

1) $\displaystyle\int xe^{-x^2}dx, \quad (t=x^2),$

2) $\displaystyle\int \frac{\sin(\ln x)}{x}dx, \quad (t=\ln x),$

3) $\displaystyle\int_0^1 \frac{dx}{e^x+1}, \quad (t=e^x),$

2-3-2 Evaluate the following integrals:

1) $\displaystyle\int \frac{x-2}{x^2-4x+9}dx$,

2) $\displaystyle\int \frac{(2+\ln(x))^{10}}{x}dx$,

3) $\displaystyle\int \frac{\tan(e^{-3x})}{e^{3x}}dx$,

4 The General Exponential Functions

Definition 4.1

For $a > 0$, the function $f(x) = e^{x\ln(a)}$ defined for $x \in \mathbb{R}$ is called the exponential function with base a and denoted by a^x.

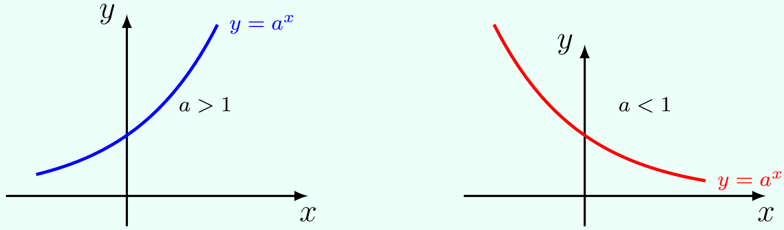
Theorem 4.2

Let $a > 0$ and $b > 0$, x and y two real numbers, then

1. $a^{x+y} = a^x a^y$,
2. $a^{x-y} = \dfrac{a^x}{a^y}$,
3. $(a^x)^y = a^{xy}$,
4. $(ab)^x = a^x b^x$,
5. $(\dfrac{a}{b})^x = \dfrac{a^x}{b^x}$,
6. $\dfrac{d}{dx}(a^x) = a^x\ln(a)$,
7. $\dfrac{d}{dx}(a^{u(x)}) = a^{u(x)}\ln(a)u'(x)$, if u is differentiable.

Properties 4.3: (Properties of the General Exponential Functions)

1. If $a > 1$, $\dfrac{d}{dx}(a^x) = a^x\ln(a) > 0$, and the function a^x is increasing on $\mathbb{R}$.
   If $0 < a < 1$, $\dfrac{d}{dx}(a^x) = a^x\ln(a) < 0$, and the function a^x is decreasing on $\mathbb{R}$.



2. If $a > 0$ and $a \neq 1$, $\displaystyle\int a^u du = \frac{a^u}{\ln(a)} + c$.

Examples 1 :

1. $\dfrac{d}{dx}(5^x) = 5^x \ln(5)$,
2. $\dfrac{d}{dx}(6^{\sqrt{x}}) = 6^{\sqrt{x}} \ln(6) \dfrac{1}{2\sqrt{x}}$.
3. $\displaystyle\int 3^x dx = \frac{3^x}{\ln(3)} + c$,
4. $\displaystyle\int_{-1}^{0} 3^x dx = \left[\frac{3^x}{\ln(3)}\right]_{-1}^{0} = \frac{1-\frac{1}{3}}{\ln(3)} = \frac{2}{3\ln(3)}$,
5. $\displaystyle\int \frac{5^{\tan(x)}}{\cos^2(x)} dx = \int 5^{\tan(x)} \sec^2(x) dx = \frac{5^{\tan(x)}}{\ln(5)} + c$.

Theorem 4.4

$$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = \lim_{x\to\infty}(1+\frac{1}{x})^x = e.$$

Proof .
$(1+h)^{\frac{1}{h}} = e^{\frac{\ln(1+h)}{h}}$. Since $1 = \ln'(1) = \displaystyle\lim_{h\to 0}\frac{\ln(1+h)}{h}$, then
$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e$.
For the second equality, take $x = \frac{1}{h}$. ☐

Exercise 2 :
Find $f'(x)$ if

1. $2x = 4^{f(x)}$,
2. $f(x) = 7^{\sqrt[3]{x}}$,
3. $f(x) = \pi^{3x}$,
4. $f(x) = |\sin(x)|^x$,
5. $f(x) = (1+x^2)^{2x+1}$,

Solution

1. Differentiate both sides:
   $2 = 4^{f(x)} f'(x) \ln 4 \Rightarrow f'(x) = \dfrac{2}{4^{f(x)} \ln 4} = \dfrac{2}{2x \ln 4} = \dfrac{1}{x \ln 4}$.
2. $f(x) = 7^{\sqrt[3]{x}} = e^{\sqrt[3]{x} \ln 7}$, then $f'(x) = 7^{\sqrt[3]{x}} \dfrac{1}{3} x^{\frac{-2}{3}} \ln 7$.
3. $f'(x) = 3\pi^{3x} \ln \pi = 3\pi^{3x} \ln \pi$.
4. $f(x) = |\sin(x)|^x = e^{x \ln|\sin(x)|}$, then
   $f'(x) = [\ln|\sin(x)| + x\cot(x)]\, |\sin(x)|^x$.
5. $f(x) = (1+x^2)^{2x+1} = e^{(2x+1)\ln(1+x^2)}$, then
   $f'(x) = \left(2\ln(1+x^2) + \dfrac{2x(2x+1)}{1+x^2}\right)(1+x^2)^{2x+1}$.

Examples 2 :

1. $\displaystyle\int x^2 6^{x^3} dx \overset{t=x^3}{=} \frac{6^{x^3}}{3\ln 6} + c.$
2. $\displaystyle\int \frac{2^x}{2^x+1} dx \overset{t=2^x}{=} \frac{1}{\ln 2}\int \frac{dt}{t+1} = \frac{\ln(2^x+1)}{\ln 2} + c.$
3. $\displaystyle\int \frac{3^{-\cot(x)}}{\sin^2(x)} dx \overset{t=-\cot(x)}{=} \int 3^t dt = \frac{3^{-\cot(x)}}{\ln 3} + c$
4. $\displaystyle\int 2^{x\ln x}(1+\ln x) dx \overset{t=x\ln x}{=} \int 2^t dt = \frac{2^{x\ln x}}{\ln 2} + c$
5. $\displaystyle\int 4^x 5^{4^x} dx \overset{t=4^x}{=} \frac{1}{\ln 4}\int 5^t dt = \frac{5^{4^x}}{\ln 4\ \ln 5} + c$

6.

$$\begin{aligned}\int 3^x\,(1+\sin(3^x))dx &= \int 3^x dx + \int 3^x \sin(3^x)dx\\ &= \frac{3^x}{\ln 3} + \frac{1}{\ln 3}\int \sin(3^x)3^x \ln 3 dx\\ &= \frac{3^x}{\ln 3} - \frac{\cos(3^x)}{\ln 3} + c.\end{aligned}$$

7. $\displaystyle \int \frac{3^{\sqrt{x}}}{\sqrt{x}}dx \overset{u=\sqrt{x}}{=} 2\int 3^u du = \frac{2}{\ln 3}3^{\sqrt{x}} + c.$

4.1 Exercises

2-4-1 Find the derivative of the following functions:

1) $f(x) = 10^{x^2}$,
2) $f(x) = 2^{(x^3+1)}$,
3) $f(x) = 5^{(x^4+x^2)}$,
4) $f(x) = 6^{\sqrt{x}}$
5) $f(x) = (x^2+1)^{\sin(2x)}$,
6) $f(x) = (x^2+4)^{(x^3+1)}$,
7) $f(x) = (\sin(x)+3)^{(4\cos(x)+7)}$,
8) $f(x) = (e^{x^2}+1)^{(2x+1)}$,
9) $f(x) = x^2(x^2+1)^{(x^3+1)}$,

2-4-2 Evaluate the following integrals:

1) $\displaystyle \int \frac{(2^x+1)^2}{2^x}dx$,
2) $\displaystyle \int e^{3x}\sec^2(2+e^{3x})dx$,
3) $\displaystyle \int 10^{\cos(x)}\sin(x)dx$,
4) $\displaystyle \int x10^{x^2+3}dx$,
5) $\displaystyle \int_1^8 \left(\sqrt[3]{\frac{5}{x}}\right) dx$,
6) $\displaystyle \int x3^{2x^2}(3^{2x^2}+1)^{-4}dx$.

5 The General Logarithmic Function

Definition 5.1: (The General Logarithmic Function)

If $a \in (0,\infty)$ and $a \neq 1$, the function $f\colon \mathbb{R} \longrightarrow (0,\infty)$ defined by $f(x) = a^x$ is bijective. Its inverse function f^{-1} is denoted by $\log_a$ and called the logarithmic function with base a. For $y \in (0,\infty)$ and $x \in \mathbb{R}$,

$$x = \log_a(y) \iff y = a^x. \tag{5.1}$$

Examples 3 :

1. $9 = 3^2 \iff 2 = \log_3(9)$,
2. $16 = 4^2 \iff 2 = \log_4(16)$,
3. $64 = 4^3 \iff 3 = \log_4(64)$.
4. $\log_2 x = 3 \Leftrightarrow x = 2^3 = 8$.
5. $\log_a 125 = 3 \Leftrightarrow 125 = a^3 \iff a = \sqrt[3]{125} = 5$.
6.

$$\begin{aligned} 2\log|x| = \log 2 + \log|3x-4| &\iff \log x^2 = \log|2(3x-4)| \\ &\iff x^2 = 2|3x-4| \\ &\iff \begin{cases} x^2 = 2(3x-4) & \text{if } x \geq \frac{4}{3} \\ x^2 = 2(4-3x) & \text{if } x \leq \frac{4}{3} \end{cases} \\ &\iff x = 4,\ 2,\ -3+\sqrt{17} \text{ or } -3-\sqrt{17}. \end{aligned}$$

Theorem 5.2

For all $a \in (0,\infty) \setminus \{1\}$,

1. $\dfrac{d}{dx}\log_a(x) = \dfrac{1}{x\ln(a)}$,

2. $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}, \ \forall x > 0,$

3. $\log_e(x) = \ln(x).$

Proof .

1. Since $a^{\log_a(x)} = e^{\log_a(x)\ln a} = x$, then by differentiation, $1 = x(\ln a)\frac{d}{dx}\log_a(x) \iff \frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$.

2. From the previous result it is deduced that there exists $c \in \mathbb{R}$ such that $\log_a(x) = \dfrac{\ln(x)}{\ln(a)} + c$. Since $\log_a(a) = 1$, then $c = 0$.

3. $\log_e(x) = \dfrac{\ln(x)}{\ln(e)} = \ln(x).$

□

Notation. For $a = 10$ the function $\log_{10}$ is denoted by Log.

Examples 4 :

1. If $f(x) = \mathrm{Log}|\ln(x)| = \frac{\ln|\ln(x)|}{\ln(10)}$, then $f'(x) = \dfrac{1}{x\ln(10)\ln(x)}$,

2. If $f(x) = \ln|\mathrm{Log}(x)| = \ln(|\ln(x)|) - \ln(\ln(10))$, then $f'(x) = \dfrac{1}{x\ln(x)}$.

3. If $f(x) = x^{4+x^2}$, then $\ln(f(x)) = (4+x^2)\ln(x)$. Differentiating both sides with respect to x, we have $\dfrac{f'(x)}{f(x)} = 2x\ln(x) + \dfrac{4+x^2}{x}$ and $f'(x) = \left(2x\ln(x) + (4+x^2)\dfrac{1}{x}\right)x^{4+x^2}$.

4. $\dfrac{d}{dx}\pi^x = \pi^x \ln\pi,$

5. $\dfrac{d}{dx}x^\pi = \pi x^{\pi-1},$

6. $\dfrac{d}{dx}x^x = \dfrac{d}{dx}e^{x\ln x} = (1+\ln x)x^x,$

Properties 5.3

For $a > 0$, $b > 0$, $a \neq 1$ and $b \neq 1$, we have

1. $\log_b(b) = 1$, $\log_b(1) = 0$, and $\log_b(b^x) = x$, $\forall x \in \mathbb{R}$,
2. $\log_b(xy) = \log_b x + \log_b y$, $\forall x > 0$, $y > 0$,
3. $\log_b(\frac{x}{y}) = \log_b x - \log_b y$, $\forall x > 0$, $y > 0$,
4. $\log_b(x^y) = y\log_b x$, $x > 0$, $x \neq 1$, $\forall y \in \mathbb{R}$,
5. $(\log_b x)(\log_a b) = \log_a x$.
6. $y = b^{(\log_b y)}$, for $y > 0$,
7. $b^{\ln a} = a^{\ln b}$.

Proof .

$$\log_a x = y \iff x = a^y \iff x = e^{y\ln a} \iff y\ln a = \ln x \iff y = \frac{\ln x}{\ln a},$$

1. From the previous property, $\log_b(b) = \dfrac{\ln b}{\ln b} = 1$,
   $\log_b(1) = \dfrac{\ln 1}{\ln b} = 0$, and $\log_b(b^x) = \dfrac{\ln b^x}{\ln b} = \dfrac{x\ln b}{\ln b} = x$,
2. $\log_b(xy) = \dfrac{\ln xy}{\ln b} = \dfrac{\ln x + \ln y}{\ln b} = \log_b x + \log_b y$,
3. $\log_b(\frac{x}{y}) = \frac{\ln x - \ln y}{\ln b} = \log_b x - \log_b y$,
4. $\log_b(x^y) = \frac{\ln x^y}{\ln b} = \frac{y\ln x}{\ln b} = y\log_b x$,
5. $(\log_b x)(\log_a b) = \frac{\ln x}{\ln b}\cdot\frac{\ln b}{\ln a} = \frac{\ln x}{\ln a} = \log_a x$,
6. $b^{(\log_b y)} = e^{(\log_b y)\ln b} = e^{(\frac{\ln y}{\ln b})\ln b} = y$,

7. $b^{\ln a} = e^{(\ln a)(\ln b)} = a^{\ln b}$.

□

5.1 Exercises

2-5-1 Solve the following equations for x:
$\log_3(x^4) + \log_3(x^3) - 2\log_3(x^{\frac{1}{2}}) = 5$.

2-5-2 Find the derivative of the following functions:

1) $f(x) = \log_5(x^3 + 1)$;
2) $f(x) = \sqrt{1 + \text{Log}(1 + x^2)}\,\log_3(1 + x^4)$.

6 Inverse Trigonometric Functions

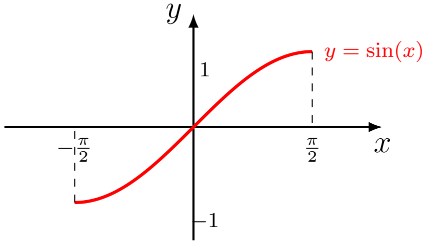
Theorem 6.1

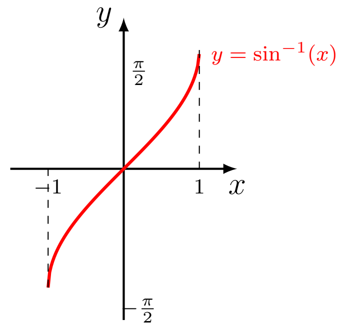
Let $f\colon I \longrightarrow J$ be a bijective function where I and J are intervals, then

1. If f is continuous, then f^{-1} is also continuous.
2. If f is differentiable and $f'(x) \neq 0$ for all $x \in I$, then f^{-1} is differentiable on J and $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$.

6.1 The Sine Function

The function $f\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$ defined by $f(x) = \sin(x)$ is continuous and bijective. The inverse function f^{-1} is denoted by $\sin^{-1}(x)$ or Arcsin$x$. The inverse function is continuous on $[-1, 1]$.





Remark 4 :

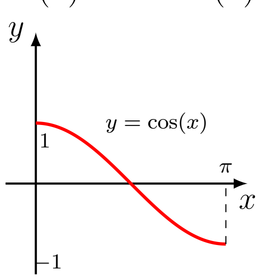
1. $\sin^{-1}(\sin(x)) = x$ only for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
2. $\sin\left(\sin^{-1}(x)\right) = x$; $\forall x \in [-1, 1]$.
3. Since $\sin^{-1}(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for all $x \in [-1, 1]$, then $\cos(\sin^{-1}(x)) = \sqrt{1 - \sin^2(\sin^{-1}(x))} = \sqrt{1 - x^2}$.
4. $\dfrac{d}{dx}(\sin^{-1})(x) = \dfrac{1}{\cos(\sin^{-1}(x))} = \dfrac{1}{\sqrt{1 - x^2}}$, for all $x \in]-1, 1[$.

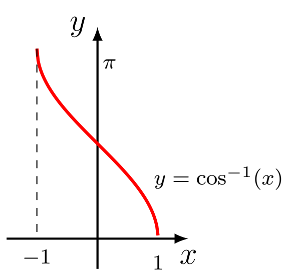
Example 5 :

1. $\tan(\sin^{-1}(x)) = \dfrac{\sin(\sin^{-1}(x))}{\cos(\sin^{-1}(x))} = \dfrac{x}{\sqrt{1 - x^2}}$, for $x \in]-1, 1[$.
2. $\cot(\sin^{-1}(x)) = \dfrac{\sqrt{1 - x^2}}{x}$ for $x \in [-1, 1], x \neq 0$.

6.2 The Cosine Function

The function $f\colon [0, \pi] \longrightarrow [-1, 1]$ defined by $f(x) = \cos(x)$ is continuous and bijective. The inverse function f^{-1} is denoted by $f^{-1}(x) = \cos^{-1}(x)$ or $f^{-1}(x) = \text{Arccos}(x)$.



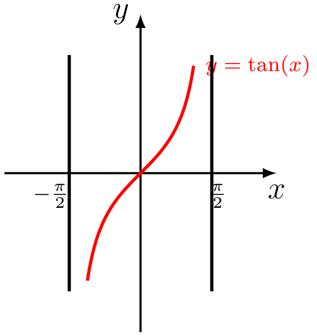


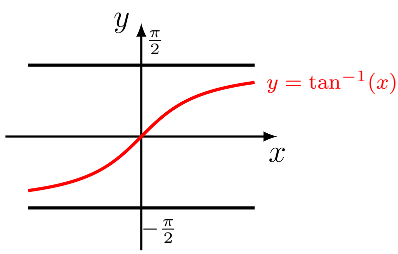
Remark 5 :

1. $\cos(\cos^{-1}(x)) = x$, if $x \in [-1, 1]$,
2. $\cos^{-1}(\cos(x)) = x$, if $x \in [0, \pi]$.
3. $\sin(\cos^{-1}(x)) = \sqrt{1 - x^2}$, if $x \in [-1, 1]$.
4. $\dfrac{d}{dx}(\cos^{-1})(x) = \dfrac{-1}{\sin(\cos^{-1}(x))} = \dfrac{-1}{\sqrt{1 - x^2}}$, for $x \in]-1, 1[$.

6.3 The Tangent Function

The function $f :]-\frac{\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$ defined by $f(x) = \tan(x)$ is increasing, continuous and differentiable, $(f'(x) = 1 + \tan^2(x) = \sec^2(x))$. The inverse function f^{-1} is denoted by $\tan^{-1}(x)$, for $x \in \mathbb{R}$.





Remark 6 :

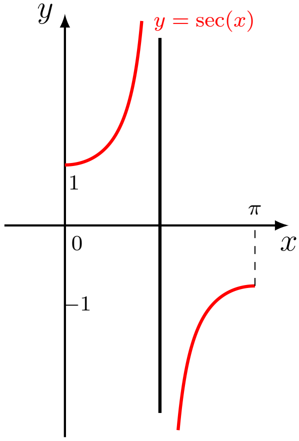
1. $y = \tan^{-1}(x) \iff x = \tan y$, $\forall x \in \mathbb{R}$ and $\forall y \in]-\frac{\pi}{2}, \frac{\pi}{2}[$,
2. $\tan(\tan^{-1}(x)) = x$, $\forall x \in \mathbb{R}$,
3. $\tan^{-1}(\tan(x)) = x$; $\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$,
4. $\dfrac{d}{dx}(\tan^{-1})(x) = \dfrac{1}{1 + \tan^2(\tan^{-1}(x))} = \dfrac{1}{1 + x^2}$, for all $x \in \mathbb{R}$.

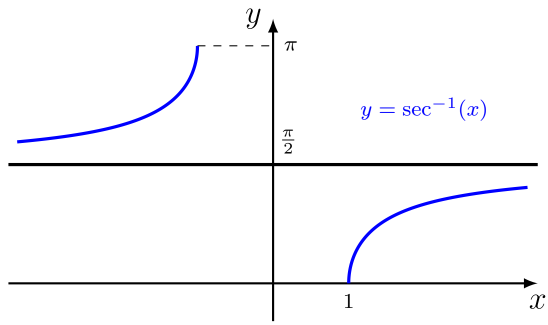
In the same way we define the function $\cot^{-1} : \mathbb{R} \longrightarrow]0, \pi[$, as the inverse function of $\cot :]0, \pi[\longrightarrow \mathbb{R}$.

$$(\cot^{-1})'(x) = \frac{-1}{1 + \cot^2(\cot^{-1}(x))} = \frac{-1}{1 + x^2}.$$

6.4 The Secant Function

The function $f\colon [0,\frac{\pi}{2}[\cup]\frac{\pi}{2},\pi]$ defined by $f(x) = \dfrac{1}{\cos(x)} = \sec(x)$ is increasing and C^∞. Its inverse function is denoted by $f^{-1}(x) = \sec^{-1}(x)$, for $x \in]-\infty,-1] \cup [1,+\infty[$.



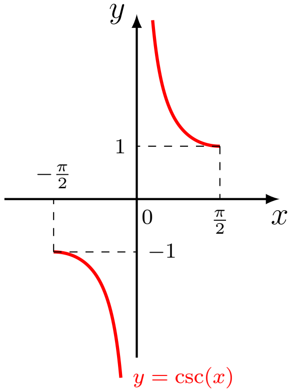


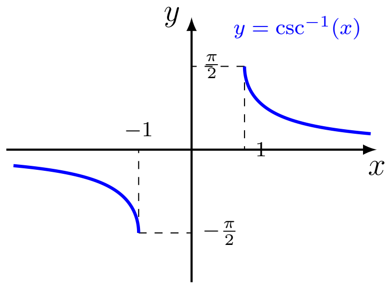
Remark 7 :

1. $\sec'(x) = \sec(x)\tan(x)$, $\sec^2(x) = 1 + \tan^2(x)$,
2. $\tan^2(\sec^{-1}(x)) = x^2 - 1$ and $\tan(\sec^{-1}(x)) = \sqrt{x^2-1}$, if $x \in]1,+\infty[$
3. $\tan(\sec^{-1}(x)) = -\sqrt{x^2-1}$, if $x \in]-\infty,-1[$,
4. $\dfrac{d}{dx}(\sec^{-1})(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$, for all $x \in]-\infty,-1[\cup]1,+\infty[$.

6.5 The Cosecant Function

The function $f\colon [-\frac{\pi}{2},0[\cup]0,\frac{\pi}{2}]$ defined by $f(x) = \dfrac{1}{\sin(x)} = \csc(x)$ is decreasing and C^∞, $(f'(x) = -\csc(x)\cot(x) = -\dfrac{\cos(x)}{\sin^2(x)})$. Its inverse function is denoted by $f^{-1}(x) = \csc^{-1}(x)$ for $x \in]-\infty,-1] \cup [1,+\infty[$.





Remark 8 :

1. $\csc'(x) = -\csc(x)\cot(x)$, $\csc^2(x) = 1 + \cot^2(x)$,
2. $\cot^2(\csc^{-1}(x)) = x^2 - 1$,
3. $\cot(\csc^{-1}(x)) = \sqrt{x^2 - 1}$, if $x \in]1, +\infty[$,
4. $\cot(\csc^{-1}(x)) = -\sqrt{x^2 - 1}$, if $x \in]-\infty, -1[$,
5. $\dfrac{d}{dx}(\csc^{-1})(x) = \dfrac{-1}{|x|\sqrt{x^2 - 1}}$, for all $x \in]-\infty, -1[\cup]1, +\infty[$.

Exercise 3 :
Prove that

1. $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$.
2. $\tan^{-1}(x) + \tan^{-1}(\dfrac{1}{x}) = \dfrac{\pi}{2}$, if $x > 0$ and $\tan^{-1}(x) + \tan^{-1}(\dfrac{1}{x}) = -\dfrac{\pi}{2}$, if $x < 0$.

Solution

1. $\dfrac{d}{dx}\left(\sin^{-1}(x) + \cos^{-1}(x)\right) = 0$ and since $\sin^{-1} 0 + \cos^{-1} 0 = \frac{\pi}{2}$, then $\sin^{-1}(x) + \cos^{-1}(x) = \dfrac{\pi}{2}$.

2. $\frac{d}{dx}\left(\tan^{-1}(x)+\tan^{-1}(\frac{1}{x})\right)=\frac{1}{1+x^2}+\frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}=0$ and since $\tan^{-1}(1)+\tan^{-1}(1)=\frac{\pi}{2}$, then $\tan^{-1}(x)+\tan^{-1}(\frac{1}{x})=\frac{\pi}{2}$, if $x>0$ and the function $\tan^{-1}(x)+\tan^{-1}(\frac{1}{x})$ is odd, then $\tan^{-1}(x)+\tan^{-1}(\frac{1}{x})=-\frac{\pi}{2}$, if $x<0$.

Theorem 6.2

1. $\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}},\ \forall\ |x|<1,$
2. $\frac{d}{dx}\cos^{-1}(x)=\frac{-1}{\sqrt{1-x^2}},\ \forall\ |x|<1,$
3. $\frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+x^2},\ \forall\ x\in\mathbb{R},$
4. $\frac{d}{dx}\cot^{-1}(x)=\frac{-1}{1+x^2},\ \forall\ x\in\mathbb{R},$
5. $\frac{d}{dx}\sec^{-1}(x)=\frac{1}{|x|\sqrt{x^2-1}},\ \forall\ |x|>1.$
6. $\frac{d}{dx}\csc^{-1}(x)=\frac{-1}{|x|\sqrt{x^2-1}},\ \forall\ |x|>1.$

Theorem 6.3

For $a>0$,

1. $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+c\ \ ,\ \ (|x|<a)$
2. $\int\frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx=\sin^{-1}\left(\frac{f(x)}{a}\right)+c\ \ ,\ \ (|f|<a))$

3. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

4. $\displaystyle\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right) + c$

5. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c \quad , \quad (x>a)$

6. $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left(\frac{f(x)}{a}\right) + c \quad , \quad (f>a))$

Examples 6 :

1. $\displaystyle\int \frac{x^2}{5+x^6}dx \overset{u=x^3,a=\sqrt{5}}{=} \frac{1}{3}\int \frac{du}{a^2+u^2} = \frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{x^3}{\sqrt{5}}\right) + c,$

2. $\displaystyle\int \frac{3x}{\sqrt{9-x^4}}dx \overset{u=x^2,a=3}{=} \frac{3}{2}\int \frac{du}{\sqrt{a^2-u^2}} = \frac{3}{2}\sin^{-1}\left(\frac{x^2}{3}\right) + c,$

3. $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}} \overset{u=\ln x}{=} \int \frac{du}{1-u^2} = \sin^{-1}(\ln x) + c,$

4. $\displaystyle\int \frac{dx}{1+3x^2} \overset{u=\sqrt{3}x}{=} \frac{1}{\sqrt{3}}\int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}x\right) + c,$

5. $\displaystyle\int \frac{e^{2x}}{e^{4x}+16}dx \overset{u=e^{2x},a=4}{=} \frac{1}{2}\int \frac{du}{a^2+u^2} = \frac{1}{8}\tan^{-1}\left(\frac{e^{2x}}{4}\right) + c,$

6. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-36}} \overset{u=e^x,a=6}{=} \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{6}\sec^{-1}\left(\frac{e^x}{6}\right) + c,$

7.
$$\begin{aligned}\int \frac{\sin(x)}{\sqrt{25-\cos^2(x)}}dx \quad &\overset{u=\cos(x),a=5}{=} \quad -\int \frac{du}{\sqrt{a^2-u^2}} \\ &= \quad -\sin^{-1}\left(\frac{\cos(x)}{5}\right) + c,\end{aligned}$$

8. $$\int \frac{2^x}{\sqrt{4-4^x}}dx \overset{u=2^x, a=2}{=} \frac{1}{\ln 2}\int \frac{du}{\sqrt{a^2-u^2}} = \frac{1}{\ln 2}\sin^{-1}\left(\frac{2^x}{2}\right)+c,$$

9. $$\int \frac{dx}{x^2+6x+25} \overset{u=x+3, a=4}{=} \int \frac{du}{u^2+a^2} = \frac{1}{4}\tan^{-1}\left(\frac{x+3}{4}\right)+c,$$

10.

$$\begin{aligned}\int \frac{x+2}{\sqrt{4-x^2}}dx &= \int\left(\frac{x}{\sqrt{4-x^2}}+\frac{2}{\sqrt{4-x^2}}\right)dx \\ &= \frac{-1}{2}\int\left(4-x^2\right)^{-\frac{1}{2}}(-2x)dx+2\int\frac{dx}{\sqrt{(2)^2-(x)^2}} \\ &= -\left(4-x^2\right)^{\frac{1}{2}}+2\sin^{-1}\left(\frac{x}{2}\right)+c,\end{aligned}$$

11.

$$\begin{aligned}\int \frac{x+\tan^{-1}x}{1+x^2}dx &= \int\left(\frac{x}{1+x^2}+\frac{\tan^{-1}x}{1+x^2}\right)dx \\ &= \frac{1}{2}\ln(1+x^2)+\frac{\left(\tan^{-1}x\right)^2}{2}+c.\end{aligned}$$

12.

$$\begin{aligned}\int \frac{\sin(x)}{\sqrt{e^{\cos(x)}-1}}dx &\overset{t=\cos(x)}{=} -\int\frac{dt}{\sqrt{e^t-1}} \\ &\overset{u^2=e^t-1}{=} -\int\frac{2du}{1+u^2} \\ &= -2\tan^{-1}(\sqrt{e^{\cos(x)}-1})+c.\end{aligned}$$

Or

$$\begin{aligned}\int \frac{\sin(x)}{\sqrt{e^{\cos(x)}-1}}dx &\overset{t=e^{\frac{1}{2}\cos(x)}}{=} -2\int\frac{dt}{t\sqrt{t^2-1}} \\ &= -2\sec^{-1}(e^{\frac{1}{2}\cos(x)})+c.\end{aligned}$$

Exercise 4 :
Compute the following integrals:

1. $\displaystyle\int \frac{x+1}{x^2+1}dx,$
2. $\displaystyle\int \frac{e^x}{\sqrt{e^{2x}-1}}dx,$
3. $\displaystyle\int \frac{e^x}{1+e^{2x}}dx,$
4. $\displaystyle\int_0^1 \frac{e^x}{1+e^{2x}}dx,$
5. $\displaystyle\int_0^{2^{-\frac{1}{4}}} \frac{x}{\sqrt{1-x^4}}dx,$
6. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}dx,$
7. $\displaystyle\int \frac{\cos(x)}{\sqrt{1-\sin^2(x)}}dx,$
8. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos(x)}{\sqrt{1-\sin^2(x)}}dx,$
9. $\displaystyle\int \frac{1}{x\sqrt{x^6-1}}dx,$
10. $\displaystyle\int \frac{x+\sin^{-1}(x)}{\sqrt{1-x^2}}dx,$

6.6 Exercises

2-6-1 Compute $\frac{dy}{dx}$ for each of the following:

1) $y=\sin^{-1}(\frac{x}{2}),$
2) $y=\cos^{-1}(\frac{x}{3}),$
3) $y=\tan^{-1}(\frac{x}{5}),$
4) $y=\cot^{-1}(\frac{x}{7}),$
5) $y=\sec^{-1}(\frac{x}{2}),$
6) $y=\csc^{-1}(\frac{x}{3}),$

2-6-2 Find the exact value of y in each of the following

1) $y=3\sin^{-1}(\frac{1}{2})$
2) $y=2\cos^{-1}(\frac{\sqrt{3}}{2}),$
3) $y=4\tan^{-1}(\frac{1}{\sqrt{3}})$
4) $y=5\cot^{-1}(\frac{1}{\sqrt{3}}),$
5) $y=2\sec^{-1}(-2),$
6) $y=3\cos^{-1}(-\frac{\sqrt{3}}{2}),$
7) $y=\cos(2\cos^{-1}(x)),$
8) $y=\sin(2\cos^{-1}(x)).$
9) $y=\cos^{-1}(-\frac{1}{2}),$
10) $y=\sin^{-1}(\frac{\sqrt{3}}{2}),$

11) $y=\tan^{-1}(-\sqrt{3})$,

12) $y=\cot^{-1}(-\frac{\sqrt{3}}{3})$,

13) $y=\sec^{-1}(-\sqrt{2})$,

14) $y=\csc^{-1}(-\sqrt{2})$

15) $y=\sec^{-1}(-\frac{2}{\sqrt{3}})$,

16) $y=\csc^{-1}(-\frac{2}{\sqrt{3}})$,

17) $y=\sec^{-1}(-2)$,

18) $y=\csc^{-1}(-2)$,

19) $y=\tan^{-1}(\frac{-1}{\sqrt{3}}$,

20) $y=\cot^{-1}(-\sqrt{3})$.

2-6-3 Evaluate the following integrals.

1) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}dx$,

2) $\displaystyle\int_0^{2^{-\frac{1}{4}}} \frac{x}{\sqrt{1-x^4}}dx$,

3) $\displaystyle\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$,

4) $\displaystyle\int \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}dx$,

5) $\displaystyle\int e^{\sin(2x)}\cos(2x)dx$,

6) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}dx$,

7) $\displaystyle\int e^x\cos(1+2e^x)dx$,

8) $\displaystyle\int \frac{4^{\sec^{-1}(x)}}{x\sqrt{x^2-1}}dx$.

7 Hyperbolic and Inverse Hyperbolic Functions

7.1 The Hyperbolic Functions

Definition 7.1

1. The function $\sinh(x)=\dfrac{e^x-e^{-x}}{2}$, for $x\in\mathbb{R}$ is called the hyperbolic sine function.

2. The function $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$, for $x\in\mathbb{R}$, is called the hyperbolic cosine function.

3. The function $\tanh(x)=\dfrac{\sinh(x)}{\cosh(x)}=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, for $x\in\mathbb{R}$, is

called the hyperbolic tangent function.

4. The function $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, for $x \in \mathbb{R} \setminus \{0\}$, is called the hyperbolic cotangent function.

5. The function $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)} = \dfrac{2}{e^x + e^{-x}}$, for $x \in \mathbb{R}$, is called the hyperbolic secant function:

6. The function $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)} = \dfrac{2}{e^x - e^{-x}}$, for $x \in \mathbb{R} \setminus \{0\}$, is called the hyperbolic cosecant function:

Some properties of the hyperbolic functions:

Theorem 7.2

1. $\cosh^2(x) - \sinh^2(x) = 1, \quad \forall x \in \mathbb{R}$,
2. $1 - \tanh^2(x) = \operatorname{sech}^2(x), \ \forall x \in \mathbb{R}$,
3. $\coth^2(x) - 1 = \operatorname{csch}^2(x), \ \forall x \in \mathbb{R} \setminus \{0\}$,
4. $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$,
5. $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$.

Proof .

1. $\cosh(x) + \sinh(x) = e^x$, $\cosh(x) - \sinh(x) = e^{-x}$, then $\cosh^2(x) - \sinh^2(x) = 1$,

2. $\operatorname{sech}^2(x) = \dfrac{1}{\cosh^2(x)} = \dfrac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = 1 - \tanh^2(x)$,

3. $\operatorname{csch}^2(x) = \dfrac{1}{\sinh^2(x)} = \dfrac{\cosh^2(x) - \sinh^2(x)}{\sinh^2(x)} = \coth^2(x) - 1$,

4. $\cosh(x)\cosh(y) = \dfrac{e^{x+y} + e^{-x-y} + e^{x-y} + e^{-x+y}}{4}$ and

   $\sinh(x)\sinh(y) = \dfrac{e^{x+y} + e^{-x-y} - e^{x-y} - e^{-x+y}}{4}$, then

$$\cosh(x)\cosh(y) + \sinh(x)\sinh(y) = \frac{e^{x+y} + e^{-x-y}}{2} = \cosh(x+y)$$

5. $\cosh(x)\sin(y) = \dfrac{e^{x+y} - e^{-x-y} - e^{x-y} + e^{-x+y}}{4}$ and

   $\sinh(x)\cosh(y) = \dfrac{e^{x+y} - e^{-x-y} + e^{x-y} - e^{-x+y}}{4}$, then

$$\cosh(x)\sinh(y) + \sinh(x)\cosh(y) = \frac{e^{x+y} - e^{-x-y}}{2} = \sinh(x+y).$$

□

Theorem 7.3: (Derivative of Hyperbolic Functions)

1. $\dfrac{d}{dx}(\sinh(x)) = \cosh(x)$
2. $\dfrac{d}{dx}(\cosh(x)) = \sinh(x)$
3. $\dfrac{d}{dx}(\tanh(x)) = \text{sech}^2(x)$
4. $\dfrac{d}{dx}(\coth(x)) = -\text{csch}^2(x)$
5. $\dfrac{d}{dx}(\text{sech}(x)) = -\text{sech}(x)\tanh(x)$
6. $\dfrac{d}{dx}(\text{csch}(x)) = -\text{csch}(x)\coth(x)$.

Theorem 7.4: (Integration of Hyperbolic Functions)

1. $\displaystyle\int \sinh(x)dx = \cosh(x) + c$
2. $\displaystyle\int \cosh(x)dx = \sinh(x) + c$

3. $\displaystyle\int \operatorname{sech}^2(x)dx = \tanh(x) + c$

4. $\displaystyle\int \operatorname{csch}^2(x)dx = -\coth(x) + c$

5. $\displaystyle\int \operatorname{sech}(x)\tanh(x)dx = -\operatorname{sech}(x) + c$

6. $\displaystyle\int \operatorname{csch}(x)\coth(x)dx = -\operatorname{csch}(x) + c$

Examples 1 :

1. $\displaystyle\int \frac{\sinh(\sqrt{x})}{\sqrt{x}}dx \overset{u=\sqrt{x}}{=} 2\int \sinh(u)du = 2\cosh(u) + c = 2\cosh(\sqrt{x}) + c.$

2. $\displaystyle\int \cosh(x)\operatorname{csch}^2(x)dx = \int \frac{\cosh(x)}{\sinh^2(x)}dx = -\frac{1}{\sinh(x)} + c.$

Exercise 5 :
Find the points on the graph of the function $f(x) = \sinh(x)$ at which the tangent line has slope 2.
Solution
The slope of the tangent line is $m = \dfrac{dy}{dx} = f'(x) = \cosh(x)$.
Then $m = 2 \iff \cosh(x) = 2 \iff e^{2x} - 4e^x + 1 = 0$.
There are two solutions: $x_1 = \ln(2 + \sqrt{3})$ and $x_2 = \ln(2 - \sqrt{3})$.
$f(x_1) = \frac{2+\sqrt{3}-\frac{1}{2+\sqrt{3}}}{2} = \sqrt{3}$ and $f(x_2) = \frac{2-\sqrt{3}-\frac{1}{2-\sqrt{3}}}{2} = -\sqrt{3}$.
Then $\Big(\ln(2+\sqrt{3}), \sqrt{3}\Big)$ and $\Big(\ln(2-\sqrt{3}), -\sqrt{3}\Big)$ are the required points.

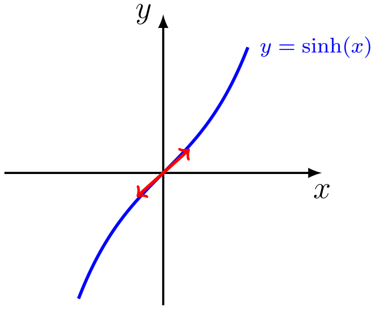
7.2 The Sine Hyperbolic Function and its Inverse

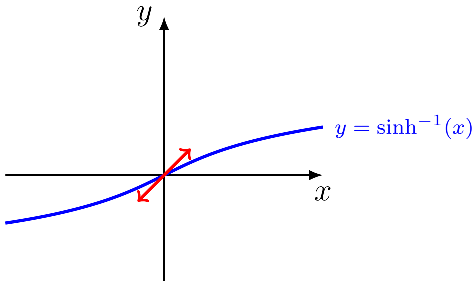
1. The function $f(x) = \sinh(x)$ is odd and $f'(x) = \cosh(x) > 0$,

2. $\displaystyle\lim_{x\to+\infty} \sinh(x) = +\infty$ and $\displaystyle\lim_{x\to+\infty} \frac{\sinh(x)}{x} = +\infty.$

3. f is continuous and bijective. The inverse function f^{-1} is denoted $\sinh^{-1}$. This function is continuous,

4. $x, y \in \mathbb{R}$, $y = \sinh^{-1}(x) \iff x = \sinh(y)$.





Theorem 7.5

1. $\dfrac{d}{dx}\sinh^{-1}(x) = \dfrac{1}{\sqrt{x^2+1}}, \ \forall x \in \mathbb{R}$,

2. $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}), \ \forall x \in \mathbb{R}$.

Proof .

1. $\dfrac{d}{dx}\sinh^{-1}(x) = \dfrac{1}{\cosh(\sinh^{-1}(x))} = \dfrac{1}{\sqrt{1+\sinh^2(\sinh^{-1}(x))}} = \dfrac{1}{\sqrt{x^2+1}}$, $\forall x \in \mathbb{R}$,

2. $y = \sinh^{-1}(x) \iff \sinh(y) = x$, then $\cosh(y) = \sqrt{1+\sinh^2(y)} = \sqrt{1+x^2}$ and $e^y = \cosh(y) + \sinh(y)$. Hence $y = \sinh^{-1}(x) = \ln(x + \sqrt{x^2+1}), \ \forall x \in \mathbb{R}$.

□

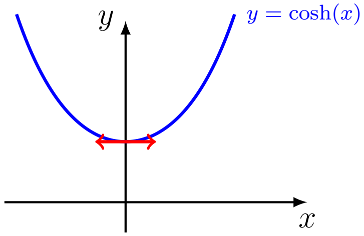
7.3 The Cosine Hyperbolic Function

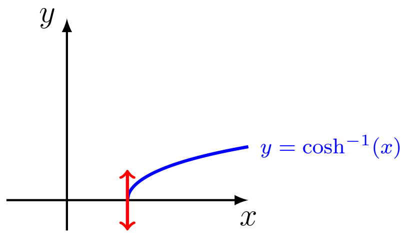
1. The function $f(x) = \cosh(x)$ defined on $\mathbb{R}$ is even and $f'(x) = \sinh(x)$,

2. The restriction of the function f on the interval $[0,+\infty[$ is continuous and increasing. Then $f\colon [0,+\infty[\longrightarrow [1,+\infty[$ is bijective. The inverse function $f^{-1}\colon [1,+\infty[\longrightarrow [0,+\infty[$ is denoted by $\cosh^{-1}$ is continuous on $[1,+\infty[$.

3. $\lim\limits_{x\to+\infty} \cosh(x) = +\infty$ and $\lim\limits_{x\to+\infty} \dfrac{\cosh(x)}{x} = +\infty$,

4. If $x \in [1,\infty)$ and $y \in [0,\infty)$, $y = \cosh^{-1}(x) \iff x = \cosh(y)$.





Theorem 7.6

1. $\dfrac{d}{dx}\cosh^{-1}(x) = \dfrac{1}{\sqrt{x^2-1}}, \ \forall x \in]1,+\infty[,$

2. $\cosh^{-1}(x) = \ln(x+\sqrt{x^2-1}), \ \forall x \in [1,+\infty[.$

Proof .

1. $\dfrac{d}{dx}\cosh^{-1}(x) = \dfrac{1}{\sinh(\cosh^{-1}(x))} = \dfrac{1}{\sqrt{\cosh^2(\cosh^{-1}(x))-1}} = \dfrac{1}{\sqrt{x^2-1}},$
$\forall x \in \mathbb{R}$,

2. $y = \cosh^{-1}(x) \iff \cosh(y) = x$, then
$\sinh(y) = \sqrt{\cosh^2(y)-1} = \sqrt{x^2-1}$ and $e^y = \cosh(y) + \sinh(y)$.
Hence $y = \cosh^{-1}(x) = \ln(x+\sqrt{x^2-1})$, $\forall x \in]1,+\infty[$.

$\square$

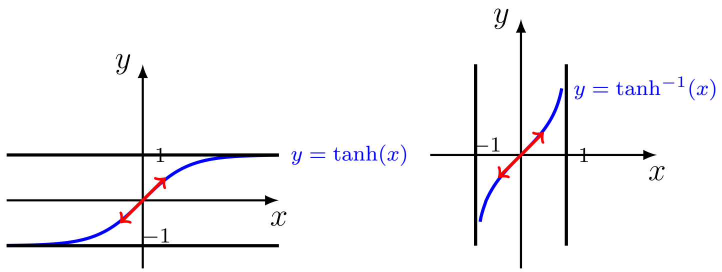
7.4 The Tangent Hyperbolic Function

1. The function $f(x) = \tanh(x)$ defined on $\mathbb{R}$ is odd and $f'(x) = 1 - \tanh^2(x) = \operatorname{sech}^2(x) > 0$,

2. The function $f\colon \mathbb{R} \longrightarrow]-1,1[$ is continuous and increasing. Then f is bijective. The inverse function f^{-1} denoted by $\tanh^{-1}$ is continuous on $]-1,1[$.

3. $\lim\limits_{x\to+\infty} \tanh(x) = 1$,

4. $y = \tanh^{-1}(x) \iff x = \tanh(y)$ for all $y \in \mathbb{R}$ and $x \in]-1,1[$.



Theorem 7.7

1. $\dfrac{d}{dx}\tanh^{-1}(x) = \dfrac{1}{1-x^2}, \ \forall x \in]-1,1[$,

2. $\tanh^{-1}(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), \ \forall x \in]-1,1[$.

Proof .

1. $\dfrac{d}{dx}\tanh^{-1}(x) = \dfrac{1}{1-\tanh^2(\tanh^{-1}(x))} = \dfrac{1}{1-x^2}, \ \forall x \in]-1,1[$,

2.

$$\begin{aligned} y = \tanh^{-1}(x) \quad &\iff \quad \tanh(y) = x \\ &\iff \quad \frac{e^{2y}-1}{e^{2y}+1} = x \iff e^{2y} = \frac{1+x}{1-x} \\ &\iff \quad y = \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \end{aligned}$$

□

7.5 The Inverse Hyperbolic Cotangent Function

1. The function $f(x) = \coth(x)$ defined on $\mathbb{R}^*$ is odd and $f'(x) = 1 - \coth^2(x) = -\text{csch}^2(x) < 0$. The function f is continuous and decreasing, then f is bijective. The inverse function f^{-1} is denoted by $f^{-1} = \coth^{-1}$ and it is also continuous.
   $\coth^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$.
2. $\lim_{x \to +\infty} \coth(x) = 1,$
3. $y = \coth^{-1}(x) \iff x = \coth(y)$ for all $y \in]0, +\infty[$ and $x \in]0, 1[$.
4. $(f^{-1})'(x) = \frac{-1}{1-x^2}$.
5. $\int \frac{dx}{1-x^2} = -\coth^{-1}(x) + c$ for $|x| > 1$.
6. $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$ on $\mathbb{R} \setminus \{\pm 1\}$.

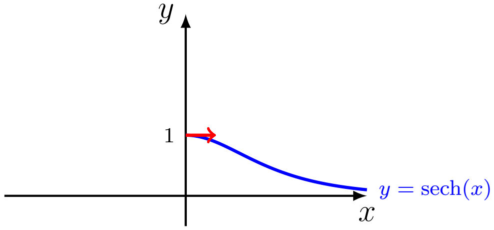
7.6 The Inverse Hyperbolic Secant Function

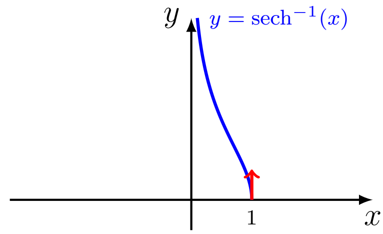
1. The function $f\colon [0, +\infty[\longrightarrow]0, 1]$ defined by:
   $f(x) = \text{sech}(x) = \frac{2}{e^x + e^{-x}}$ is bijective and decreasing since $f'(x) = -\text{sech}(x)\tanh(x) < 0$.

2. The inverse function f^{-1} is denoted by $f^{-1} = \mathrm{sech}^{-1}$ and it is continuous,

3. $\lim_{x \to +\infty} \mathrm{sech}(x) = 0$,

4. For all $x \in]0,1]$ and $y \in [0,+\infty[$, $y = \mathrm{sech}^{-1}(x) \iff x = \mathrm{sech}(y)$.





Theorem 7.8

1. $(\mathrm{sech}^{-1})'(x) = \dfrac{-1}{x\sqrt{1-x^2}}$, $\forall x \in]0,1[$,

2. $\mathrm{sech}^{-1}(x) = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$, $\forall x \in]0,1[$.

Proof .

1. $\dfrac{d}{dx}\mathrm{sech}^{-1}(x) = -\dfrac{1}{\mathrm{sech}(\mathrm{sech}^{-1}(x)).\tanh(\mathrm{sech}^{-1}(x))} = -\dfrac{1}{x\sqrt{1-x^2}}$, $\forall x \in]0,1[$,

2. $y = \mathrm{sech}^{-1}(x) \iff \mathrm{sech}(y) = x \iff \cosh(y) = \frac{1}{x}$, then $\sinh(y) = \dfrac{\sqrt{1-x^2}}{x}$ and $y = \mathrm{sech}^{-1}(x) = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$.

□

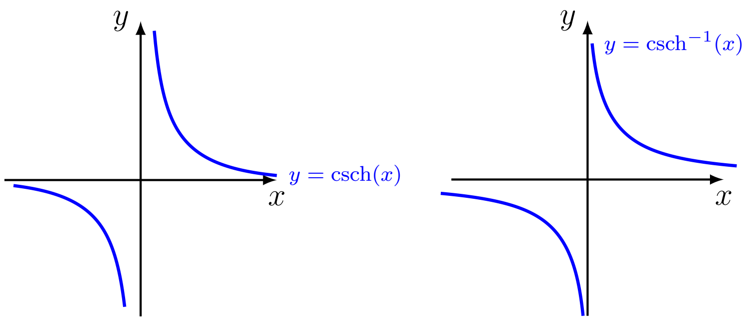
7.7 The Inverse Cosecant Hyperbolic Function

1. The function $f\colon \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}\setminus\{0\}$ defined by:
$f(x) = \mathrm{csch}(x) = \dfrac{2}{e^x - e^{-x}}$ is bijective and decreasing since
$f'(x) = -\mathrm{csch}(x)\coth(x) < 0$.

2. If $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \{0\}$, $y = \operatorname{csch}^{-1}(x) \iff x = \operatorname{csch}(y)$.

3. $\lim_{x \to +\infty} \operatorname{csch}(x) = 0$,

4. For all $x, y \in \mathbb{R} \setminus \{0\}$, $y = \operatorname{csch}^{-1}(x) \iff x = \operatorname{csch}(y)$.



Theorem 7.9

1. $(\operatorname{csch}^{-1})'(x) = \dfrac{-1}{x\sqrt{1+x^2}}, \qquad \forall x \in]0, +\infty[$,

2. $\operatorname{csch}^{-1}(x) = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right), \qquad \forall x \in]0, +\infty[$.

Proof .

1. $\dfrac{d}{dx}\operatorname{csch}^{-1}(x) = -\dfrac{1}{\operatorname{csch}(\operatorname{csch}^{-1}(x)).\coth(\operatorname{csch}^{-1}(x))} = -\dfrac{1}{x\sqrt{1+x^2}}$, $\forall x \in]0, +\infty[$,

2. $y = \operatorname{csch}^{-1}(x) \iff \operatorname{csch}(y) = x \iff \sinh(y) = \frac{1}{x}$, then $\cosh(y) = \frac{\sqrt{1+x^2}}{x}$ and $y = \operatorname{csch}^{-1}(x) = \ln\left(\dfrac{1+\sqrt{1+x^2}}{x}\right)$.

□

7.8 General Summary

Theorem 7.10: [Derivatives of Hyperbolic Functions]

1. $\dfrac{d}{dx}\sinh(x) = \cosh(x)$,
2. $\dfrac{d}{dx}\sinh(f(x)) = \cosh(f(x))\ f'(x)$,
3. $\dfrac{d}{dx}\cosh(x) = \sinh(x)$,
4. $\dfrac{d}{dx}\cosh(f(x)) = \sinh(f(x))\ f'(x)$,
5. $\dfrac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$,
6. $\dfrac{d}{dx}\tanh(f(x)) = \operatorname{sech}^2(f(x))\ f'(x)$,
7. $\dfrac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$,
8. $\dfrac{d}{dx}\coth(f(x)) = -\operatorname{csch}^2(f(x))\ f'(x)$,
9. $\dfrac{d}{dx}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x)$,
10. $\dfrac{d}{dx}\operatorname{sech}(f(x)) = -\operatorname{sech}(f(x))\tanh(f(x))\ f'(x)$,
11. $\dfrac{d}{dx}\operatorname{csch}(x) = -\operatorname{csch}(x)\coth(x)$,
12. $\dfrac{d}{dx}\operatorname{csch}(f(x)) = -\operatorname{csch}(f(x))\coth(f(x))\ f'(x)$.

Theorem 7.11: [Integration of Some Hyperbolic Functions]

1. $\int \sinh(x)dx = \cosh(x) + c,$

2. $\int \sinh\left(f(x)\right) f'(x)dx = \cosh\left(f(x)\right) + c,$

3. $\int \cosh(x)dx = \sinh(x) + c,$

4. $\int \cosh\left(f(x)\right) f'(x)dx = \sinh\left(f(x)\right) + c,$

5. $\int \operatorname{sech}^2(x)dx = \tanh(x) + c,$

6. $\int \operatorname{sech}^2\left(f(x)\right) f'(x)dx = \tanh\left(f(x)\right) + c,$

7. $\int \operatorname{csch}^2(x)dx = -\coth(x) + c,$

8. $\int \operatorname{csch}^2\left(f(x)\right) f'(x)dx = -\coth\left(f(x)\right) + c,$

9. $\int \operatorname{sech}(x)\tanh(x)dx = -\operatorname{sech}(x) + c,$

10. $\int \operatorname{sech}\left(f(x)\right)\tanh\left(f(x)\right) f'(x)dx = -\operatorname{sech}\left(f(x)\right) + c$

11. $\int \operatorname{csch}(x)\coth(x)dx = -\operatorname{csch}(x) + c,$

12. $\int \operatorname{csch}\left(f(x)\right)\coth\left(f(x)\right) f'(x)dx = -\operatorname{csch}\left(f(x)\right) + c,$

13. $\int \tanh(x)dx = \ln\left|\cosh(x)\right| + c,$

14. $\int \tanh(f(x))\, f'(x)dx = \ln|\cosh(f(x))| + c,$

15. $\int \coth(x)dx = \ln|\sinh(x)| + c,$

16. $\int \coth(f(x))\, f'(x)dx = \ln|\sinh(f(x))| + c.$

Theorem 7.12: [Derivatives of Inverse Hyperbolic Functions]

1. $\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}},$

   $\frac{d}{dx}\sinh^{-1}(f(x)) = \frac{f'(x)}{\sqrt{1+(f(x))^2}},$

2. $\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$, where $x > 1$,

   $\frac{d}{dx}\cosh^{-1}(f(x)) = \frac{f'(x)}{\sqrt{(f(x))^2-1}}$, where $f > 1$,

3. $\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$, where $|x| < 1$,

   $\frac{d}{dx}\tanh^{-1}(f(x)) = \frac{f'(x)}{1-(f(x))^2}$, where $|f| < 1$,

4. $\frac{d}{dx}\coth^{-1}(x) = \frac{1}{1-x^2}$, where $|x| > 1$,

   $\frac{d}{dx}\coth^{-1}(f(x)) = \frac{f'(x)}{1-(f(x))^2}$, where $|f| > 1$,

5. $\frac{d}{dx}\text{sech}^{-1}(x) = \frac{-1}{x\sqrt{1-x^2}}$, where $0 < x < 1$.

$$\frac{d}{dx}\text{sech}^{-1}(f(x)) = \frac{-f'(x)}{f(x)\sqrt{1-(f(x))^2}}, \text{ where } 0 < f < 1,$$

6. $$\frac{d}{dx}\text{csch}^{-1}(x) = \frac{-1}{|x|\sqrt{1+x^2}}, \text{ where } x \neq 0,$$

$$\frac{d}{dx}\text{csch}^{-1}(f(x)) = \frac{-f'(x)}{|f(x)|\sqrt{1+(f(x))^2}}, \text{ where } f \neq 0.$$

Theorem 7.13

1. $$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + c,$$

2. $$\int \frac{f'(x)}{\sqrt{a^2+[f(x)]^2}}dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c,$$

3. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + c \ , \ (x > a),$$

4. $$\int \frac{f'(x)}{\sqrt{[f(x)]^2-a^2}}dx = \cosh^{-1}\left(\frac{f(x)}{a}\right) + c \ , \ (f > a),$$

5. $$\int \frac{dx}{a^2-x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c & \text{if } |x| < a \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + c & \text{if } |x| > a \end{cases},$$

6. $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + c, \ \forall x \in \mathbb{R} \setminus \{-a, a\},$$

7. $$\int \frac{f'(x)}{a^2-[f(x)]^2}dx = \frac{1}{a}\tanh^{-1}\left(\frac{f(x)}{a}\right) + c \ , \ (|f| < a)),$$

8. $$\int \frac{f'(x)}{a^2-[f(x)]^2}dx = \frac{1}{2a}\ln\left|\frac{f(x)+a}{f(x)-a}\right| + c \ , \ (|f| \neq a)),$$

9. $$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ \text{sech}^{-1}\left(\frac{x}{a}\right) + c \ , \ (0 < x < a),$$

10. $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{a^2-[f(x)]^2}}dx = -\frac{1}{a}\,\text{sech}^{-1}\left(\frac{f(x)}{a}\right)+c \quad , \quad (0<f<a)),$

11. $\displaystyle\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\,\text{csch}^{-1}\left(\frac{x}{a}\right)+c \quad , \quad (x\neq 0),$

12. $\displaystyle\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2+a^2}}dx = -\frac{1}{a}\,\text{csch}^{-1}\left(\frac{f(x)}{a}\right)+c \quad , \quad (|f|\neq 0).$

Examples 2 :

1. $\displaystyle\frac{d}{dx}\tanh^{-1}(3x) = \frac{3}{1-(3x)^2} = \frac{3}{1-9x^2},$

2. $\displaystyle\frac{d}{dx}\sinh^{-1}(\sqrt{x}) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1+(\sqrt{x})^2}} = \frac{1}{2\sqrt{x}\sqrt{1+x}},$

3. $\displaystyle\frac{d}{dx}\text{sech}^{-1}(\cos(2x)) = \frac{-(-2\sin(2x))}{\cos(2x)\sqrt{1-(\cos(2x))^2}} = \frac{2\sin(2x)}{\cos(2x)|\sin(2x)|},$

4. $\displaystyle\frac{d}{dx}\cosh^{-1}(\sqrt{x}) = \frac{1}{2\sqrt{x}}\frac{1}{\sqrt{x-1}},$

5. $\displaystyle\frac{d}{dx}\tanh^{-1}(\sin(3x)) = \frac{3\cos(3x)}{(\cos(3x))^2} = \frac{3}{\cos(3x)},$

6.
$$\begin{aligned}\frac{d}{dx}\ln(\cosh^{-1}(4x)) &= \frac{(\cosh^{-1}(4x))'}{\cosh^{-1}(4x)} = \frac{\frac{4}{\sqrt{(4x)^2-1}}}{\cosh^{-1}(4x)} \\ &= \frac{4}{\sqrt{(4x)^2-1}\cosh^{-1}(4x)},\end{aligned}$$

7. $\displaystyle\frac{d}{dx}\ln[\cosh(3x)] = \frac{3\sinh(3x)}{\cosh(3x)} = 3\tanh(3x),$

8. $\displaystyle\frac{d}{dx}\ln|1+\sinh(x)| = \frac{\cosh(x)}{1+\sinh(x)},$

9. $\frac{d}{dx}e^{\sinh(x)} = e^{\sinh(x)}\cosh(x)$,

10.

$$\begin{aligned}\int \frac{dx}{\sqrt{81+16x^2}} &\overset{u=4x}{=} \frac{1}{4}\int \frac{du}{\sqrt{9^2+u^2}} = \frac{1}{4}\sinh^{-1}(\frac{u}{9}) + c\\ &= \frac{1}{4}\sinh^{-1}(\frac{4x}{9}) + c,\end{aligned}$$

11.

$$\begin{aligned}\int \frac{dx}{\sqrt{5-e^{2x}}} &\overset{u=e^x}{=} \int \frac{du}{u\sqrt{\sqrt{5}^2-u^2}} = -\frac{1}{\sqrt{5}}\operatorname{sech}^{-1}(\frac{|u|}{\sqrt{5}}) + c\\ &= -\frac{1}{\sqrt{5}}\operatorname{sech}^{-1}(\frac{e^x}{\sqrt{5}}) + c,\end{aligned}$$

12. $\int x^2\cosh(x^3)dx \overset{u=x^3}{=} \frac{1}{3}\int \cosh(u)du = \frac{1}{3}\sinh(x^3) + c$,

13. $\int \frac{\operatorname{csch}\left(\frac{1}{x}\right)\coth\left(\frac{1}{x}\right)}{x^2}dx \overset{u=\frac{1}{x}}{=} -\int \operatorname{csch}(u)\coth(u)du = \operatorname{csch}(\frac{1}{x}) + c$,

14. $\int \left(e^x - e^{-x}\right)\operatorname{sech}^2\left(e^x+e^{-x}\right)dx \overset{u=e^x+e^{-x}}{=} \int \operatorname{sech}^2(u)du = \tanh\left(e^x+e^{-x}\right) + c$,

15. $\int \frac{\sinh(x)}{1+\sinh^2(x)}dx = \int \frac{\sinh(x)}{\cosh^2(x)}dx = -\operatorname{sech}(x) + c$,

16. $\int \frac{\sinh(x)}{1+\cosh(x)}dx = \ln(1+\cosh(x)) + c$,

17. $\int \frac{\sinh(x)}{1+\cosh^2(x)}dx = \int \frac{\sinh(x)}{(1)^2+(\cosh(x))^2}dx = \tan^{-1}(\cosh(x)) + c$,

7.9 Exercises

2-7-1 Find the derivative of the following functions:

1) $f(x) = 4\operatorname{csch}^2(2x-1)$,

2) $f(x) = \sinh(2x)\text{csch}(3x)$,
3) $f(x) = \log_2(\sec(x) + \tan(x))$,
4) $f(x) = (3\sinh(x) + \cos(x) + 5)^{(x^3+1)}$,
5) $\text{sech}\,(1 + \sqrt{x})$,
6) $\tan^{-1}(\sinh(x))$,
7) $\ln|\sinh(1 - x^2)|$,
8) $x^{\cosh(x)}$.

2-7-2 Compute the following integrals:

1) $\displaystyle\int \frac{1}{\text{sech}(x)\sqrt{4 - \sinh^2(x)}}dx$,
2) $\displaystyle\int \frac{e^x}{1 - e^{2x}}dx$,
3) $\displaystyle\int \frac{e^x}{\sqrt{4e^{2x} + 9}}dx$,
4) $\displaystyle\int \frac{dx}{\sqrt{x}\sqrt{4 + x}}$,
5) $\displaystyle\int \frac{dx}{\sqrt{16 - e^{2x}}}$.
6) $\displaystyle\int \frac{dx}{\sqrt{1 + e^{2x}}}$,
7) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 2x - 8}}$,
8) $\displaystyle\int \frac{dx}{(x - 1)\sqrt{-x^2 + 2x + 3}}$,

8 Indeterminate Forms and L'Hôpital Rule

8.1 Indeterminate Forms

The indeterminate forms arise from the fact that $(\bar{\mathbb{R}}, +, .)$ is not a field, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The only operations that are wrong are $0.\infty$

and $+\infty + (-\infty)$. These operations are obtained for example within the real sequences or the limits of functions. For example if a sequence $(u_n)_n$ converges to 0 and the sequences $(v_n)_n$ tends to ∞, we can not decide if the limit of the sequence $(u_n.v_n)_n$ exists.
The only indeterminate forms are $0.\infty$ and $+\infty + -\infty$. The other indeterminate forms can be transformed to these two forms. For examples we have

$$\frac{0}{0} = 0.\infty, \quad \frac{\infty}{\infty} = 0.\infty, \quad 1^{\infty} = e^{\infty \ln(1)} = e^{0.\infty}, \quad 0^0 = e^{0 \ln(0)} = e^{0.\infty}.$$

Example 1 :

$$\begin{aligned}
``\frac{0}{0}" &= \lim_{x\to 2} \frac{(x-2)^2}{(x-2)} &= \lim_{x\to 2}(x-2) &= 0 \\
``\frac{0}{0}" &= \lim_{x\to 2} \frac{3(x-2)}{(x-2)} &= \lim_{x\to 2} 3 &= 3 \\
``\frac{0}{0}" &= \lim_{x\to 2} \frac{(x-2)^2}{(x-2)^4} &= \lim_{x\to 2} \frac{1}{(x-2)^2} &= \infty.
\end{aligned}$$

In each of above cases the functions are undefined at $x = 2$. And both numerator and denominator in each example approach to 0 as $x \to 0$.

Example 2 :
$\lim\limits_{x\to 0} \dfrac{\ln(\cos(x))}{\sin(x)}$, $\lim\limits_{x\to\infty} e^{3x} \ln(1 + \dfrac{1}{x})$, $\lim\limits_{x\to\infty} (1+x)^2 - \sqrt{x^4 + x + 2}$, $\lim\limits_{x\to\infty} (1 + \dfrac{1}{x})^x$ are all indeterminate forms.

8.2 The Hôpital's Rule

Theorem 8.1

Let f and g be two continuous functions on the interval $[a, b]$ and differentiable on $]a, b[$. We assume that $g'(x) \neq 0$ for all $x \in]a, b[$.

Then there exists $c \in]a,b[$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof .
We remark that $g(b)-g(a) \neq 0$, otherwise there is $a < d < b$ such that $g'(d) = 0$. We consider the function

$$h(x) = f(x) - f(a) - (f(b)-f(a))\frac{g(x)-g(a)}{g(b)-g(a)}.$$

h fulfills the conditions of Rolle's lemma, then there exists $c \in]a,b[$ such that $0 = h'(c) = f'(c) - g'(c)\dfrac{f(b)-f(a)}{g(b)-g(a)}$, which proves the Theorem. □

Theorem 8.2: [The Hôpital's Rule]

Let f, g be two differentiable functions on $]a,b[\setminus\{c\}$. Assume that $g'(x) \neq 0$ for all $x \in]a,b[\setminus\{c\}$ and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$.
If $\lim_{x\to c} \dfrac{f'(x)}{g'(x)} = \ell \in \mathbb{R} \cup \{-\infty,+\infty\}$, then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \ell$.

Proof .
Define $f(c) = 0$ and $g(c) = 0$. Let $x \in]c,b[$. Then f and g are continuous on $[c,x]$, differentiable on $]c,x[$ and $g'(y) \neq 0$ on $]c,x[$. By Theorem (8.2) there is $y \in]c,x[$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f'(y)}{g'(y)}.$$

Then $\lim_{x\to c^+} \dfrac{f(x)}{g(x)} = \lim_{y\to c^+} \dfrac{f'(y)}{g'(y)} = \ell$. Similarly, we prove that $\lim_{x\to c^-} \dfrac{f(x)}{g(x)} = \ell$. Therefore, $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \lim_{x\to c} \dfrac{f'(x)}{g'(x)} = \ell$.

□

Remark 9 :

1. Theorem (8.2) is valid for one-sided limits as well as the two sided limit. This theorem is also true if $c = +\infty$ or $c = -\infty$.

2. Theorem (8.2) is valid for the case, $\lim_{x \to c} f(x) = \infty$ or $-\infty$ and $\lim_{x \to c} g(x) = \infty$ or $-\infty$.

Examples 3 :

1. $\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \cos(x) = 1,$

2. $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{\sin(x)}{2x} = \lim_{x \to 0} \frac{\cos(x)}{2} = \frac{1}{2},$

3. $\lim_{x \to 0} \frac{\sin(3x)}{\sin(5x)} = \lim_{x \to 0} \frac{3\cos(3x)}{5\cos(5x)} = \frac{3}{5},$

4. $\lim_{x \to 0} \frac{\tan(2x)}{\tan(3x)} = \lim_{x \to 0} \frac{2\sec^2(2x)}{3\sec^2(3x)} = \frac{2}{3},$

5. $\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0} -x = 0,$

6. $\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin(x)}{\cos(x)} = \lim_{x \to \frac{\pi}{2}} \frac{\cos(x)}{\sin(x)} = 0,$

7. $\lim_{x \to 5} \frac{\sqrt{x-1} - 2}{x^2 - 25} = \lim_{x \to 5} \frac{1}{4x\sqrt{x-1}} = \frac{1}{40},$

8. $\lim_{x \to 1} \frac{x^3 - 3x + 2}{x^2 - 2x + 1} = \lim_{x \to 1} \frac{3x^2 - 3}{2x - 2} = \lim_{x \to 1} \frac{6x}{2} = 3,$

9. $\lim_{x \to 1} \frac{\sqrt{x} - 1}{\ln x} = \lim_{x \to 1} \frac{\left(\frac{1}{2\sqrt{x}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to 1} \frac{x}{2\sqrt{x}} = \frac{1}{2},$

10. $\lim_{x \to 0} \frac{\sin(x)\sqrt{1 - \sin(x)}}{x} = \lim_{x \to 0} \frac{\sin(x)}{x}\sqrt{1 - \sin(x)} = 1\sqrt{1 - 0} = 1,$

11. $\lim_{x\to 0} \frac{\int_0^x \sqrt{1+\sin(t)}\; dt}{x} = \lim_{x\to 0} \frac{\sqrt{1+\sin(x)}}{1} = \frac{\sqrt{1+0}}{1} = 1,$

12. $\lim_{x\to 1} \frac{\tan^{-1} x - \frac{\pi}{4}}{x-1} = \lim_{x\to 1} \frac{\left(\frac{1}{1+x^2}\right)}{1} = \lim_{x\to 1} \frac{1}{1+x^2} = \frac{1}{1+1} = \frac{1}{2},$

13. $\lim_{x\to\infty} \frac{x}{\ln(x)} = \lim_{x\to\infty} x = +\infty,$

14. $\lim_{x\to\infty} (1+\frac{1}{x})^{5x} = \lim_{x\to\infty} e^{5\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}} = e^5,$

15. $\lim_{x\to\infty} x^x = \lim_{x\to\infty} e^{x\ln(x)} = +\infty,$

16. $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$

17. $\lim_{x\to\infty} \frac{x+e^x}{1+e^{3x}} = \lim_{x\to\infty} \frac{1+e^x}{3e^{3x}} = \lim_{x\to\infty} \frac{e^x}{9e^{3x}} = \lim_{x\to\infty} \frac{1}{9e^{2x}} = 0,$

18. $\lim_{x\to\infty} (x^2-1)e^{-x^2} = \lim_{x\to\infty} \frac{x^2-1}{e^{x^2}} = \lim_{x\to\infty} \frac{2x}{2x\; e^{x^2}} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0,$

19. $\lim_{x\to\infty} \left(1+e^{2x}\right)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln\left(1+e^{2x}\right)}{x}} = \lim_{x\to\infty} e^{\frac{2e^{2x}}{1+e^{2x}}} = \lim_{x\to\infty} e^{\frac{4e^{2x}}{2e^{2x}}} = e^2,$

20. $\lim_{x\to\infty} \left(1+\frac{\ln 3}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{\ln 3}{x}\right)} = \lim_{x\to\infty} e^{\frac{\ln\left(1+\frac{\ln 3}{x}\right)}{x^{-1}}} = \lim_{x\to\infty} e^{\frac{\ln 3}{\left(1+\frac{\ln 3}{x}\right)}} = 3,$

Note that: $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$, for all $a \in \mathbb{R}$.

8.3 Exercises

2-8-1 Use L'Hospital's rule when appropriate. When not appropriate, say so.

1) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)},$

2) $\lim_{x\to+\infty} \sqrt{x}e^{-\frac{x}{2}},$

3) $\lim_{x\to+\infty} x\sin(\frac{1}{x}),$

4) $\lim_{x\to 0}(\cot(x) - \frac{1}{x})$,

5) $\lim_{x\to 0}(\csc(x) - \cot(x))$,

6) $\lim_{x\to 0^+} (\tan(2x))^x$,

7) $\lim_{x\to 0} \frac{\tan(x) - x}{x^3}$,

8) $\lim_{x\to (\frac{\pi}{2})^-} \frac{2 - \sec(x)}{3\tan(x)}$,

9) $\lim_{x\to 0^+} x^x$,

10) $\lim_{x\to 0} \frac{2\sinh(x) - \sinh(2x)}{2x(\cos(x) - 1)}$,

11) $\lim_{x\to 0} \frac{\sin(3x) - 3\sin(x)}{3x^3}$,

12) $\lim_{x\to 0^+} (2x + 1)^{\cot(x)}$,

13) $\lim_{x\to 0} \frac{x - \tan(x)}{1 - \cos(x)}$,

14) $\lim_{x\to 0^+} \big(\sec(x)+\tan(x)\big)^{\csc(x)}$,

15) $\lim_{x\to 1^+} (\frac{3}{\ln(x)} - \frac{2}{x - 1})$,

16) $\lim_{x\to +\infty} (e^x + 1)^{\frac{1}{x}}$,

17) $\lim_{x\to \infty} \frac{4e^x}{x^2}$,

18) $\lim_{x\to \infty} \frac{e^{2x} - 1}{x}$,

19) $\lim_{x\to \infty} e^{-x}\sqrt{x}$,

20) $\lim_{x\to \infty} (1 + 4x)^{\frac{1}{x^2}}$.

2-8-2 Use L'Hospital's rule to find the sum $\sum_{k=1}^{n} k$.

CHAPTER 3

TECHNIQUES OF INTEGRATION

1 Integration By Parts

The product rule of differentiation yields an integration technique known as integration by parts. Let us begin with the product rule:

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{dv(x)}{dx} + \frac{du(x)}{dx}v(x).$$

Integrating each term with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b \frac{d}{dx}(u(x)v(x))dx = \int_a^b u(x)\left(\frac{dv(x)}{dx}\right)dx + \int_a^b v(x)\left(\frac{du(x)}{dx}\right)dx.$$

Using the differential notation and the fundamental theorem of calculus, we get

$$[u(x)v(x)]_a^b = (u(b)v(b) - u(a)v(a)) = \int_a^b u(x)v'(x)dx + \int_a^b v(x)u'(x)dx.$$

The standard form of this integration by parts formula is written as

$$\int_a^b u(x)v'(x)dx = (u(b)v(b) - u(a)v(a)) - \int_a^b v(x)u'(x)dx.$$

and

$$\int udv = uv - \int vdu.$$

We state this result in the following Theorem

Theorem 1.1: (Integration by Parts)

If u and v are two continuously differentiable functions on the interval $[a, b]$, then we have

$$\int_a^b u(x)v'(x)dx = (u(b)v(b) - u(a)v(a)) - \int_a^b v(x)u'(x)dx$$

and for the indefinite integrals

$$\int udv = uv - \int vdu.$$

Examples 4 :
Using integration by parts, we have

1. $\int \ln(x)dx \overset{u=\ln(x),v'=1}{=} x\ln(x) - x + c.$

2. $\int \ln^2(x)dx \overset{u=\ln^2(x),v'=1}{=} x\ln^2(x) - 2(x\ln(x) - x) + c.$

3. $\int xe^x dx \overset{u=x,v'=e^x}{=} xe^x - e^x + c.$

4. $\int x^2e^x dx = (x^2 - 2x + 2)e^x + c.$

5. $\int_0^\pi x\sin(x)\ dx \overset{u=x,v'=\sin(x)}{=} [-x\cos(x)]_0^\pi + \int_0^\pi \cos(x)\ dx = [-x\cos(x)]_0^\pi = \pi.$

6. $\int x\cos(x)dx \overset{u=x,v'=\cos(x)}{=} x\sin(x) + \cos(x) + c.$

7.

$$\begin{aligned}\int x^2\sin(x)dx &\overset{u=x^2 v'=\sin(x)}{=} -x^2\cos(x)+\int 2x\cos(x)dx\\ &= -x^2\cos(x)+2x\sin(x)+2\cos(x)+c.\end{aligned}$$

8. $\int x^2\cos(x)dx=(x^2-2)\sin(x)+2x\cos(x)+c.$

9.

$$\begin{aligned}\int e^x\sin(x)dx &\overset{u=e^x,v'=\sin(x)}{=} -e^x\cos(x)+\int e^x\cos(x)dx\\ &\overset{u=e^x,v'=\cos(x)}{=} -e^x\cos(x)+e^x\sin(x)-\int e^x\sin(x)dx\\ &= \frac{e^x}{2}(\sin(x)-\cos(x))+c,\end{aligned}$$

10.

$$\begin{aligned}\int e^x\cos(x)dx &\overset{u=\cos(x),v'=e^x}{=} e^x\cos(x)+\int e^x\sin(x)dx\\ &\overset{u=\sin(x),v'=e^x}{=} e^x\cos(x)+e^x\sin(x)-\int e^x\cos(x)dx\\ &= \frac{e^x}{2}(\cos(x)+\sin(x))+c,\end{aligned}$$

11.

$$\begin{aligned}\int \cosh(x)\cos(x)dx &\overset{u=\cosh(x),v'=\cos(x)}{=} \cosh(x)\sin(x)-\int \sinh(x)\sin(x)dx\\ &\overset{u=\sinh(x),v'=\sin(x)}{=} \cosh(x)\sin(x)+\sinh(x)\cos(x)\\ &\quad -\int \cosh(x)\cos(x)dx.\end{aligned}$$

Then $\int \cosh(x)\cos(x)dx=\frac{1}{2}(\sin(x)\cosh(x)+\cos(x)\sinh(x))+c.$

12. $\int x\sec^2(x)dx \overset{u=x,v'=\sec^2(x)}{=} x\tan(x)-\int \tan(x)dx = x\tan(x)-\ln|\sec(x)|+c.$

13.

$$\begin{aligned}\int \sinh^{-1}(x)dx &\overset{u=\sinh^{-1}(x),v'=1}{=} x\sinh^{-1}(x) - \int \frac{x}{\sqrt{1+x^2}}dx \\ &= x\sinh^{-1}(x) - \sqrt{1+x^2} + c.\end{aligned}$$

14.

$$\begin{aligned}\int \cosh^{-1}(x)dx &\overset{u=\cosh^{-1}(x),v'=1}{=} x\cosh^{-1}(x) - \int \frac{x}{\sqrt{x^2-1}}dx \\ &= x\cosh^{-1}(x) - \sqrt{x^2-1} + c.\end{aligned}$$

15.

$$\begin{aligned}\int \tanh^{-1}(x)dx &\overset{u=\tanh^{-1}(x),v'=1}{=} x\tanh^{-1}(x) - \int \frac{x}{1-x^2}dx \\ &= x\tanh^{-1}(x) + \frac{1}{2}\ln(1-x^2) + c.\end{aligned}$$

16.

$$\begin{aligned}\int \sin^{-1}(x)dx &\overset{u=\sin^{-1}(x),v'=1}{=} x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx \\ &= x\sin^{-1}(x) + \sqrt{1-x^2} + c.\end{aligned}$$

Or

$$\begin{aligned}\int \sin^{-1}(x)dx &\overset{u=\sin^{-1}(x)}{=} \int u\cos(u)du = u\sin(u) + \cos(u) + c \\ &= x\sin^{-1}(x) + \sqrt{1-x^2} + c.\end{aligned}$$

17.

$$\begin{aligned}\int \tan^{-1}(x)dx &\overset{u=\tan^{-1}(x),v'=1}{=} x\tan^{-1}(x) - \int \frac{x}{1+x^2}dx \\ &= x\tanh^{-1}(x) - \frac{1}{2}\ln(1+x^2) + c.\end{aligned}$$

Or

18.

$$\int \tan^{-1}(x)dx \overset{u=\tan^{-1}(x)}{=} \int u\sec^2(u)du = u\tan(u) + \ln|\cos(u)| + c$$
$$= x\tanh^{-1}(x) - \frac{1}{2}\ln(1+x^2) + c.$$

19.

$$\int \ln(x^2+1)dx \overset{u=\ln(x^2+1),v'=1}{=} x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}dx$$
$$= x\ln(x^2+1) - \int \frac{(2x^2+2)-2}{x^2+1}dx$$
$$= x\ln(x^2+1) - 2x + 2\tan^{-1}(x) + c.$$

1.1 Exercises

2-9-1 Evaluate the following integrals:

1) $\int \ln^3(x)dx$,

2) $\int \ln(x^2-1)dx$,

3) $\int \ln\left(x^2+x+1\right)dx$,

4) $\int x^3e^x dx$,

5) $\int x^3\cos(x)dx$,

6) $\int x^4\cos(x)dx$,

7) $\int x^3\sin(x)dx$,

8) $\int x^4\sin(x)dx$,

9) $\int \cos(x)\cosh(x)dx$,

10) $\int e^{ax}\cos(bx)dx$,

11) $\int e^{ax}\sin(bx)dx$,

12) $\int x\tan^{-1}(x)dx$,

13) $\int x\sinh^{-1}(x)dx$,

2 Integrals Involving Trigonometric and Hyperbolic Functions

Some Important Trigonometric Formulas

a	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	radians
	0	30^o	45^o	60^o	90^o	degree
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	

table 3.1: Exact values of the sine, cosine and tangent functions for important angles.

1. $\cos^2(x) + \sin^2(x) = 1,$
2. $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$
3. $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y),$
4. $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y),$
5. $\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y),$
6. $\sin(x)\cos(y) = \dfrac{1}{2}(\sin(x+y) + \sin(x-y)),$
7. $\sin(x)\sin(y) = \dfrac{1}{2}(\cos(x-y) - \cos(x+y)),$
8. $\cos(x)\cos(y) = \dfrac{1}{2}(\cos(x+y) + \cos(x-y)),$
9. $\cos(x) + \cos(y) = 2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2}),$
10. $\cos(x) - \cos(y) = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2}),$
11. $\sin(x) + \sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2}),$

12. $\sin(x) - \sin(y) = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$,
13. $\cos(2x) = \cos^2(x) - \sin^2(x)$,
14. $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$,
15. $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$,
16. $\sec^2(x) = 1 + \tan^2(x)$,
17. $\csc^2(x) = 1 + \cot^2(x)$,
18. $\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$,
19. $\tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$,

Some Important Hyperbolic Formulas

1. $\cosh^2(x) - \sinh^2(x) = 1$,
2. $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$,
3. $\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$,
4. $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$,
5. $\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$,
6. $\sinh(x)\cosh(y) = \dfrac{1}{2}(\sinh(x+y) + \sinh(x-y))$,
7. $\sinh(x)\sinh(y) = \dfrac{1}{2}(\cosh(x+y) + \cosh(x-y))$,
8. $\cosh(x)\cosh(y) = \dfrac{1}{2}(\cosh(x+y) - \cosh(x-y))$,

Integrals of Type $\int \cos^n(x)\sin^m(x)dx,\ m, n \in \mathbb{N}$

1. If $m = 2q + 1$, we set $u = \cos(x)$, then $du = -\sin(x)dx$ and

$$\begin{aligned}\int \cos^n(x)\sin^{2q+1}(x)dx &= \int \cos^n(x)\sin^{2q}(x).\sin(x)dx\\ &= \int \cos^n(x)\Big(\sin^2(x)\Big)^q.\sin(x)dx\\ &= -\int \cos^n(x)\Big(1-\cos^2(x)\Big)^q.(-\sin(x))dx\\ &= -\int u^n\Big(1-u^2\Big)^q du.\end{aligned}$$

2. If $n = 2p + 1$, we set $u = \sin(x)$, then $du = \cos(x)dx$ and

$$\begin{aligned}\int \cos^{2p+1}(x)(x)\sin^m(x)dx &= \int \cos^{2p}(x)\sin^m(x).\cos(x)dx\\ &= \int \Big(\cos^2(x)\Big)^p\sin^m(x).\cos(x)dx\\ &= \int \Big(1-\sin^2(x)\Big)^p\sin^m(x).\cos(x))dx\\ &= \int \Big(1-u^2\Big)^p u^m du.\end{aligned}$$

3. If $n = 2p$ and $m = 2q$,

$$\int \cos^{2p}(x)(x)\sin^{2q}(x)dx = \int \cos^{2p}(x)(1-\cos^2(x))^q dx.$$

We compute the integral $J_n = \displaystyle\int \cos^{2n}(x)dx$ by induction and by parts: We set $u = \cos^{2n-1}(x)$ and $v' = \cos(x)$, then

$$\begin{aligned}J_n &= \sin(x)\cos^{2n-1}(x) + (2n-1)\int \cos^{2n-2}(x)\sin^2(x)dx\\ &= \sin(x)\cos^{2n-1}(x) + (2n-1)J_{n-1} - (2n-1)J_n.\end{aligned}$$

Thus $J_n = \frac{1}{2n}\sin(x)\cos^{2n-1}(x) + \frac{2n-1}{2n}J_{n-1}$.

Examples 5 :

1.
$$\begin{aligned}\int \sin^3(x)dx &\overset{u=\cos(x)}{=} -\int (1-u^2)du \\ &= -\cos(x) + \frac{1}{3}\cos^3(x) + c,\end{aligned}$$

2. $\int \cos^3(x)dx \overset{u=\sin(x)}{=} \int (1-u^2)du = \sin(x) - \frac{1}{3}\sin^3(x) + c.,$

3. $\int \sin^5(x)\cos^4(x)dx \overset{u=\cos(x)}{=} -\int u^4(1-u^2)^2du = -\frac{\cos^5(x)}{5} - \frac{\cos^9(x)}{9} + \frac{2\cos^7(x)}{7} + c,$

4. $\int \sin^4(x)\cos^3(x)dx \overset{u=\sin(x)}{=} \int u^4(1-u^2)du = \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + c.$

5. $\int \sqrt{\sin(x)}\ \cos^3(x)dx \overset{u=\sin(x)}{=} \int u^{\frac{1}{2}}(1-u^2)\ du = \frac{2}{3}\sin^{\frac{3}{2}}(x) - \frac{2}{7}\sin^{\frac{7}{2}}(x) + c,$

6. $\int \frac{\sin^3(x)}{\cos^2(x)}dx \overset{u=\cos(x)}{=} -\int (1-u^2)\ u^{-2}\ du = \sec(x) + \cos(x) + c,$

7. $\int \sin^7(x)\ \cos^3(x)dx \overset{u=\sin(x)}{=} \int u^7\ (1-u^2)\ du = \frac{\sin^8(x)}{8} - \frac{\sin^{10}(x)}{10} + c,$

8. $\int \cos^2(x)dx = \frac{\sin(x)\cos(x)}{2} + \frac{x}{2} + c = \frac{\sin(2x)}{4} + \frac{x}{2} + c,$

$$\int \cos^4(x)dx = \frac{\sin(x)\cos^3(x)}{4} + \frac{3\sin(x)\cos(x)}{8} + \frac{3x}{8} + c,$$

9. $\int \sin^4(x)dx = \int (1-\cos^2(x))^2dx = -\frac{5\sin(2x)}{16} + \frac{3}{4}x + \frac{\sin(x)\cos^3(x)}{4} + c,$

10.
$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \int (1-\cos^2(x))\cos^2(x)dx \\ &= \frac{\sin(x)\cos(x)}{8} + \frac{x}{8} - \frac{\sin(x)\cos^3(x)}{4} + c.\end{aligned}$$

Another solution

$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \frac{1}{4}\int \sin^2(2x)dx = \frac{1}{8}\int (1-\cos(4x))dx \\ &= \frac{x}{8} - \frac{\sin(4x)}{32} + c.\end{aligned}$$

Integrals of Type $\int \sec^m(x)\tan^n(x)dx,\ m, n \in \mathbb{N}$

1. If $m = 2q$ and $q \neq 0$, we set $u = \tan(x)$, then $du = \sec^2(x)dx$ and

$$\int \sec^{2q}(x)\tan^n(x)dx = \int u^n(1+u^2)^{q-1}du.$$

2. If $m = 0$,

$$\int \tan(x)dx = \ln|\sec(x)| + c.$$

$$\int \tan^2(x)dx = \int (\sec^2(x) - 1)dx = \tan(x) - x + c.$$

For $n \geq 3$,

$$\begin{aligned}L_n = \int \tan^n(x)dx &= \int \tan^{n-2}(x)\tan^2(x)dx \\ &= \int \tan^{n-2}(x)\sec^2(x)dx - L_{n-2} \\ &= \frac{\tan^{n-1}(x)}{n-1} - L_{n-2}.\end{aligned}$$

3. If $m \neq 0$, $n = 2p+1$, we set $u = \sec(x)$, then $du = \sec(x)\tan(x)dx$.

$$\int \sec^m(x)\tan^{2p+1}(x)dx = \int u^{m-1}(u^2-1)^p du.$$

4. If $m = 2q+1$ and $n = 2p$, the result is obtained using integration by parts and induction.

Examples 6 :

1. $\displaystyle\int \sec^4(x)dx \overset{u=\tan(x)}{=} \int (1+u^2)du = \tan(x) + \frac{\tan^3(x)}{3} + c;$

2. $\displaystyle\int \tan^4(x)\sec^2(x)dx \overset{u=\tan(x)}{=} \int u^4 du = \frac{\tan^5(x)}{5} + c;$

3. $\displaystyle\int \sec^4(x)\tan^7(x)dx \overset{u=\tan(x)}{=} \int u^7(1+u^2)du = \frac{\tan^8(x)}{8} + \frac{\tan^{10}(x)}{10} + c;$

4. $\displaystyle\int \frac{\sec^4(x)}{\sqrt{\tan(x)}}dx \overset{u=\tan(x)}{=} \int (1+u^2)u^{-\frac{1}{2}}\ du = 2\tan^{\frac{1}{2}}(x) + \frac{2}{5}\tan^{\frac{5}{2}}(x) + c;$

5. $\displaystyle\int \tan^3(x)dx = \int \tan(x)(\sec^2(x)-1)dx = \frac{\tan^2(x)}{2} - \ln|\sec(x)| + c;$

6. $$\begin{aligned}\int \tan^5(x)dx &= \int \tan^3(x)(\sec^2(x)-1)dx\\ &= \frac{\tan^4(x)}{4} - \int \tan^3(x)dx\\ &= \frac{\tan^4(x)}{4} - \frac{\tan^2(x)}{2} + \ln|\sec(x)| + c;\end{aligned}$$

7. $\displaystyle\int \tan^3(x)\ \sec^3(x)dx \overset{u=\sec(x)}{=} \int (u^2-1)u^2\ du = \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + c;$

8. $\displaystyle\int \sec^5(x)\tan^3(x)dx \overset{u=\sec(x)}{=} \int u^4(u^2-1)du = \frac{\sec^7(x)}{7} - \frac{\sec^5(x)}{5} + c;$

9. By integration by parts, $u(x) = \sec(x)$, $v'(x) = \sec^2(x)$, we get

$$\begin{aligned}\int \sec^3(x)dx &= \int \sec(x)\sec^2(x)dx\\ &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)dx\\ &= \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx;\end{aligned}$$

Therefore

$$\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)| + c.$$

By integration by parts, $u(x) = \sec^3(x)$, $v'(x) = \sec^2(x)$, we get

$$\begin{aligned}\int \sec^5(x)dx &= \sec^3(x)\tan(x) - 3\int \sec^3(x)\tan^2(x)dx\\ &= \sec^3(x)\tan(x) - 3\int \sec^5(x)dx + 3\int \sec^3(x)dx.\end{aligned}$$

Then

$$\begin{aligned}\int \sec^5(x)dx &= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x)\\ &\quad + \frac{3}{8}\ln|\sec(x)+\tan(x)| + c.\end{aligned}$$

Integrals of Type $\int \csc^m(x)\cot^n(x)dx$, $m, n \in \mathbb{N}$

1. If $m = 2q$ and $q \neq 0$, we set $u = \cot(x)$, then $du = -\csc^2(x)dx$ and

$$\int \csc^{2q}(x)\cot^n(x)dx = -\int u^n(1+u^2)^{q-1}du.$$

2. If $m = 0$,
$$\int \cot(x)dx = \ln|\sin(x)| + c.$$
$$\int \cot^2(x)dx = \int (\csc^2(x) - 1)dx = -\cot(x) - x + c.$$
For $n \geq 3$,

$$\begin{aligned}L_n = \int \cot^n(x)dx &= \int \cot^{n-2}(x)\cot^2(x)dx\\ &= \int \cot^{n-2}(x)\csc^2(x)dx - L_{n-2}\\ &= -\frac{\cot^{n-1}(x)}{n-1} - L_{n-2}.\end{aligned}$$

3. If $m \neq 0$ and $n = 2p+1$, we set $u = \csc(x)$, then $du = -\csc(x)\cot(x)dx$

$$\int \csc^m(x)\cot^{2p+1}(x)dx = -\int u^{m-1}(u^2-1)^p du.$$

4. If $m = 2q+1$ and $n = 2p$. The result is obtained by integration by parts and induction.

Examples 7 :

1. $\displaystyle\int \csc(x)dx = \ln|\csc(x) - \cot(x)| + c,$

2. Let $u = \csc(x), v' = \csc^2(x)$,
$\displaystyle\int \csc^3(x)dx = -\csc(x)\cot(x) + \ln|\csc(x) - \cot(x)| - \int \csc^3(x)dx,$
then $\displaystyle\int \csc^3(x)dx = \frac{1}{2}(-\csc(x)\cot(x) + \ln|\csc(x) - \cot(x)|) + c.$

3.
$$\begin{aligned}\int \csc(x)\ \cot^2(x)dx &= \int \csc^3(x)dx - \int \csc(x)dx \\ &= \frac{1}{2}(-\csc(x)\cot(x) - \ln|\csc(x) - \cot(x)|) + c,\end{aligned}$$

4.
$$\begin{aligned}\int \csc^4(x)\ \cot^4(x)dx &\overset{u=\cot(x)}{=} -\int (1+u^2)u^4\ du = -\int (u^4+u^6)\ du \\ &= -\frac{\cot^5(x)}{5} - \frac{\cot^7(x)}{7} + c.\end{aligned}$$

Integrals of Type $\int \sin(ax)\cos(bx)dx$, $\int \sin(ax)\sin(bx)dx$, $\int \cos(ax)\cos(bx)dx$

1. $\displaystyle\int \sin(ax)\sin(bx)dx = \frac{1}{2}\int \cos((a-b)x) - \cos((a+b)x)dx,$

2. $\displaystyle\int \sin(ax)\cos(bx)dx = \frac{1}{2}\int \sin((a+b)x) + \sin((a-b)x)dx,$

3. $\displaystyle\int \cos(ax)\cos(bx)dx = \frac{1}{2}\int \cos((a+b)x) + \cos((a-b)x)dx,$

   For $a, b \in \mathbb{R}$ such that $|a| \neq |b|$, we have

4. $\displaystyle 2\int \sin(ax)\sin(bx)dx = \frac{\sin((a-b)x)}{a-b} - \frac{\sin((a+b)x)}{a+b} + c,$

5. $\displaystyle 2\int \cos(ax)\cos(bx)dx = \frac{\sin((a+b)x)}{a+b} + \frac{\sin((a-b)x)}{a-b} + c,$

6. $\displaystyle 2\int \sin(ax)\cos(bx)dx = -\frac{\cos((a+b)x)}{a+b} - \frac{\cos((a-b)x)}{a-b} + c,$

Examples 8 :

1.

$$\begin{aligned}\int \sin(5x)\sin(3x)dx &= \frac{1}{2}\int \cos(2x) - \cos(8x)dx \\ &= \frac{\sin(2x)}{4} - \frac{\sin(8x)}{16} + c.\end{aligned}$$

2.

$$\begin{aligned}\int \sin(4x)\cos(3x)dx &= \frac{1}{2}\int \sin(7x) + \sin(x)dx \\ &= -\frac{\cos(x)}{2} - \frac{\cos(7x)}{14} + c.\end{aligned}$$

3.

$$\begin{aligned}\int \cos(5x)\ \cos(2x)dx &= \int \frac{1}{2}\left[\cos(7x) + \cos(3x)\right]dx \\ &= \frac{1}{14}\sin(7x) + \frac{1}{6}\sin(3x) + c.\end{aligned}$$

Integrals of Type $\int \sinh(ax)\cosh(bx)dx$, $\int \sinh(ax)\sinh(bx)dx$, $\int \cosh(ax)\cosh(bx)dx$

$$\int \sinh(ax)\sinh(bx)dx = \frac{1}{2}\int \cosh((a+b)x) - \cosh((a-b)x)dx,$$

$$\int \sinh(ax)\cosh(bx)dx = \frac{1}{2}\int \sinh((a+b)x) + \sinh((a-b)x)dx,$$

$$\int \cosh(ax)\cosh(bx)dx = \frac{1}{2}\int \cosh((a+b)x) + \cosh((a-b)x)dx,$$

Examples 9 :

1.

$$\begin{aligned}\int \sinh(5x)\sinh(3x)dx &= \frac{1}{2}\int \cosh(8x) - \cosh(2x)dx \\ &= \frac{1}{16}\sinh(8x) - \frac{1}{4}\sinh(2x) + c,\end{aligned}$$

2.

$$\begin{aligned}\int \sinh(4x)\cosh(3x)dx &= \frac{1}{2}\int \sinh(7x) + \sinh(x)dx \\ &= \frac{1}{2}\cosh(x) + \frac{1}{14}\cosh(7x) + c,\end{aligned}$$

3.

$$\begin{aligned}\int \cosh(5x)\cosh(2x)dx &= \int \frac{1}{2}[\cosh(7x) + \cosh(3x)]dx \\ &= \frac{1}{14}\sinh(7x) + \frac{1}{6}\sinh(3x) + c.\end{aligned}$$

Integrals of Type $\int \cosh^n(x)\sinh^m(x)dx$, $m, n \in \mathbb{N}$

1. If $m = 2q+1$, we set $u = \cosh(x)$, then $du = \sinh(x)dx$ and

$$\begin{aligned} \int \cosh^n(x)\sinh^{2q+1}(x)dx &= \int \cosh^n(x)\sinh^{2q}(x)\sinh(x)dx \\ &= \int \cosh^n(x)\Big(\cosh^2(x)-1\Big)^q \sinh(x)dx \\ &= \int u^n\Big(u^2-1\Big)^q du. \end{aligned}$$

2. If $n = 2p+1$, we set $u = \sinh(x)$, then $du = \cosh(x)dx$ and

$$\begin{aligned} \int \cosh^{2p+1}(x)(x)\sinh^m(x)dx &= \int \cosh^{2p}(x)\sinh^m(x)\cosh(x)dx \\ &= \int \Big(1+\sinh^2(x)\Big)^p \sinh^m(x)\cosh(x))dx \\ &= \int \Big(1+u^2\Big)^p u^m du. \end{aligned}$$

3. If $n = 2p$ and $m = 2q$,

$$\int \cosh^{2p}(x)(x)\sinh^{2q}(x)dx = \int \cosh^{2p}(x)(\cosh^2(x)-1)^q dx.$$

We compute the integral $I_n = \displaystyle\int \cosh^{2n}(x)dx$ by induction and by parts: We set $u = \cosh^{2n-1}(x)$ and $v' = \cosh(x)$, then

$$\begin{aligned} I_n &= \sinh(x)\cosh^{2n-1}(x) - (2n-1)\int \cosh^{2n-2}(x)\sinh^2(x)dx \\ &= \sinh(x)\cosh^{2n-1}(x) - (2n-1)I_n + (2n-1)I_{n-1}. \end{aligned}$$

Thus $I_n = \frac{1}{2n}\sinh(x)\cosh^{2n-1}(x) - \frac{2n-1}{2n}I_{n-1}$. In particular

$$I_1 = \int \cosh^2(x)dx = \frac{\sinh(x)\cosh(x)}{2} - \frac{x}{2} + c = \frac{\sinh(2x)}{4} - \frac{x}{2} + c.$$

$$I_2 = \int \cosh^4(x)dx = \frac{\sinh(x)\cosh^3(x)}{4} - \frac{3\sinh(x)\cosh(x)}{8} + \frac{3x}{8} + c.$$

Examples 10 :

1.
$$\begin{aligned}\int \sinh^5(x)\cosh^4(x)dx &\overset{u=\cosh(x)}{=} \int u^4(u^2-1)^2 du \\ &= \frac{\cosh^5(x)}{5} + \frac{\cosh^9(x)}{9} - \frac{2\cosh^7(x)}{7} + c,\end{aligned}$$

2. $\int \sinh^4(x)\cosh^3(x)dx \overset{u=\sinh(x)}{=} \int u^4(u^2-1)du = \frac{\sinh^7(x)}{7} - \frac{\sinh^5(x)}{5} + c.$

3.
$$\begin{aligned}\int \sqrt{\sinh(x)}\ \cosh^3(x)dx &\overset{u=\sinh(x)}{=} \int u^{\frac{1}{2}}(u^2-1)\ du \\ &= -\frac{2\,(\sinh(x))^{\frac{3}{2}}}{3} + \frac{2\,(\sinh(x))^{\frac{7}{2}}}{7} + c,\end{aligned}$$

4. $\int \frac{\sinh^3(x)}{\cosh^2(x)}dx \overset{u=\cosh(x)}{=} \int (u^2-1)\ u^{-2}\ du = \operatorname{sech}(x) + \cosh(x) + c,$

5.
$$\begin{aligned}\int \sinh^2(x)\cosh^2(x)dx &= \int (\cosh^2(x)-1)\cosh^2(x)dx \\ &= \frac{\sinh(x)\cosh^3(x)}{4} - \frac{7}{8}\sinh(x)\cosh(x) + \frac{7}{8}x + c.\end{aligned}$$

Another solution

$$\begin{aligned}\int \sinh^2(x)\cosh^2(x)dx &= \frac{1}{4}\int \sinh^2(2x)dx = \frac{1}{8}\int (\cosh(4x)-1)dx \\ &= \frac{\sinh(4x)}{32} - \frac{x}{8} + c.\end{aligned}$$

Integrals of Type $\int \operatorname{sech}^m(x)\tanh^n(x)dx,\ m,n\in\mathbb{N}$

1. If $m = 2q$ and $q \neq 0$, we set $u = \tanh(x)$, then $du = \operatorname{sech}^2(x)dx$ and

$$\int \operatorname{sech}^{2q}(x)\tanh^n(x)dx = \int u^n(1-u^2)^{q-1}du.$$

2. If $m = 0$,

$$\int \tanh(x)dx = \ln|\cosh(x)| + c.$$

$$\int \tanh^2(x)dx = \int (1-\operatorname{sech}^2(x))dx = x - \tanh(x) + c.$$

For $n \geq 3$,

$$\begin{aligned} L_n = \int \tanh^n(x)dx &= \int \tanh^{n-2}(x)\tanh^2(x)dx \\ &= L_{n-2} - \int \tanh^{n-2}(x)\operatorname{sech}^2(x)dx \\ &= L_{n-2} - \frac{\tanh^{n-1}(x)}{n-1}. \end{aligned}$$

3. If $m = 2q+1$ and $n = 2p+1$, we set $u = \operatorname{sech}(x)$, then $du = -\operatorname{sech}(x)\tanh(x)dx$.

$$\int \operatorname{sech}^{2q+1}(x)\tanh^{2p+1}(x)dx = -\int u^{2q}(1-u^2)^p du.$$

4. If $m = 2q+1$ and $n = 2p$. The result is obtained using integration by parts and induction.

Examples 11 :

1. $\displaystyle\int \tanh^4(x)\operatorname{sech}^2(x)dx = \int (\tanh(x))^4\operatorname{sech}^2(x)dx = \frac{\tanh^5(x)}{5} + c$

2.

$$\begin{aligned}\int \tanh^3(x)\,\operatorname{sech}(x)dx &= \int \tanh^2(x)\,\operatorname{sech}(x)\tanh(x)dx\\ &= \int (1-\operatorname{sech}^2(x))\,\operatorname{sech}(x)\tanh(x)dx\\ &\overset{u=\operatorname{sech}(x)}{=} \int \tanh^3(x)\,\operatorname{sech}(x)dx\\ &= -\operatorname{sech}(x)+\frac{\operatorname{sech}^3(x)}{3}+c,\end{aligned}$$

3. $\int \operatorname{sech}^4(x)\tanh^7(x)dx \overset{u=\tanh(x)}{=} \int u^7(1-u^2)du = \frac{\tanh^8(x)}{8} - \frac{\tanh^{10}(x)}{10} + c,$

4. $\int \operatorname{sech}^4(x)dx \overset{u=\tanh(x)}{=} \int (1-u^2)du = \tanh(x) - \frac{\tanh^3(x)}{3} + c,$

5. $\int \operatorname{sech}(x)dx = \int \frac{\cosh(x)}{\cosh^2(x)}dx = \int \frac{\cosh(x)}{1+\sinh^2(x)}dx = \tan^{-1}(\sinh(x)) + c.$

6. By integration by parts, $u=\operatorname{sech}(x)$, $v'(x)=\operatorname{sech}^2(x)$, we get

$$\begin{aligned}\int \operatorname{sech}^3(x)dx &= \operatorname{sech}(x)\tanh(x) + \int \operatorname{sech}(x)\tanh^2(x)dx\\ &= \operatorname{sech}(x)\tanh(x) - \int \operatorname{sech}^3(x)dx + \int \operatorname{sech}(x)dx.\end{aligned}$$

Therefore

$$\int \operatorname{sech}^3(x)dx = \frac{1}{2}\operatorname{sech}(x)\tanh(x) + \frac{1}{2}\tan^{-1}(\sinh(x)) + c.$$

7. By integration by parts, $u=\operatorname{sech}^3(x)$, $v'(x)=\operatorname{sech}^2(x)$, we get

$$\begin{aligned}\int \operatorname{sech}^5(x)d &= \operatorname{sech}^3(x)\tanh(x) + 3\int \operatorname{sech}^3(x)\tanh^2(x)dx\\ &= \operatorname{sech}^3(x)\tanh(x) - 3\int \operatorname{sech}^5(x)dx + 3\int \operatorname{sech}^3(x)dx.\end{aligned}$$

Then

$$\begin{aligned}\int \operatorname{sech}^5(x)dx &= \frac{1}{4}\operatorname{sech}^3(x)\tanh(x) + \frac{3}{8}\operatorname{sech}(x)\tanh(x)\\ &\quad +\frac{3}{8}\tan^{-1}(\sinh(x)) + c.\end{aligned}$$

8.

$$\begin{aligned}\int \frac{\operatorname{sech}^4(x)}{\sqrt{\tanh(x)}}dx &\overset{u=\tanh(x)}{=} \int (1-u^2)u^{-\frac{1}{2}}\ du\\ &= 2\tanh^{\frac{1}{2}}(x) - \frac{2}{5}\tanh^{\frac{5}{2}}(x) + c,\end{aligned}$$

Integrals of Type $\int \operatorname{csch}^m(x)\coth^n(x)dx,\ m,n \in \mathbb{N}$

1. If $m = 2q$ and $q \neq 0$, we set $u = \coth(x)$, then $du = -\operatorname{csch}^2(x)dx$ and

$$\int \operatorname{csch}^{2q}(x)\coth^n(x)dx = -\int u^n(u^2-1)^{q-1}du.$$

2. If $m = 0$,

$$\int \coth(x)dx = \ln|\cosh(x)| + c.$$

$$\int \coth^2(x)dx = \int (\operatorname{csch}^2(x)+1)dx = -\coth(x) + x + c.$$

For $n \geq 3$,

$$\begin{aligned}L_n = \int \coth^n(x)dx &= \int \coth^{n-2}(x)\coth^2(x)dx\\ &= \int \coth^{n-2}(x)\operatorname{csch}^2(x)dx + L_{n-2}\\ &= -\frac{\coth^{n-1}(x)}{n-1} + L_{n-2}.\end{aligned}$$

3. If $m = 2q+1$ and $n = 2p+1$, we set $u = \operatorname{csch}(x)$, then $du = -\operatorname{csch}(x)\coth(x)dx$.

$$\int \operatorname{csch}^{2q+1}(x)\coth^{2p+1}(x)dx = -\int u^{2q}(u^2+1)^p du.$$

4. If $m = 2q+1$ and $n = 2p$. The result is obtained using integration by parts and induction.

2.1 Exercises

3-2-1 Evaluate the following integrals:

1) $\displaystyle\int \sinh(ax)\cosh(bx)dx$, for $|a| \neq |b|$

2) $\displaystyle\int \cosh^3(x)dx$,

3) $\displaystyle\int \sinh^3(x)dx$,

4) $\displaystyle\int \sinh^7(x)\ \cosh^3(x)dx$,

5) $\displaystyle\int \sinh^5(x)\ \cosh^4(x)dx$,

6) $\displaystyle\int \sinh^3(x)\ \cosh^2(x)dx$,

7) $\displaystyle\int \sinh^2(x)dx$,

8) $\displaystyle\int \sinh^4(x)dx$,

9) $\displaystyle\int \operatorname{sech}^5(x)\tanh^3(x)dx$,

10) $\displaystyle\int \tanh^3(x)\ \operatorname{sech}^3(x)dx$,

3 Integral of Rational Functions

In this section, we study the integrals of the form $\int F(x)dx$, where F is a rational function:

$$F(x) = \frac{P(x)}{Q(x)}, \ P, \ Q \in \mathbb{R}[X].$$

We shall describe a method for computing this type of integrals. The method is to decompose a given rational function into a sum of simpler fractions (called partial fractions) which is easier to integrate.

3.1 The Irreducible Polynomials in $\mathbb{R}[X]$

Definition 3.1

1. A rational function has the form $R(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials.
2. A rational function is called proper if the degree of the numerator is less than the degree of the denominator, and improper otherwise.
3. A rational function $R(x) = \dfrac{P(x)}{Q(x)}$ is called irreducible if there is no polynomial $S \in \mathbb{R}[X]$ which divide P and Q.

Remark 10 :
Let $F = \dfrac{P}{Q}$ be rational functions with $\deg Q \leq \deg P$, then by using polynomial long division, there are two polynomials R and S such that

$$\frac{P}{Q} = R + \frac{S}{Q},$$

and $\deg S < \deg Q$. (i.e. the rational function $\dfrac{S(x)}{Q(x)}$ is proper).

For example $\dfrac{x^4+5x^2+3}{x^3-x} = x + \dfrac{6x^2+3}{x^3-x}$.

In what follows, the rational functions are taken irreducible and proper.

Definition 3.2

1. The irreducible linear polynomials are the polynomials of the form
$$R(x) = x - \alpha, \quad \alpha \in \mathbb{R}.$$
2. The irreducible quadratic polynomials are the polynomials of the form
$$R(x) = ax^2 + bx + c, \quad a,\ b,\ c \in \mathbb{R}:\ b^2 - 4ac < 0.$$

Examples 1 :

1. x^2+9 and x^2+x+1 are examples of irreducible quadratic polynomials.
2. x^2-x and x^2-1 are reducible quadratics polynomials since $x^2-x = x(x-1)$ and $x^2-1 = (x-1)(x+1)$.

Theorem 3.3

The only irreducible polynomials in $\mathbb{R}[X]$ are the irreducible linear polynomials and the irreducible quadratic polynomials.
Any polynomial $Q \in \mathbb{R}[X]$ has the following decomposition:
$$Q(x) = \prod_{j=1}^{m} L_j^{m_j}(x) \prod_{k=1}^{n} Q_k^{n_k}(x), \tag{3.1}$$
where $L_j(x) = a_j x + b_j$ and $Q_k(x) = c_k x^2 + d_k x + e_k$, where $a_k \neq 0$, $d_k^2 - 4c_k e_k < 0$ for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$.

3.2 Decomposition of Rational Functions

In what follows, $F = \frac{P}{Q}$ is a rational function such that $\deg P < \deg Q$. We look for the right form of the decomposition of F. This right form is a sum of terms $\dfrac{A}{(ax+b)^n}$, $a \neq 0$ and $\dfrac{Ax+B}{(ax^2+bx+c)^n}$, where $b^2-4ac<0$.

1. If $Q(x) = (ax+b)Q_1(x)$, with $a \neq 0$ and $Q_1(-\frac{b}{a}) \neq 0$, the rational function F has a decomposition in the form

$$F(x) = \frac{A}{ax+b} + \frac{P_1}{Q_1}, \qquad \deg P_1 < \deg Q_1.$$

$$\int F(x)dx = \frac{A}{a}\ln|ax+b| + \int \frac{P_1(x)}{Q_1(x)}dx.$$

2. For a repeated linear term, such as $Q(x) = (ax+b)^n Q_1(x)$, with $a \neq 0$, $n \geq 2$ and $Q_1(-\frac{b}{a}) \neq 0$, the rational function F has a decomposition in the form

$$F(x) = \sum_{j=1}^{n} \frac{A_j}{(ax+b)^j} + \frac{P_1}{Q_1}, \qquad \deg P_1 < \deg Q_1.$$

$$\int F(x)dx = \frac{A_1}{a}\ln|ax+b| + \sum_{j=2}^{n} \frac{A_j}{(1-j)(ax+b)^{j-1}} + \int \frac{P_1(x)}{Q_1(x)}dx.$$

3. For a quadratic term ax^2+bx+c, such as $Q(x) = (ax^2+bx+c)^n Q_1(x)$, with $b^2-4ac<0$ and Q_1 is premier with (ax^2+bx+c), the rational function F has a decomposition in the form

$$F(x) = \sum_{j=1}^{n} \frac{A_j x + B_j}{(ax^2+bx+c)^j} + \frac{P_1}{Q_1}, \qquad \deg P_1 < \deg Q_1.$$

Examples 2 :

1. $\dfrac{3x+11}{x^2-2x-3} = \dfrac{5}{x-3} - \dfrac{2}{x-1}.$

2.
$$\begin{aligned}\frac{2x^2-5x-8}{x^3-x^2-8x+12} &= \frac{2x^2-5x-8}{(x-2)^2(x+3)}\\ &= \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+3}\\ &= \frac{1}{x-2}-\frac{2}{(x-2)^2}+\frac{1}{x+3}\end{aligned}$$

3.
$$\begin{aligned}\frac{3x+1}{(x^2+x+2)(x+3)} &= \frac{Ax+B}{x^2+x+2}+\frac{C}{x+3}\\ &= \frac{x+1}{x^2+x+2}-\frac{1}{x+3}\end{aligned}$$

4.
$$\begin{aligned}\frac{1}{(x-1)(x+1)(x^2-x+1)} &= \frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}\\ &= \frac{1}{2(x-1)}-\frac{1}{6(x+1)}-\frac{1}{3}\frac{x+1}{x^2-x+1}.\end{aligned}$$

5.
$$\begin{aligned}\frac{1}{(x^2-3)^2} &= \frac{1}{(x-\sqrt{3})^2(x+\sqrt{3})^2}\\ &= \frac{A}{x-\sqrt{3}}+\frac{B}{(x-\sqrt{3})^2}+\frac{C}{x+\sqrt{3}}+\frac{D}{(x+\sqrt{3})^2}\\ &= -\frac{\sqrt{3}}{36(x-\sqrt{3})}+\frac{1}{12(x-\sqrt{3})^2}+\frac{\sqrt{3}}{36(x+\sqrt{3})}+\frac{1}{12(x+\sqrt{3})^2}.\end{aligned}$$

6. $\dfrac{x^2+1}{(x-1)(x+1)(x-2)} = -\dfrac{1}{x-1}+\dfrac{1}{3(x+1)}+\dfrac{5}{3(x-2)}.$

7. $\dfrac{2x^2+3x-7}{x^3-3x^2+x-3} = \dfrac{2}{x-3} + \dfrac{3}{x^2+1}.$

8. $\dfrac{2x-1}{(x+2)^2(x-3)} = \dfrac{-1}{5(x+2)} + \dfrac{1}{(x+2)^2} + \dfrac{1}{5(x-3)}.$

9. $\dfrac{x+3}{(x^2-1)(x+5)} = \dfrac{1}{3}\dfrac{1}{x-1} - \dfrac{1}{4}\dfrac{1}{x+1} + \dfrac{1}{12}\dfrac{1}{x+5}.$

10. $\dfrac{2x-2}{(x^2+x+4)(x+2)} = \dfrac{x+1}{x^2+x+4} - \dfrac{1}{x+2}.$

11. $\dfrac{2x+6}{x^2-2x-3} = \dfrac{2x+6}{(x-3)(x+1)} = \dfrac{3}{x-3} - \dfrac{1}{x+1}$

12. $\dfrac{x+5}{x^2+4x+4} = \dfrac{x+5}{(x+2)^2} = \dfrac{1}{x+2} + \dfrac{3}{(x+2)^2}$

13. $\dfrac{x^2+1}{x^4+4x^2} = \dfrac{x^2+1}{x^2(x^2+4)} = \dfrac{A_1}{x}+\dfrac{A_2}{x^2}+\dfrac{B_1x+C_1}{x^2+4} = \dfrac{1}{4(x^2)}-\dfrac{1}{4(x^2+4)}.$

14.

$$\begin{aligned}\frac{x}{(x-1)(x^2-1)} &= \frac{x}{(x+1)(x-1)^2} = \frac{A_1}{x+1} + \frac{A_2}{x-1} + \frac{A_3}{(x-1)^2} \\ &= -\frac{1}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}.\end{aligned}$$

15.

$$\begin{aligned}\frac{x^4+2x^3+1}{x^4+x^3+x^2} &= \frac{(x^4+x^3+x^2)+(x^3-x^2+1)}{x^4+x^3+x^2} \\ &= 1+\frac{x^3-x^2+1}{x^4+x^3+x^2} = 1+\frac{x^3-x^2+1}{x^2(x^2+x+1)} \\ &= 1+\frac{1}{2x}+\frac{1}{x^2}+\frac{x-2}{2(x^2+x+1)}\end{aligned}$$

3.3 Integral of Rational Functions

3.3.1 Case of Irreducible Linear Factor of Denominator:

Examples 3 :
Evaluation of some integrals

1.
$$\begin{aligned}\int \frac{6x^2+3}{x(x-1)(x+1)}dx &= -3\int \frac{dx}{x} + \frac{9}{2}\int \frac{dx}{x-1} + \frac{9}{2}\int \frac{dx}{x+1} \\ &= -3\ln|x| + \frac{9}{2}\ln|x-1| + \frac{9}{2}\ln|x+1| + c.\end{aligned}$$

2.
$$\begin{aligned}\int \frac{x^4+5x^2+3}{x^3-x}dx &= \int \left(x - \frac{3}{x} + \frac{9}{2(x-1)} + \frac{9}{2(x+1)}\right)dx \\ &= \frac{x^2}{2} - 3\ln|x| + \frac{9}{2}\ln|x-1| + \frac{9}{2}\ln|x+1| + c.\end{aligned}$$

3.
$$\begin{aligned}\int \frac{x+3}{(x^2-1)(x+5)}dx &= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{12}\int \frac{dx}{x+5} \\ &= \frac{1}{3}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{12}\ln|x+5| + c.\end{aligned}$$

4.
$$\begin{aligned}\int \frac{4}{x^4-x^3}dx &= -4\int \frac{1}{x}dx - 4\int x^{-2}dx - 4\int x^{-3}dx + 4\int \frac{1}{x-1}dx \\ &= -4\ln|x| + \frac{4}{x} + \frac{2}{x^2} + 4\ln|x-1| + c.\end{aligned}$$

5. Using the change of variable $u = \sin(x)$, we get
$$\begin{aligned}\int \frac{3\cos(x)}{\sin^2(x)+\sin(x)-2}dx &= \int \frac{3}{u^2+u-2}\,du \\ &= \int \frac{3}{(u-1)(u+2)}du \\ &= \int \frac{du}{u-1} - \int \frac{du}{u+2} \\ &= \ln|u-1| - \ln|u+2| + c \\ &= \ln|\sin(x)-1| - \ln|\sin(x)+2| + c,\end{aligned}$$

3.3.2 Case of Irreducible Quadratic Factor of Denominator:

Examples 4 :

1.

$$\begin{aligned}\int \frac{8}{(x^2+1)(x^2+9)}dx &= \int \frac{dx}{x^2+1} - \int \frac{dx}{x^2+9} \\ &= \tan^{-1}(x) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c,\end{aligned}$$

2.

$$\begin{aligned}\int \frac{2x+5}{x^2+x+1}dx &= \int \frac{2x+1}{x^2+x+1}dx + \int \frac{4}{x^2+x+1}dx \\ &= \ln(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} \\ &= \ln(x^2+x+1) + \frac{8}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c,\end{aligned}$$

3.

$$\begin{aligned}\int \frac{2x^2-x+2}{x(x^2+1)^2}dx &= 2\int \frac{dx}{x} - 2\int \frac{x}{x^2+1}dx - \int \frac{dx}{(x^2+1)^2} \\ &= \ln x^2 - \ln(x^2+1) + \frac{1}{2}\tan^{-1}(x) + \frac{x}{2(x^2+1)} + c,\end{aligned}$$

4.

$$\begin{aligned}\int \frac{2x-2}{(x^2+x+4)(x+2)}dx &= \int \left(\frac{x+1}{x^2+x+4} + \frac{-1}{x+2}\right)dx \\ &= \int \frac{x+1}{x^2+x+4}dx - \int \frac{dx}{x+2} \\ &= \int \frac{x+1}{x^2+x+4}dx - \ln|x+2| + c,\end{aligned}$$

$$\begin{aligned}\int \frac{x+1}{x^2+x+4}dx &= \int \frac{x+1}{(x+\frac{1}{2})^2+\frac{15}{4}}dx \\ &\overset{u=x+\frac{1}{2}}{=} \int \frac{u+\frac{1}{2}}{u^2+\frac{15}{4}}du \\ &= \frac{1}{2}\ln\left|u^2+\frac{15}{4}\right| + \frac{1/2}{\sqrt{15/4}}\tan^{-1}\frac{u}{\sqrt{15/4}} + c \\ &= \frac{1}{2}\ln(x^2+x+4) + \frac{1}{\sqrt{15}}\tan^{-1}(\frac{2x+1}{\sqrt{15}}) + c,\end{aligned}$$

The final answer is

$$\int \frac{2x-2}{(x^2+x+4)(x+2)}dx = \frac{1}{2}\ln(x^2+x+4) + \frac{1}{\sqrt{15}}\tan^{-1}(\frac{2x+1}{\sqrt{15}}) - \ln|x+2| + c,$$

5.

$$\begin{aligned}
\int \frac{x^2+3x+1}{x^4+5x^2+4}dx &= \int \left(\frac{x}{x^2+1} + \frac{-x+1}{x^2+4}\right) dx \\
&= \int \left(\frac{1}{2}\frac{2x}{x^2+1} - \frac{1}{2}\frac{2x}{x^2+4} + \frac{1}{x^2+2^2}\right) dx \\
&= \frac{1}{2}\ln(x^2+1) - 12\ln(x^2+4) + \frac{1}{2}\tan^{-1}(\frac{x}{2}) + c,
\end{aligned}$$

6.

$$\begin{aligned}
\int \frac{8x^3+13x}{(x^2+2)^2}dx &= 4\int \frac{2x}{x^2+2}dx - \frac{3}{2}\int \frac{2x}{(x^2+2)^2}dx \\
&= 4\ln(x^2+2) - \frac{3}{2}\ \frac{(x^2+2)^{-1}}{-1} + c \\
&= 4\ln(x^2+2) + \frac{3}{2}\ \frac{1}{x^2+2} + c,
\end{aligned}$$

7.

$$\begin{aligned}
\int \frac{x^3+1}{x^3+4x}dx &= \int dx + \frac{1}{4}\int \frac{dx}{x} - \frac{1}{8}\int \frac{2x}{x^2+4}dx - 4\int \frac{dx}{x^2+4} \\
&= x + \frac{1}{4}\ln|x| - \frac{1}{8}\ln(x^2+4) - 2\tan^{-1}\left(\frac{x}{2}\right) + c,
\end{aligned}$$

3.4 Exercises

3-3-1 Compute the following integrals:

1) $\displaystyle\int \frac{x-3}{x+5}dx,$

2) $\displaystyle\int \frac{x^2+x-5}{x^2+2x-35}dx,$

3) $\displaystyle\int \frac{dx}{x^2(x-1)^2},$

4) $\displaystyle\int \frac{3x^2+x+4}{x^4+3x^2+2}dx.$

5) $\displaystyle\int \frac{dx}{1+x+x^2}$,

6) $\displaystyle\int \frac{dx}{(1+x+x^2)^2}$,

7) $\displaystyle\int \frac{dx}{(x+1)(x^2+x+1)}$,

8) $\displaystyle\int \frac{dx}{(x-1)^2(1+x+x^2)^2}$,

4 Trigonometric Substitutions

4.1 Integral Involving $\sqrt{a^2 - x^2}$

We use the substitution $x = a\sin(\theta)$ where $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ to solve the integral. We can also take $x = a\cos(\theta)$, with $\theta \in]0, \pi[$.

4.2 Integral Involving $\sqrt{a^2 + x^2}$

We use the substitution $x = a\tan(\theta)$ where $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ to solve the integral. We can also use the substitution $x = a\sinh(t)$, with $t \in \mathbb{R}$ to solve the integral.

4.3 Integral Involving $\sqrt{x^2 - a^2}$

We use the substitution $x = a\sec(\theta)$ where $\theta \in \left[0, \frac{\pi}{2}\right[$ to solve the integral. We can also use the substitution $x = a\cosh(t)$, with $t \in]0, +\infty[$ to solve the integral.

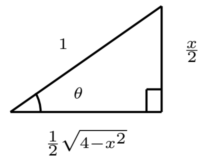
Examples 5 :

1.

$$\begin{aligned}
\int \frac{x+3}{\sqrt{4-x^2}}dx &= \int \frac{x}{\sqrt{4-x^2}} + \frac{3}{\sqrt{4-x^2}}dx \\
&= -\frac{1}{2}\int \left(\frac{-2x}{\sqrt{4-x^2}} + \frac{3}{\sqrt{2^2-x^2}}\right) dx \\
&= -\sqrt{4-x^2} + 3\sin^{-1}(\frac{x}{2}) + c.
\end{aligned}$$

2.

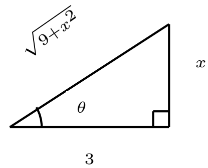
$$\begin{aligned}\int \frac{dx}{x\sqrt{4-x^2}} &= \int \frac{d\theta}{2\sin(\theta)} \quad (x = 2\sin(\theta)) \\ &= -\frac{1}{2}\ln(\csc(\theta)+\cot(\theta)) + c \\ &= -\frac{1}{2}\ln(\frac{2+\sqrt{4-x^2}}{x}) + c.\end{aligned}$$



We know that $\int \frac{dx}{x\sqrt{4-x^2}} = -\frac{1}{2}\text{sech}^{-1}(\frac{x}{2}) + c = -\frac{1}{2}\ln(\frac{2+\sqrt{4-x^2}}{x}) + c.$

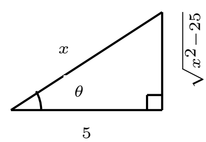
3.

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+9}} &= \int \frac{\sec(\theta)}{9\tan^2(\theta)}d\theta \quad (x = 3\tan(\theta)) \\ &= \frac{1}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}d\theta \\ &= -\frac{\csc(\theta)}{9} + c = -\frac{\sqrt{9+x^2}}{9x} + c.\end{aligned}$$



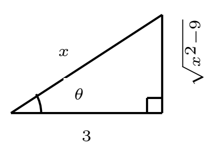
4.

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-25}} &= \int \frac{d\theta}{25\sec(\theta)} \quad (x = 5\sec(\theta)) \\ &= \frac{1}{25}\int \cos(\theta)d\theta \\ &= \frac{1}{25}\sin(\theta) + c = \frac{\sqrt{x^2-25}}{25x} + c.\end{aligned}$$



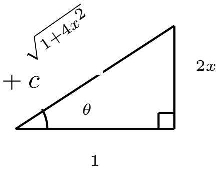
5.

$$\begin{aligned}\int \frac{\sqrt{x^2-9}}{x}dx &= 3\int \tan^2(\theta)d\theta \quad (x = 3\sec(\theta)) \\ &= 3\int (\sec^2(\theta)-1)d\theta \\ &= 3\tan(\theta) - \theta + c \\ &= \sqrt{9-x^2} - \sec^{-1}(\frac{x}{3}) + c.\end{aligned}$$



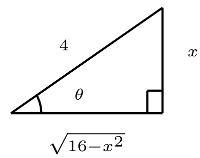
6.

$$
\begin{aligned}
\int \sqrt{1+4x^2}dx &= \frac{1}{2}\int \sec^3(\theta)d\theta \quad (x = \frac{\tan(\theta)}{2}) \\
&= \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta)+\tan(\theta) + c \\
&= \frac{x}{\sqrt{1+4x^2}} + \frac{1}{2}\ln(\frac{1}{\sqrt{1+4x^2}} + 2x) + c.
\end{aligned}
$$



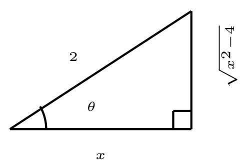
7.

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{16-x^2}}dx &= \int \frac{4\cos(\theta)}{(4\sin(\theta))^2\sqrt{16-(4\sin(\theta))^2}}\ d\theta \quad (x = 4\sin(\theta)) \\
&= \frac{1}{16}\int \frac{1}{\sin^2(\theta)}\ d\theta \\
&= \frac{1}{16}\int \csc^2(\theta)\ d\theta \\
&= -\frac{1}{16}\cot(\theta) + c = -\frac{\sqrt{16-x^2}}{16x} + c.
\end{aligned}
$$



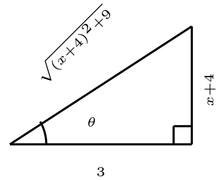
8.

$$
\begin{aligned}
\int \frac{\sqrt{x^2-4}}{x^2}dx &= \int \frac{\sqrt{4\sec^2(\theta)-4}\ 2\sec(\theta)\tan(\theta)}{4\sec^2(\theta)}\ d\theta \quad (x = 2\sec(\theta)) \\
&= \int \frac{\tan^2(\theta)}{\sec(\theta)}\ d\theta \\
&= \int \sec(\theta)\ d\theta - \int \cos(\theta)\ d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| - \sin(\theta) + c \\
&= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| - \frac{\sqrt{x^2-4}}{x} + c.
\end{aligned}
$$



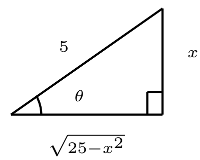
9.

$$\begin{aligned}\int \frac{1}{(x^2+8x+25)^{\frac{3}{2}}}dx &= \int \frac{1}{[(x+4)^2+3^2]^{\frac{3}{2}}}dx \quad (x+4=3\tan(\theta))\\ &= \int \frac{3\sec^2(\theta)}{\left(9\tan^2(\theta)+9\right)^{\frac{3}{2}}}\ d\theta\\ &= \frac{1}{9}\int \frac{d\theta}{\sec(\theta)} = \frac{1}{9}\sin(\theta)+c\\ &= \frac{1}{9}\ \frac{x+4}{\sqrt{x^2+8x+25}}+c.\end{aligned}$$



10.

$$\begin{aligned}\int \frac{1}{(25-x^2)^{\frac{3}{2}}}dx &\overset{x=5\sin(\theta)}{=} \int \frac{5\cos(\theta)}{(25-25\sin^2(\theta))^{\frac{3}{2}}}\ d\theta\\ &= \frac{1}{25}\int \sec^2(\theta)\ d\theta\\ &= \frac{1}{25}\tan(\theta)+c\\ &= \frac{1}{25}\ \frac{x}{\sqrt{25-x^2}}+c.\end{aligned}$$



11.

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+9}} &\overset{x=3\tan(\theta)}{=} \int \frac{\sec(\theta)}{9\tan^2(\theta)}d\theta = \frac{1}{9}\int \frac{\cos(\theta)}{\sin^2(\theta)}d\theta\\ &= -\frac{\csc(\theta)}{9}+c = -\frac{\sqrt{9+x^2}}{9x}+c,\end{aligned}$$

12.

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-25}} &\overset{x=5\sec(\theta)}{=} \int \frac{d\theta}{25\sec(\theta)} = \frac{1}{25}\int \cos(\theta)d\theta\\ &= \frac{1}{25}\sin(\theta)+c = \frac{\sqrt{x^2-25}}{25x}+c.\end{aligned}$$

13.

$$\begin{aligned}\int \frac{x^2}{\sqrt{4-x^2}}dx &\overset{x=2\sin(\theta)}{=} \int 4\sin^2(\theta)d\theta = \int 2(1-\cos(2\theta))d\theta\\ &= 2\theta - 2\sin(\theta)\cos(\theta)+c\\ &= 2\sin^{-1}(\frac{x}{2}) - \frac{x}{2}\sqrt{4-x^2}+c,\end{aligned}$$

14.

$$\begin{aligned}\int\sqrt{x^2+2x+2}dx &= \int\sqrt{(x+1)^2+1}dx, \quad (x+1=\tan(\theta))\\ &= \int \sec^3(\theta)d\theta\\ &= \frac{1}{2}\sec(\theta)\tan(\theta)+\frac{1}{2}\ln|\sec(\theta)+\tan(\theta)|+c\\ &= \frac{1}{2}(x+1)\sqrt{(x+1)^2+1}+\frac{1}{2}\ln|x+1+\sqrt{(x+1)^2+1}|+c.\end{aligned}$$

4.4 Integrals Involving Quadratic Expressions and Miscellaneous Substitutions

Examples 1 :

1. $x^2+2x+5=(x^2+2x+1)-1+5=(x+1)^2+4.$
2. $x^2+x+1=(x^2+2.\frac{1}{2}.x+(\frac{1}{2})^2)-(\frac{1}{2})^2+1,=(x+\frac{1}{2})^2+\frac{3}{4}.$
3. $x^2-x+1=(x^2-2.\frac{1}{2}.x+(-\frac{1}{2})^2)-(-\frac{1}{2})^2+1=(x-\frac{1}{2})^2+\frac{3}{4}.$
4. $x^2+5x=(x^2+2.\frac{5}{2}.x+(\frac{5}{2})^2)-(\frac{5}{2})^2=(x+\frac{5}{2})^2-\frac{25}{4}.$

Examples 2 :

We use the completing square method to evaluate the following integrals:

1. $\displaystyle\int\frac{dx}{x^2-2x+2}\overset{u=x-1}{=}\int\frac{du}{u^2+1}=\tan^{-1}(u)+c=\tan^{-1}(x-1)+c.$
2.

$$\begin{aligned}\int\frac{1}{\sqrt{7+6x-x^2}}dx &= \int\frac{1}{\sqrt{16-(x-3)^2}}dx\\ &\overset{u=x-3}{=} \int\frac{1}{\sqrt{4^2-u^2}}du\\ &= \sin^{-1}(\frac{u}{4})+c=\sin^{-1}(\frac{x-3}{4})+c.\end{aligned}$$

3.

$$\begin{aligned}
\int \frac{2x-5}{x^2-6x+13}dx &= \int \frac{2x-5}{(x-3)^2+4}dx \\
&\overset{x-3=2t}{=} \int \frac{4t+1}{2(1+t^2)}dt \\
&= \int \frac{2t}{1+t^2}dt + \frac{1}{2}\int \frac{dt}{1+t^2} \\
&= \ln(1+t^2) + \tan^{-1}(t) + c \\
&= \ln(x^2-6x+13) + \frac{1}{2}\tan^{-1}(\frac{x-3}{2}) + c.
\end{aligned}$$

4.

$$\begin{aligned}
\int \frac{1}{(x^2+6x+13)^{\frac{3}{2}}}dx &= \int \frac{dx}{((x+3)^2+4)^{\frac{3}{2}}} \quad (u=x+3) \\
&= \int \frac{du}{(u^2+2^2)^{\frac{3}{2}}} \\
&= \int \frac{2\sec^2(\theta)}{8\sec^3(\theta)}d\theta \quad (u=2\tan(\theta)) \\
&= \frac{1}{4}\int \frac{d\theta}{\sec(\theta)} = \frac{1}{4}\int \cos(\theta)d\theta \\
&= \frac{1}{4}\sin(\theta) + c = \frac{1}{4}\frac{u}{\sqrt{u^2+4}} + c \\
&= \frac{1}{4}\frac{x+3}{\sqrt{(x+3)^2+4}} + c..
\end{aligned}$$

5.

$$\begin{aligned}
\int \frac{\sin(x)}{\sqrt{5-2\cos(x)+\cos^2 x}}dx &\overset{u=\cos(x)}{=} -\int \frac{du}{\sqrt{5-2u+u^2}} \\
&= -\int \frac{du}{\sqrt{(u-1)^2+(2)^2}} \\
&= -\sinh^{-1}\left(\frac{u-1}{2}\right) + c \\
&= -\sinh^{-1}\left(\frac{\cos(x)-1}{2}\right) + c.
\end{aligned}$$

6.

$$\begin{aligned}
\int \sqrt{x^2+10x}dx &= \int \sqrt{(x+5)^2-5^2}dx \\
&= \int \sqrt{u^2-5^2}du \quad (u=x+5) \\
&= \int 5\tan(\theta)5\sec(\theta)\tan(\theta)d\theta \quad (u=5\sec(\theta)) \\
&= 25\int \sec^3(\theta)d\theta - 25\int \sec(\theta)d\theta \\
&= \frac{25}{2}\sec(\theta)\tan(\theta) - \frac{25}{2}\ln|\sec(\theta)+\tan(\theta)| + c \\
&= \frac{1}{2}u\sqrt{u^2-25} - \frac{25}{2}\ln\left|\frac{u}{5}+\frac{\sqrt{u^2-25}}{5}\right| + c \\
&= \frac{1}{2}(x+5)\sqrt{x^2+10x} - \frac{25}{2}\ln\left|\frac{x+5}{5}+\frac{\sqrt{x^2+10x}}{5}\right| + c.
\end{aligned}$$

4.5 Exercises

3-4-1 Simplify each of the following expressions by eliminating the radical by using an appropriate trigonometric substitution.

1) $\dfrac{x}{\sqrt{9-x^2}}$,

2) $\dfrac{3+x}{\sqrt{16+x^2}}$,

3) $\dfrac{x-2}{x\sqrt{x^2-25}}$,

4) $\dfrac{1+x}{\sqrt{x^2+2x+2}}$,

5) $\dfrac{2-2x}{\sqrt{x^2-2x-3}}$.

3-4-2 Evaluate the following integrals:

1) $\displaystyle\int \frac{3+x}{\sqrt{16+x^2}}dx$,

2) $\displaystyle\int \frac{x-2}{x\sqrt{x^2-25}}dx$,

3) $\displaystyle\int \sqrt{x^2+a^2}dx$, for $a>0$,

4) $\displaystyle\int \sqrt{x^2-a^2}dx$, for $a>0$,

5) $\displaystyle\int \sqrt{x^2+3x+1}dx$,

6) $\displaystyle\int \sqrt{x^2+2x+5}dx$,

7) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+3}}$,

8) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x-3}}$,

9) $\displaystyle\int \frac{dx}{(4+x^2)^{\frac{3}{2}}}$,

5 Half Angle Substitution

In this section, we treat the integrals of the following form

$$\int \frac{P\big(\cos(x), \sin(x)\big)}{Q\big(\cos(x), \sin(x)\big)} dx$$

where $P(X, Y)$ and $Q(X, Y)$ are two polynomial functions in X, Y.
Method: Generally we use the following substitution
$u = \tan(\frac{x}{2})$, $\quad du = \frac{1}{2}\sec^2(\frac{x}{2})dx$, then $dx = \frac{2du}{1+u^2}$. We have

$$\sin(x) = \frac{2u}{1+u^2}, \qquad \cos(x) = \frac{1-u^2}{1+u^2}.$$

Indeed:

$$\begin{aligned}\sin(x) &= \sin(2.\frac{x}{2}) = 2\sin(\frac{x}{2})\cos(\frac{x}{2}) \\ &= \frac{\tan(\frac{x}{2})}{\sec^2(\frac{x}{2})} = \frac{2u}{1+u^2}.\end{aligned}$$

and

$$\begin{aligned}\cos(x) &= \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) = \cos^2(\frac{x}{2})(1-\tan^2(\frac{x}{2})) \\ &= \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{1-u^2}{1+u^2}.\end{aligned}$$

Examples 1 :
Evaluation of the following integrals

1.

$$\begin{aligned}\int \frac{dx}{2+\sin(x)} &\overset{u=\tan(\frac{x}{2})}{=} \int \frac{1}{2+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du \\ &= \int \frac{du}{u^2+u+1} = \int \frac{du}{(u+\frac{1}{2})^2+\frac{3}{4}} \\ &= \frac{2}{\sqrt{3}} \tan^{-1}\Big(\frac{2\tan(\frac{x}{2})+1}{\sqrt{3}}\Big) + c.\end{aligned}$$

2.

$$\int \frac{1}{2+\cos(x)}dx \overset{u=\tan\left(\frac{x}{2}\right)}{=} \int \frac{1}{2+\left(\frac{1-u^2}{1+u^2}\right)} \frac{2}{1+u^2}\, du$$

$$= \int \frac{2}{3+u^2}\, du = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right) + c.$$

3.

$$\begin{aligned}
\int \frac{1}{3\sin(x)+4\cos(x)}dx &= \int \frac{1}{3\left(\frac{2u}{1+u^2}\right)+4\left(\frac{1-u^2}{1+u^2}\right)} \frac{2}{1+u^2}\, du \quad (u=\tan(\frac{x}{2})) \\
&= \int \frac{2}{-2(2u^2-3u-2)}\, du \\
&= -\int \frac{1}{(2u+1)(u-2)}\, du \\
&= -\frac{1}{5}\int \frac{1}{u-2}\, du + \frac{1}{5}\int \frac{2}{2u+1}\, du \\
&= -\frac{1}{5}\ln|u-2| + \frac{1}{5}\ln|2u+1| + c \\
&= -\frac{1}{5}\ln\left|\tan\left(\frac{x}{2}\right)-2\right| + \frac{1}{5}\ln\left|2\tan\left(\frac{x}{2}\right)+1\right| + c.
\end{aligned}$$

4.

$$\int \frac{1}{5+3\cos(x)}dx \overset{t=\tan(\frac{x}{2})}{=} \int \frac{dt}{t^2+4} = \frac{1}{2}\tan^{-1}(\frac{t}{2}) + c$$

$$= \frac{1}{2}\tan^{-1}(\frac{\tan(\frac{x}{2})}{2}) + c.$$

5.

$$\int \frac{dx}{1-\sin(x)} \overset{u=\tan(\frac{x}{2})}{=} \int \frac{2du}{1+u^2-2u}$$

$$= -\frac{2}{u-1} + c = -\frac{2}{\tan(\frac{x}{2})-1} + c.$$

6. $\displaystyle\int \frac{1}{\sin(x)-\cos(x)-1}dx \overset{t=\tan(\frac{x}{2})}{=} \int \frac{dt}{t-1} = \ln|t-1|+c = \ln|\tan(\frac{x}{2})-1|+c.$

We can also compute the integral as follows

$$\begin{aligned}
\int \frac{dx}{1-\sin(x)} &= \int \frac{1}{1-\sin(x)}\,\frac{1+\sin(x)}{1+\sin(x)}dx \\
&= \int \frac{1+\sin(x)}{1-\sin^2(x)}dx = \int \frac{1+\sin(x)}{\cos^2(x)}dx \\
&= \int \left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2 x}\right)dx \\
&= \int \sec^2(x)dx + \int \sec(x)\tan(x)dx \\
&= \tan(x) + \sec(x) + c.
\end{aligned}$$

7.

$$\begin{aligned}
\int \frac{dx}{\cos(x)+\sin(x)} &\overset{t=\tan(\frac{x}{2})}{=} -\int \frac{dt}{2-(t-1)^2} \\
&= \sqrt{2}\left|\frac{t+\sqrt{2}-1}{t-\sqrt{2}-1}\right| + c \\
&= \sqrt{2}\left|\frac{\tan(\frac{x}{2})+\sqrt{2}-1}{\tan(\frac{x}{2})-\sqrt{2}-1}\right| + c.
\end{aligned}$$

We can also do the following

$$\begin{aligned}
\int \frac{dx}{\cos(x)+\sin(x)} &= \int \frac{dx}{\sqrt{2}(\sin(x+\frac{\pi}{4}))} \\
&= \frac{1}{\sqrt{2}}\ln\left|\csc(x+\frac{\pi}{4}))+\cot(x+\frac{\pi}{4}))\right| + c.
\end{aligned}$$

Remark 11 :

However it was interesting to make another change of variables.

1. If the function $F(x) = R(\sin(x), \cos(x))$ is odd, then we can set $t = \cos(x)$,
2. If $F(x+\pi) = F(x)$, we set $t = \tan(x)$,
3. If $F(\pi - x) = -F(x)$, we set $t = \sin(x)$.

5.1 Exercises

3-5-1 Compute the following integrals:

$$1)\ \int \frac{dx}{\sin^2(x)\cos(x)}, \quad 2)\ \int \frac{\sin(x)dx}{\sin(x)-\cos(x)}.$$

6 Miscellaneous Substitutions

6.1 Integrals Involving Fraction Powers of x

Examples 1 :
Evaluation of the integrals

1.

$$\begin{aligned}
\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}dx &\overset{x=u^6}{=} \int \frac{u^3}{1+u^2}6u^5du = 6\int \frac{u^8}{1+u^2}du \\
&= 6\int u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}du \\
&= \frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} + 2x^{\frac{1}{2}} - 6x^{\frac{1}{6}} + 6\tan^{-1}(x^{\frac{1}{6}}) + c.
\end{aligned}$$

2.

$$\begin{aligned}
\int \frac{2x+3}{\sqrt{1+2x}}dx &\overset{x=\frac{u-1}{2}}{=} \frac{1}{2}\int \frac{2(\frac{u-1}{2})+3}{u^{\frac{1}{2}}}du \\
&= \frac{1}{2}\int \frac{u+2}{u^{\frac{1}{2}}}du \\
&= \frac{1}{3}(1+2x)^{\frac{3}{2}} + (1+2x)^{\frac{1}{2}} + c.
\end{aligned}$$

Therefore

$$\int_0^4 \frac{2x+3}{\sqrt{1+2x}}dx = \Big[\frac{1}{3}(1+2x)^{\frac{3}{2}} + (1+2x)^{\frac{1}{2}}\Big]_0^4 = \frac{32}{3}.$$

3.
$$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx \overset{u=x^{\frac{1}{6}}}{=} \int \frac{6u^5}{u^3+u^2}\ du = \int \frac{6u^3}{u+1}\ du.$$

Using long division of polynomials,

$$\begin{aligned}\int \frac{6u^3}{u+1}\ du &= \int \left(6u^2-6u+6-\frac{6}{u+1)}\right)\ du\\ &= 2u^3-3u^2+6u-6\ln|u+1|+c.\end{aligned}$$

$$\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx = 2x^{\frac{1}{2}}-3x^{\frac{1}{3}}+6x^{\frac{1}{6}}-6\ln\left|x^{\frac{1}{6}}+1\right|+c.$$

4.
$$\begin{aligned}\int \frac{x^{\frac{1}{6}}}{x^{\frac{1}{3}}+1}dx &\overset{u=x^{\frac{1}{6}}}{=} \int \frac{u\ 6u^5}{u^2+1}\ du = \int \frac{6u^6}{u^2+1}\ du\\ &= \int \left(6u^4-6u^2+6-\frac{6}{u^2+1}\right)\ du\\ &= \frac{6u^5}{5}-2u^3+6u-6\tan^{-1}u+c\\ &= \frac{6}{5}x^{\frac{5}{6}}-2x^{\frac{1}{2}}+6x^{\frac{1}{6}}-6\tan^{-1}\left(x^{\frac{1}{6}}\right)+c.\end{aligned}$$

6.2 Integrals Involving a Square Root of a Linear Factor

Examples 2 :

1.
$$\begin{aligned}\int \frac{1}{(x+1)\sqrt{x-2}}dx &\overset{u=\sqrt{x-2}}{=} \int \frac{2u}{(u^2+3)\ u}\ du = \int \frac{2}{u^2+3}\ du\\ &= 2\int \frac{1}{(u)^2+(\sqrt{3})^2}\ du\\ &= 2\ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+c\\ &= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{x-2}}{\sqrt{3}}\right)+c.\end{aligned}$$

2.

$$\int \frac{1}{\sqrt{1+\sqrt{x}}}dx \overset{u=\sqrt{1+\sqrt{x}}}{=} \int \frac{4u(u^2-1)}{u}\, du$$
$$= 4\int (u^2-1)\, du = 4\left[\frac{u^3}{3} - u\right] + c$$
$$= 4\left[\frac{(\sqrt{1+\sqrt{x}})^3}{3} - \sqrt{1+\sqrt{x}}\right] + c.$$

3.

$$\int \frac{1-\sqrt{x}}{1+\sqrt{x}}dx \overset{u=\sqrt{x}}{=} \int \frac{(1-u)2u}{1+u}\, du = \int \frac{-2u^2+2u}{u+1}\, du$$
$$= \int \left(-2u + 4 - \frac{4}{u+1}\right)\, du$$
$$= -u^2 + 4u - 4\ln|u+1| + c$$
$$= -x + 4\sqrt{x} - 4\ln\left|1+\sqrt{x}\right| + c.$$

4. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx = 2\int \left(1+\sqrt{x}\right)^{\frac{1}{2}} \frac{1}{2\sqrt{x}}dx = \frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}} + c.$

6.3 Exercises

3-6-1 Evaluate the following integrals:

1) $\displaystyle\int \frac{\sqrt{2x-1}}{2x+3}dx,$

2) $\displaystyle\int_0^1 x\sqrt{2-\sqrt{1-x^2}}dx,$

3) $\displaystyle\int \frac{dx}{\sqrt{x}+x\sqrt{x}},$

4) $\displaystyle\int_{1/3}^3 \frac{\sqrt{x}}{x^2+x}dx,$

5) $\displaystyle\int \frac{dx}{x^2\sqrt{4x^2-1}}.$

7 Improper Integrals

Definition 7.1

Let f be a continuous function on the interval $[a, b[$, where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$.
We say that the integral of f on the interval $[a, b[$ is convergent if the function

$$F(x) = \int_a^x f(t)dt$$

defined on $[a, b[$ has a finite limit when x tends to b $(x < b)$. This limit is called the improper integral of f on $[a, b[$ and will be denoted by: $\displaystyle\int_a^b f(x)dx$.

Examples 1 :

1.

$$\begin{aligned}\int_0^1 \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}dx &= \lim_{t\to 1^-}\int_0^t \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}dx \\ &\overset{u=\sin^{-1}(x)}{=} \lim_{t\to 1^-}\int_0^{\sin^{-1}(t)} e^u du = e^{\frac{\pi}{2}} - 1.\end{aligned}$$

This integral is convergent.

2. $\displaystyle\int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{1-(x-1)^2}}dx = \sin^{-1}(x-1)\Big]_0^1 = \frac{\pi}{2}.$
The integral is convergent.

3.

$$\begin{aligned}\int_0^{\infty} xe^{-x}\,dx &= \lim_{t\to\infty}\left([-xe^{-x}]_0^t - \int_0^t -e^{-x}\,dx\right) \\ &= \lim_{t\to\infty}\left([-xe^{-x}]_0^t - [e^{-x}]_0^t\right) \\ &= \lim_{t\to\infty}\left([-te^{-t}] - [(e^{-t}-1]\right) = 1.\end{aligned}$$

This integral is convergent.

4. $\displaystyle\int_0^{+\infty} \frac{x}{1+x^2}dx = \lim_{t\to+\infty}\int_0^t \frac{x}{1+x^2}dx = \lim_{t\to+\infty}\frac{1}{2}\ln(1+t^2) = +\infty.$
This integral is divergent.

5. $\displaystyle\int_0^{+\infty} x^n e^{-x}dx = \lim_{t\to+\infty}\int_0^t x^n e^{-x}dx.$ By induction we prove that $\displaystyle\int_0^{+\infty} x^n e^{-x}dx = n!.$ This integral is convergent.

6. $\displaystyle\int_1^{+\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to+\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to+\infty}\frac{(\ln t)^2}{2} = +\infty.$

   Therefore, $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx$ diverges.

Definition 7.2

Let f be a continuous function on the interval $]a,b]$, where $a \in \mathbb{R}\cup\{-\infty\}$, $b\in\mathbb{R}$.
We say that the integral of f on the interval $]a,b]$ is convergent if the function $G(x) = \displaystyle\int_x^b f(t)dt$ defined on $]a,b]$ has a finite limit when x tends to a $(x > a)$. This limit is called the improper integral of f on $]a,b]$ and will be denoted by: $\displaystyle\int_a^b f(x)dx$.

Examples 2 :

1. $\displaystyle\int_0^1 \ln(x)dx = \lim_{t\to0^+}\int_t^1 \ln(x)dx = \lim_{t\to0^+}[x\ln(x)-x]_t^1 = -1.$ This integral is convergent.

2. $\displaystyle\int_{-\infty}^0 \frac{dx}{(x-3)^2} = \lim_{t\to-\infty}\int_t^0 \frac{dx}{(x-3)^2} = \lim_{t\to-\infty} = \left[\frac{-1}{x-3}\right]_t^0 = \frac{1}{3}.$ This integral is convergent.

3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = 2.$ This integral is convergent.

4. $\displaystyle\int_0^1 \frac{dx}{x\ln(x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x\ln(x)} = \lim_{t\to 0^+}[\ln|\ln(x)|]_t^1 = -\infty$. This integral is divergent.

Definition 7.3

Let f be a continuous function on the interval $]a,b[$, where $a \in \mathbb{R}\cup\{-\infty\}$, $b \in \mathbb{R}\cup\{+\infty\}$. We say that the integral of f on the interval $]a,b[$ is convergent if the integral of f is convergent on $]a,c]$ and on $[c,b[$ for any c in $]a,b[$.

Examples 3 :

1. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{\tan^{-1}(x)}}{1+x^2}dx \overset{u=\tan^{-1}(x)}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^u du = 2\sinh(\frac{\pi}{2})$. This integral is convergent.

2. $\displaystyle\int_0^1 \frac{\ln x}{(1-x)^{\frac{3}{2}}}dx \overset{x=1-t^2}{=} 2\int_0^1 \frac{\ln(1-t^2)}{t^2}dt = 2-2\ln 2$. This integral is convergent.

3. $\displaystyle\int_{-\infty}^{+\infty} e^x dx = [e^x]_{-\infty}^{+\infty} = +\infty$. This integral is divergent.

4. $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \pi$. This integral is convergent.

5. Let $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}_+^*$. The integral $\displaystyle\int_a^{+\infty} \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha > 1$ and the integral $\displaystyle\int_0^a \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha < 1$.

6. The integral $\displaystyle\int_0^{+\infty} \sin(x)dx$ is divergent since $\displaystyle\int_0^x \sin(t)dt = 1-\cos(x)$ doesn't have a limit when x tends to $+\infty$.

7. The integral $\int_0^1 \frac{\sin(x)}{x}dx$ is convergent since the function $\frac{\sin(x)}{x}$ can be considered as a continuous function on the interval $[0,1]$.

8. The integral $\int_0^1 \sin(\frac{1}{x})dx$ is convergent since the function $\sin(\frac{1}{t})$ is continuous on the interval $]0,1]$ and bounded.

9. $\int_0^{\frac{\pi}{2}} \cos(x)\ln(\tan(x))dx = \int_0^{\frac{\pi}{4}} \cos(x)\ln(\tan(x))dx - \int_0^{\frac{\pi}{4}} \sin(x)\ln(\tan(x))dx.$
Using integration by parts, we have
$$\int_0^{\frac{\pi}{4}} \cos(x)\ln(\tan(x))dx = -\ln(1+\sqrt{2}).$$
$$\begin{aligned} -\int_0^{\frac{\pi}{4}} \sin(x)\ln(\tan(x))dx &= [(\cos(x)-1)\ln(\sin(x)) + \ln(1+\cos(x))]_0^{\frac{\pi}{4}} \\ &= -\frac{\sqrt{2}}{4}\ln 2 + \ln(1+\sqrt{2}). \end{aligned}$$
Then $\int_0^{\frac{\pi}{2}} \cos(x)\ln(\tan(x))dx = -\frac{\sqrt{2}}{4}\ln 2.$

10.
$$\begin{aligned} \int_1^{+\infty} \frac{x^4+1}{x^3(x+1)(1+x^2)}dx &= \int_1^{+\infty} (-\frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x+1} + \frac{x+1}{1+x^2})dx \\ &= \frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\ln 2. \end{aligned}$$

11.
$$\begin{aligned} \int_1^{+\infty} \frac{dx}{x\,\sqrt[4]{1+x^2}} &\overset{t^4=1+x^2}{=} \int_{2^{\frac{1}{4}}}^{+\infty} \frac{2t^2dt}{(t^4-1)} \\ &= \int_{2^{\frac{1}{4}}}^{+\infty} (\frac{1}{2(t-1)} - \frac{1}{2(t+1)} + \frac{1}{1+t^2})dt \\ &= \frac{1}{2}\ln(\frac{2^{\frac{1}{4}}+1}{2^{\frac{1}{4}}-1}) + \frac{\pi}{2} - \tan^{-1}(2^{\frac{1}{4}}). \end{aligned}$$

12. $\int_{-1}^0 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}(\frac{x}{2})\right]_{-1}^0 = \frac{\pi}{6}.$

13.

$$\begin{aligned}\int_{-2}^{0} \frac{dx}{\sqrt[3]{x+1}} &= \int_{-2}^{-1} \frac{dx}{\sqrt[3]{x+1}} + \int_{-1}^{0} \frac{dx}{\sqrt[3]{x+1}} \\ &= \frac{3}{2}\left[(x+1)^{\frac{2}{3}}\right]_{-2}^{-1} + \frac{3}{2}\left[(x+1)^{\frac{2}{3}}\right]_{-1}^{0} = 0.\end{aligned}$$

14. $\displaystyle\int_{-3}^{1} \frac{dx}{x^2} = \int_{-3}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2} = +\infty.$

15.

$$\begin{aligned}\int_{0}^{1} \frac{x}{(x^2-1)^3}\, dx &= \lim_{t\to 1^-} \int_{0}^{t} \frac{x}{(x^2-1)^3}\, dx \\ &= \lim_{t\to 1^-} -\frac{1}{4}\left[(x^2-1)^{-2}\right]_{0}^{t} \\ &= \lim_{t\to 1^-} -\frac{1}{4}\left[\frac{1}{(t^2-1)^2} - 1\right] = -\infty.\end{aligned}$$

Therefore, $\displaystyle\int_{0}^{1} \frac{x}{(x^2-1)^3}\, dx$ diverges.

16.

$$\begin{aligned}\int_{1}^{e} \frac{1}{x\sqrt{\ln x}}\, dx &= \lim_{t\to 1^+} \int_{t}^{e} \frac{1}{x\sqrt{\ln x}}\, dx \\ &= \lim_{t\to 1^+} \left[2(\ln x)^{\frac{1}{2}}\right]_{t}^{e} \\ &= \lim_{t\to 1^+} 2\left[1 - \sqrt{\ln t}\right] = 2.\end{aligned}$$

Therefore, $\displaystyle\int_{1}^{e} \frac{1}{x\sqrt{\ln x}}\, dx$ converges to 2.

17.

$$\begin{aligned}\int_{1}^{\infty} \frac{1}{x\sqrt{x^2-1}}\, dx &= \int_{1}^{2} \frac{1}{x\sqrt{x^2-1}}\, dx + \int_{2}^{\infty} \frac{1}{x\sqrt{x^2-1}}\, dx \\ &= \lim_{t\to 1^+} \int_{t}^{2} \frac{1}{x\sqrt{x^2-1}}\, dx + \lim_{t\to\infty} \int_{2}^{t} \frac{1}{x\sqrt{x^2-1}}\, dx \\ &= \lim_{t\to 1^+} \left[\sec^{-1}(2) - \sec^{-1} t\right] + \lim_{t\to\infty} \left[\sec^{-1} t - \sec^{-1}(2)\right] \\ &= \frac{\pi}{2}.\end{aligned}$$

Therefore, $\displaystyle\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\, dx$ converges to $\dfrac{\pi}{2}$.

7.1 Exercises

3-7-1 Prove that the following improper integrals are convergent and compute the value of these integrals.

1) $\displaystyle\int_0^{+\infty} xe^{-2x} dx,$

2) $\displaystyle\int_0^{+\infty} e^{-x}\sin(x) dx,$

3) $\displaystyle\int_0^{1} \frac{x+1}{\sqrt{x}} dx,$

4) $\displaystyle\int_1^{+\infty} \frac{dx}{x^2\sqrt{x-1}},$

5) $\displaystyle\int_1^{2} \frac{2x^3}{\sqrt{x^4-1}} dx$

6) $\displaystyle\int_0^{1} \frac{\ln(1-x^2)}{x^2} dx,$

7) $\displaystyle\int_0^{1} \frac{x\ln x}{(1-x^2)^{3/2}} dx,$

8) $\displaystyle\int_0^{1} \frac{\ln x}{(1-x)^{\frac{3}{2}}} dx,$

9) $\displaystyle\int_1^{+\infty} \frac{x^4+1}{x^3(x+1)(1+x^2)} dx,$

10) $\displaystyle\int_1^{+\infty} \frac{dx}{x\ {}^4\sqrt{1+x^2}},$

11) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\ln(\tan(x)) dx.$

3-7-2 Determine whether the following integrals are convergent or divergent:

1) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt[4]{1+x}} dx,$

2) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt[4]{(1+x)^5}} dx,$

3) $\displaystyle\int_{-\infty}^{0} 2^x dx,$

4) $\displaystyle\int_{-\infty}^{+\infty} \cos(\pi x) dx,$

5) $\displaystyle\int_6^{8} \frac{4}{\sqrt[3]{x-6}} dx,$

6) $\displaystyle\int_1^{3} \frac{1}{\sqrt{3-x}} dx,$

7) $\displaystyle\int_1^{+\infty} \frac{\ln(x)}{x^4} dx,$

8) $\displaystyle\int_0^{1} \frac{1}{2-3x} dx,$

9) $\displaystyle\int_1^{+\infty} \frac{\tan^{-1}(x)}{x^2} dx,$

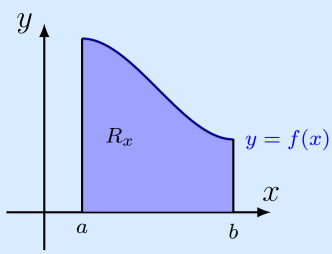
CHAPTER 4

APPLICATIONS OF DEFINITE INTEGRALS

1 Area of Plane Regions

Definition 1.1

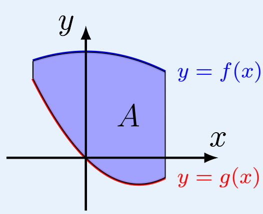
Let $f\colon [a,b] \longrightarrow \mathbb{R}^+$ be a non negative continuous function, the integral $\int_a^b f(x)dx$ represents the area of the region R_x delimited by the graphs of f, the axis of equations: $x = a$, $x = b$ and $y = 0$ (the $x-$axis).



Theorem 1.2

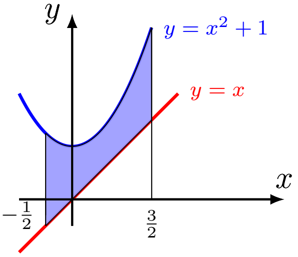
If f and g are two continuous functions on $[a,b]$ and $f(x) \geq g(x)$, $\forall x \in [a,b]$. Then the area A of the region bounded by the graphs of f,g, $x = a$ and $x = b$ is

$$A = \int_a^b f(x) - g(x)dx.$$



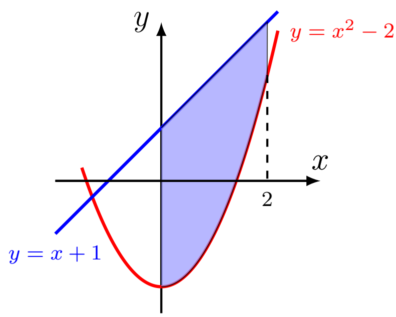
Example 1 :
Let $f(x) = x^2 + 1$ and $g(x) = x$.

The area of the shaded region is
$A = \int_{-\frac{1}{2}}^{\frac{3}{2}} (x^2 + 1 - x)dx = \frac{13}{6}$.



Example 2 :
Let $f(x) = x^2 - 2$ and $g(x) = x + 1$ on the interval $[0, 2]$.
The area of the region between the graphs of the functions f and g on the interval $[0, 2]$ is

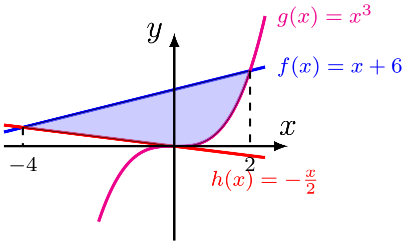
$$\begin{aligned} A &= \int_0^2 (x+1) - (x^2 - 2)dx \\ &= \int_0^2 (x + 3 - x^2)dx \\ &= \frac{16}{3}. \end{aligned}$$



Remark 12 :
If f and g are two continuous functions on $[a, b]$. Then the area A of the region bounded by the graphs of f and g is $A = \int_a^b |f(x) - g(x)|dx$.
For example if there is $c \in]a, b[$ such that $f(x) \geq g(x)$, $\forall x \in [a, c]$ and $f(x) \leq g(x)$, $\forall x \in [c, b]$, then $A = \int_a^c f(x) - g(x)dx + \int_c^b g(x) - f(x)dx$.

Example 3 :
The area A of the region R bounded by the graphs of $f(x) = x + 6$, $g(x) = x^3$ and $h(x) = -\frac{1}{2}x$:

$f(x) = h(x) \iff x = -4,$
$g(x) = h(x) \iff x = 0,$
$f(x) = g(x) \iff x^3 - x - 6 = 0$. $x = 2$ is the unique solution of this equation.
We have $f(-4) = h(-4) = 2$,
$g(0) = h(0) = 0$ and
$f(2) = g(2) = 8$.



The area of the region is equal to:

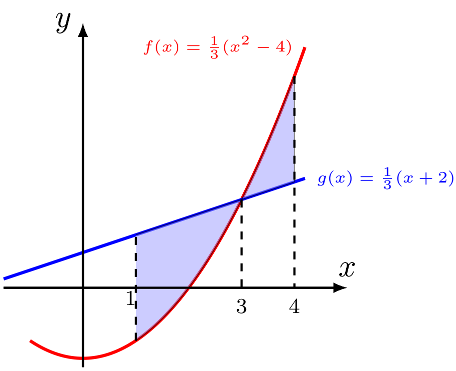
$$A = \int_{-4}^{0} (f(x) - h(x))\,dx + \int_{0}^{2} (f(x) - g(x))\,dx.$$

$$A = \int_{-4}^{0} \left((x+6) + \frac{1}{2}x\right) dx + \int_{0}^{2} \left((x+6) - x^3\right) dx = 22.$$

Example 4 :

The area of the region between the graphs of the functions:
$f(x) = \frac{1}{3}(x^2 - 4)$ and $g(x) = \frac{1}{3}(x+2)$ if x is restricted to the interval $[1, 4]$.

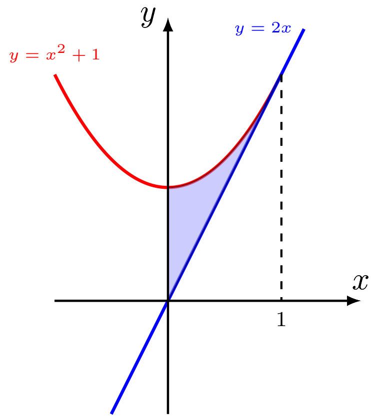
$f(x) = g(x) \iff x^2 - x - 6 = 0$.
The only solution of this equation on the interval $[1, 4]$ is $x = 3$ and we have $f(3) = g(3) = \frac{5}{3}$.
We have $f \leq g$ on the interval $[1, 3]$ and $g \leq f$ on the interval $[3, 4]$.
Then



$$\begin{aligned} A &= \int_{1}^{3} (g(x) - f(x))\,dx + \int_{3}^{4} (f(x) - g(x))\,dx \\ &= \frac{1}{3}\int_{1}^{3} \left((x+2) - (x^2 - 4)\right) dx + \frac{1}{3}\int_{3}^{4} \left((x^2 - 4) - (x+2)\right) dx = \frac{61}{18}. \end{aligned}$$

Examples 5 :

1. The area bounded by the graphs of the curves $y = x^2 + 1$, $y = 2x$ and $x = 0$:

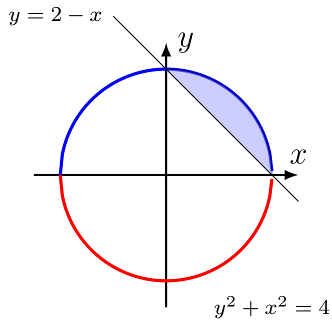


Note that $y = x^2 + 1$ is a parabola opens upward with vertex $(0, 1)$, $y = 2x$ is a straight line passing through the origin and $x = 0$ is the $y-$axis.
$x^2 + 1 = 2x \iff x = 1$.

The desired area is

$$A = \int_0^1 [(x^2 + 1) - 2x] \, dx = \int_0^1 (x - 1)^2 \, dx = \left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{1}{3}.$$

2. The area inside the graph of the curve $x^2 + y^2 = 4$ and above $y = 2 - x$. The desired area is one fourth of the area of the circle minus the area of the triangle which equals to $\pi - 2$. Note also that $x^2 + y^2 = 4$ is a circle with center $=(0, 0)$ and radius $=2$ and $y = 2 - x$ is a straight line.



$x^2 + (2 - x)^2 = 4$
$\iff (2 - x)^2 = (4 - x^2)$
$\iff x = 0$ or $x = 2$.
Note also that
$x^2 + y^2 = 4 \iff y = \pm\sqrt{4 - x^2}$, where $\sqrt{4 - x^2}$ represents the upper half of the circle and $-\sqrt{4 - x^2}$ represents the lower half of the circle.

The desired area is $A = \int_0^2 \sqrt{4 - x^2} \, dx - \int_0^2 (2 - x) \, dx = I_1 - I_2$, where $I_1 = \int_0^2 \sqrt{4 - x^2} \, dx$ and $I_2 = \int_0^2 (2 - x) \, dx$.

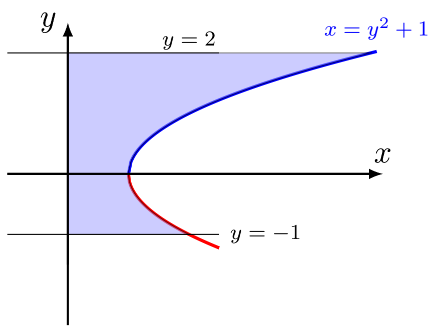
$$I_1 \overset{x=2\sin(\theta)}{=} \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2(\theta)}\; 2\cos(\theta)\; d\theta = \int_0^{\frac{\pi}{2}} 4\cos^2(\theta)\; d\theta$$
$$= 4\int_0^{\frac{\pi}{2}} \frac{1}{2}[1+\cos(2\theta)]\; d\theta = 2\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{2}} = \pi.$$

$$I_2 = \int_0^2 (2-x)\; dx = -\left[\frac{(2-x)^2}{2}\right]_0^2 = 2.$$

Hence, the desired area is $I_1 - I_2 = \pi - 2$.

3. The area bounded by the graphs of the curves of equations $x = y^2+1$, $x=0$, $y=-1$ and $y=2$.

Note that $x = y^2+1$ is a parabola opens to the right with vertex $(1,0)$, $x = 0$ is the $y-$axis, $y = 2$ is a straight line parallel to the $x-$axis and passing through the point $(0,2)$ also $y = -1$ is another straight line parallel to the $x-$axis and passing through the point $(0,-1)$. The desired area is



$$A = \int_{-1}^{2} (y^2+1)\; dy = \left[\frac{y^3}{3} + y\right]_{-1}^{2} = 6.$$

1.1 Exercises

4-1-1 Set up integrals to evaluate the areas bounded by the graphs of the following curves

1) $y = \ln x$, $y = 0$ and $x = 2$,
2) $y = e^x$, $x = \ln 4$, $x = 0$ and $y = 0$,
3) $y = x^2$ and $y = -x^2+2$,
4) $y = \frac{4}{x}$, $x = 0$, $y = 1$ and $y = 2$.

4-1-2 Find the area of the region between the graphs of the functions $y = e^x, y = 4e^{-x}$ and $y = 1$.

4-1-3 Find the area of the region bounded by the curves $x = y^2$, $x+y = 6$, $y = -4$, $y = 2$.

4-1-4 Find the area between the curves: $y = \cos(x), y = \sin(x)$, $0 \leq x \leq \frac{\pi}{2}$,

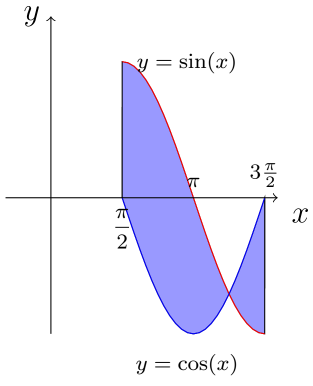
4-1-5 Sketch the region bounded by the curves and find its area

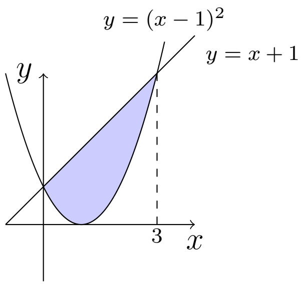
1) $4x = 4y - y^2$, $4x - y = 0$,
2) $y = e^x$, $y = e$, $y = x$, $x = 0$,
3) $4y = x^2$ and $y = \frac{8}{x^2+4}$,
4) The region in the first quadrant bounded by the x-axis, the parabola $y = \frac{x^2}{3}$, and the circle $x^2 + y^2 = 4$,
5) $y = \sin^2(x)$, $y = \tan^2 x$, $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

4-1-6 Sketch the region bounded by the curves and find its area in the following :

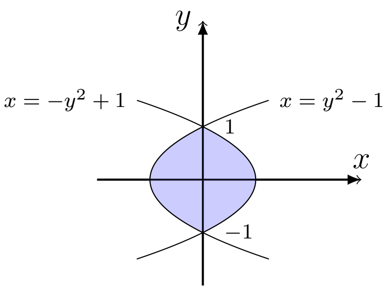
1) $y = \frac{1}{3}x^2$ and $y = 2x - \frac{1}{3}x^2$.
2) $y = -x$ and $x = y^2 + 2y$.

4-1-7 Find the area of the shaded regions:

1) 



2)



3)

4-1-8 Find the area of the region bounded by the graphs of the curves of $y = x^2 - 4x$ and $y = 0$

4-1-9 Find the area of the region bounded by the graphs of the curves $y = x^2 + 2x + 1$, $y = 1 - x$ and $y = 0$

4-1-10 Find the area of the region bounded by the graphs of the curves $y = x^2$, $y = x^2 + 1$, $x = 0$ and $x = 1$.

2 Volume of Solid of Revolution

2.1 The Disk Method

Let $f\colon [a,b] \longrightarrow \mathbb{R}^+$ be a non negative continuous function and R_x the region delimited by the graph of f and the axis: $x = a$, $x = b$ and the $x-$axis. If the region R_x is revolved around the x-axis, the resulting solid is called: the solid of revolution generated by the region R_x.

Examples 6 :

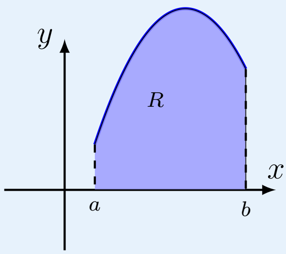
1. If $f\colon [a,b] \longrightarrow \mathbb{R}$ is a constant $c > 0$, then the region under the graph of f on the interval $[a,b]$ is a rectangle. The solid gener-

ated by revolving this region around the x-axis is a circular right cylinder.

2. Consider the region under the graph of the function $f(x) = \sqrt{4 - x^2}$ for $x \in [-2, 2]$. If we revolve the region R_x around the x-axis, the solid generated is a ball of radius $r = 2$.

Theorem 2.1

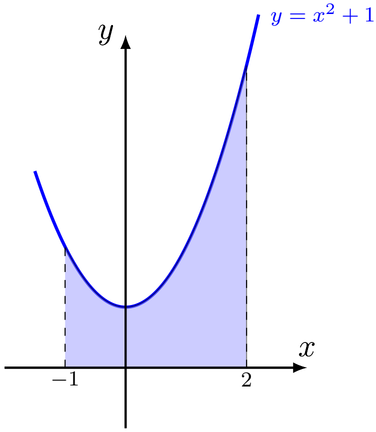
Let $f\colon [a, b] \longrightarrow \mathbb{R}^+$ be a continuous function. The volume V of the solid of revolution generated by revolving the region bounded by the graphs of f, $y = 0$ $x = a$ and $x = b$ is given by



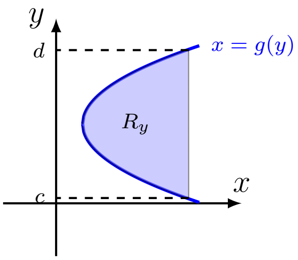
$$V = \int_a^b \pi f^2(x)dx. \tag{2.1}$$

Example 7 :

Let f be the function defined on the interval $[-1, 2]$ by $f(x) = x^2 + 1$. The volume of the solid obtained by revolving the region under the graph of f around the x-axis is
$\pi \int_{-1}^{2} (x^2 + 1)^2 dx = \frac{78\pi}{5}$.

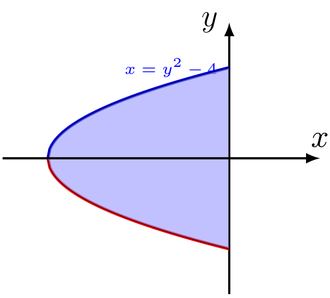


Remark 13 :
Let g be a positive continuous function on the interval $[c,d]$ and R_y the region bounded by: the graph of the function $x = g(y)$, the axis $y = c$, $y = d$ and $y-$axis. The volume of the solid of revolution of the region R_y around the y-axis is:



$$V = \pi \int_c^d g^2(y)dy. \tag{2.2}$$

Example 8 :
If $g(y) = y^2 - 4$ defined on the interval $[0,2]$. The volume of the solid obtained by revolving the region under the graph of g around the y-axis is:

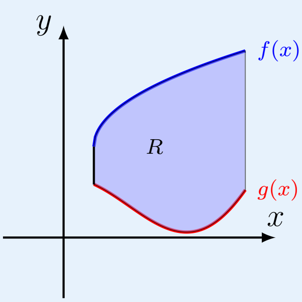


$$V = \pi \int_0^2 (y^2-4)^2 dy = \frac{256}{15}\pi.$$

2.2 Washer Method

Theorem 2.2

Let $f, g\colon [a,b] \longrightarrow \mathbb{R}^+$ be two continuous functions such that $f(x) \geq g(x) \geq 0$, $\forall x \in [a,b]$. If R is the region between the graph of f and the graph of g. The volume of the solid obtained by revolving the region R around the x-axis is equal to

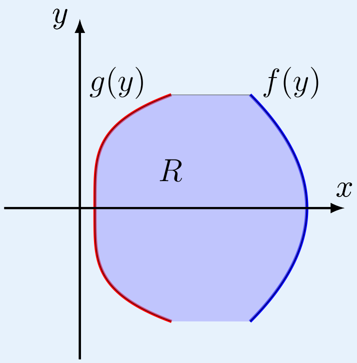


$$\pi \int_a^b \left(f^2(x) - g^2(x)\right) dx. \tag{2.3}$$

This formula can be interpreted as:

$$V = \pi \int_a^b (\text{outer radius})^2 - (\text{inner radius})^2 dx. \qquad (2.4)$$

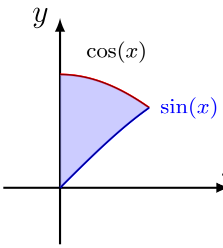
If R is the region bounded by the graphs of $x = f(y)$ and $x = g(y)$, where $f(y)$ and $g(y)$ continuous functions defined on the interval $[c, d]$ and satisfies $0 \leq g \leq f$. The volume of the solid of revolution generated by revolving the region R around the $y-$axis is



$$V = \pi \int_c^d \left[f^2(y) - g^2(y)\right] \, dy. \qquad (2.5)$$

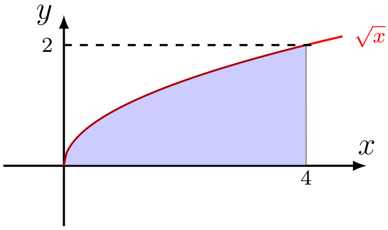
Examples 9 :

1. If $f(x) = \cos(x)$ and $g(x) = \sin(x)$ on the interval $[0, \frac{\pi}{4}]$. The volume of the solid of revolving R between the graph of f and g around the x-axis is



$$V = \pi \int_0^{\frac{\pi}{4}} \left(\cos^2(x) - \sin^2(x)\right) dx = \pi \int_0^{\frac{\pi}{4}} \cos(2x) dx = \frac{\pi}{2}.$$

2. Let $f(x) = \sqrt{x}$ defined on the interval $[0, 4]$. If R is the region under the graph of f and S the solid of revolution of R around the axis $y = 2$. The volume of S is:



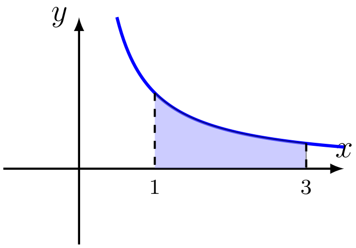
$$V = \pi \int_0^4 \left(2^2 - (2-\sqrt{x})^2\right) dx = \frac{40\pi}{3}.$$

In this example, the outer radius is 2, the inner radius is $2 - y = 2 - \sqrt{x}$.

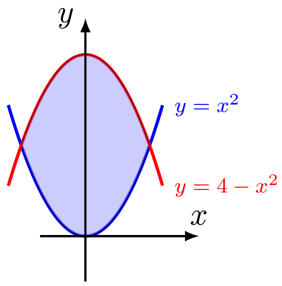
Examples 10 :
Use disk or washer method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the following curves

1. $y = \dfrac{1}{x}$, $x = 1$, $x = 3$ and $y = 0$, around the $x-$axis.

$$V = \pi \int_1^3 \frac{dx}{x^2} = \frac{2\pi}{3}.$$

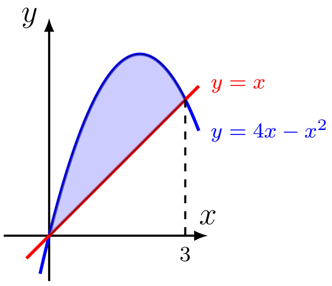


2. $y = x^2$ and $y = 4 - x^2$, around the $x-$axis.
Note that $y = x^2$ is a parabola opens upward with vertex $(0,0)$ and $y = 4 - x^2$ is a parabola opens downward with vertex $(0,4)$.
$x^2 = 4 - x^2 \iff x = \pm\sqrt{2}$. The points of intersection of $y = x^2$ and $y = 4 - x^2$ are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.
Using Washer Method



$$\begin{aligned} V &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[(4-x^2)^2 - (x^2)^2\right] \, dx \\ &= 16\pi \int_0^{\sqrt{2}} (2 - x^2) \, dx = 16\pi \left[2x - \frac{x^3}{3}\right]_0^{\sqrt{2}} = \frac{64\sqrt{2}}{3}\pi. \end{aligned}$$

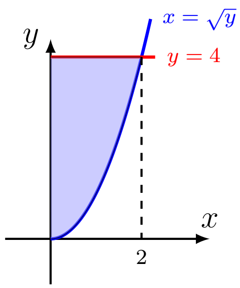
3. $y = 4x - x^2$ and $y = x$, around the $x-$axis.
   $4x - x^2 = 4 - (x-2)^2$ is a parabola opens downward with vertex (2,4) and $y = x$ is a straight line passing through the origin.
   $x = 4x - vx^2 \iff x = 0$, $x = 3$. The points of intersection of $y = 4x - x^2$ and $y = x$ are $(0,0)$ and $(3,3)$. Using Washer Method, we get



$$V = \pi \int_0^3 \left[(4x - x^2)^2 - x^2\right] \, dx$$
$$= \pi \int_0^3 \left[x^4 - 8x^3 + 15x^2\right] \, dx = \frac{108}{5}\pi.$$

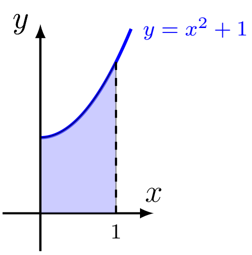
4. $x = \sqrt{y}$, $x = 0$ and $y = 4$, around the $y-$axis

   Using Disk Method, we get
   $V = \pi \int_0^4 (\sqrt{y})^2 \, dy = \pi \left[\frac{y^2}{2}\right]_0^4 = 8\pi.$



5. $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 1$, around the $y-$axis.
   Note that $y = x^2 + 1$ is a parabola opens upward with vertex $(0, 1)$, $x = 1$ is a straight line parallel to the $y-$axis and passing through the point $(1, 0)$.

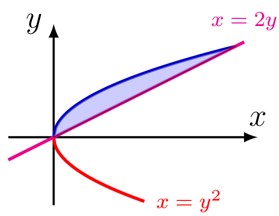
   $y = x^2 + 1 \iff x = \pm\sqrt{y-1}$, where $x = \sqrt{y-1}$ is the right half of the parabola and $y = -\sqrt{y-1}$ is the left half of the parabola.
   The point of intersection between $y = x^2 + 1$ and $x = 1$ is $(1, 2)$. Using Washer Method



$$
\begin{aligned}
V &= \pi \int_0^2 1\ dy - \pi \int_1^2 \left(\sqrt{y-1}\right)^2\ dy \\
&= 2\pi - \pi \left[\frac{1}{2}(y-1)^2\right]_1^2 = \frac{3}{2}\pi.
\end{aligned}
$$

6. $x = y^2$ and $x = 2y$, around the $y-$axis.
   Note that $x = y^2$ is a parabola opens to the right with vertex $(0,0)$ and $x = 2y$ is a straight line passing through the origin.

   $y^2 = 2y \iff y^2 - 2y = 0 \iff y = 0\ ,\ y = 2$. The points of intersection between $x = y^2$ and $x = 2y$ are $(0,0)$ and $(4,2)$.
   Using Washer Method, we get

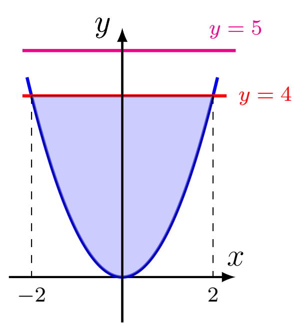


$$
\begin{aligned}
V &= \pi \int_0^2 \left[(2y)^2 - (y^2)^2\right]\ dy \\
&= \pi \int_0^2 (4y^2 - y^4)\ dy = \frac{64}{15}\pi.
\end{aligned}
$$

7. $y = x^2$ and $y = 4$, around the line $y = 5$.
   Note that $y = x^2$ is a parabola opens upward with vertex $(0,0)$ and $y = 4$ is a straight line parallel to the $x-$axis and passing through $(0,4)$.

   $x^2 = 4 \iff x = \pm 2$. The points of intersection between $y = x^2$ and $y = 4$ are $(2,4)$ and $(-2,4)$. Using Washer Method, we get

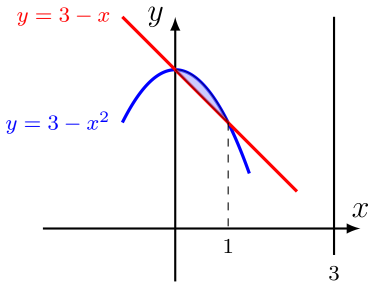
$$
\begin{aligned}
V &= \pi \int_{-2}^2 \left[(5-x^2)^2 - (5-4)^2\right]\ dx \\
&= \pi \int_{-2}^2 (24 - 10x^2 + x^4)\ dx = \frac{832}{15}\pi.
\end{aligned}
$$



8. $y + x^2 = 3$ and $y + x = 3$, around the line $x = 3$.
Note that $y = 3 - x^2$ is a parabola opens downward with vertex $(0, 3)$ and $x + y = 3$ is a straight line.

$y + x^2 = x + y \iff x = 0$, or $x = 1$. The intersection points are $(0, 3)$ and $(1, 2)$.

$y + x^2 = 3 \iff x = \pm\sqrt{3 - y}$, where $x = \sqrt{3 - y}$ is the right half of the parabola and $x = -\sqrt{3 - y}$ is the left half of the parabola. Using Washer Method, we get

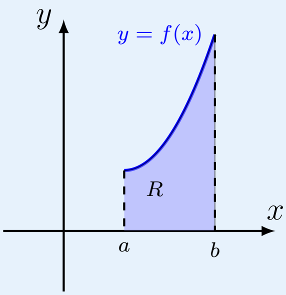


$$V = \pi \int_2^3 \left[(3-(3-y))^2 - (3-\sqrt{3-y})^2\right] dy \overset{3-y=t^2}{=} 2\pi \int_0^1 [t^5 - 7t^3 + 6t^2]dt = \frac{5}{6}\pi.$$

2.3 The Cylindrical Shells Method

Theorem 2.3

Let $f\colon [a, b] \longrightarrow \mathbb{R}^+$ be a continuous function and R the region under the graph of f on the interval $[a, b]$. The volume V of the solid of revolution generated by revolving the region R around the y-axis is given by

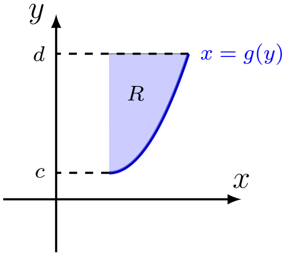


$$V = 2\pi \int_a^b x f(x) dx. \tag{2.6}$$

Example 11 :
Let $f\colon [2,11] \longrightarrow \mathbb{R}^+$ be the function defined by $\sqrt{x-2}$. The volume of the solid of revolution generated by revolving the region under the graph of f around the y-axis is

$$V = 2\pi \int_2^{11} x\sqrt{x-2}dx \overset{x-2=t^2}{=} 4\pi \int_0^3 (2t^2+t^4)dt = 12\pi\frac{111}{5}.$$

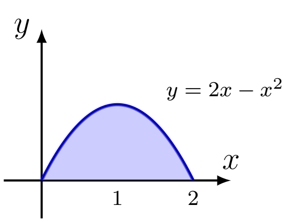
Remark 14 :
Consider the region R bounded by the graphs of the curves of $g(y)$, $y=d$, $y=c$ and the $y-$axis. Using cylindrical shells method, the volume of the solid of revolution generated by revolving the region R around the $x-$axis is
$V = 2\pi \int_c^d y\ g(y)\ dy.$



Examples 12 :
We use cylindrical shells method to find the volume of the solid of revolution generated by revolving the region bounded by the graphs of the following curves:

1. $y = 2x - x^2$ and $y = 0$, around the $y-$axis.
   $y = 2x - x^2 = -(x^2 - 2x + 1) + 1 = 1 - (x-1)^2$ is a parabola opens downward with vertex $(1,1)$.

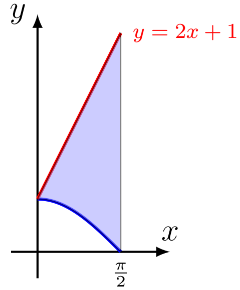
   $2x - x^2 = 0 \iff x = 0\ ,\ x = 2$, then the points of intersection between $y = 2x - x^2$ and $y = 0$ are $(0,0)$ and $(2,0)$. Using Cylindrical shells method, we get

   $$V = 2\pi \int_0^2 x(2x - x^2)\ dx = 2\pi \int_0^2 (2x^2 - x^3)\ dx = \frac{8}{3}\pi.$$

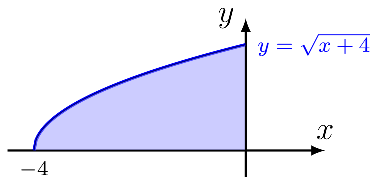
2. $y = \cos x$, $y = 2x + 1$ and $x = \frac{\pi}{2}$, around the $y-$axis.

The line $y = 2x + 1$ passes through the point $(0, 1)$. The desired region is under the line $y = 2x + 1$ and above the curve of $y = \cos x$ on the interval $\left[0, \frac{\pi}{2}\right]$.
Using Cylindrical shells method, we get



$$\begin{aligned} V &= 2\pi \int_0^{\frac{\pi}{2}} x\left[(2x+1) - \cos x\right]\, dx \\ &= 2\pi \int_0^{\frac{\pi}{2}} \left(2x^2 + x\right)\, dx - 2\pi \int_0^{\frac{\pi}{2}} \left(x\cos(x)\right)\, dx \\ &= 2\pi \left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^{\frac{\pi}{2}} - 2\pi \left[x\sin(x) + \cos(x)\right]_0^{\frac{\pi}{2}} \\ &= 2\pi \left(\frac{\pi^3}{12} + \frac{\pi^2}{8}\right) - 2\pi \left(\frac{\pi}{2} - 1\right). \end{aligned}$$

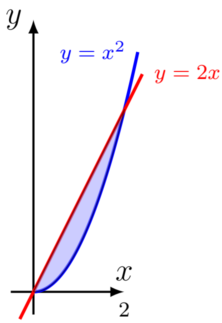
3. $y = \sqrt{x+4}$, $y = 0$ and $x = 0$, around the $x-$axis.
   $y = \sqrt{x+4}$ is the upper half of the parabola $x = y^2 - 4$ which opens to the right with vertex $(-4, 0)$. $y = \sqrt{x+4}$ intersects the $x-$axis at the point $(-4, 0)$ and intersects the $y-$axis at $(0, 2)$. Using Cylindrical shells method, we get



$$V = 2\pi \int_0^2 y[4 - y^2]\, dy = 2\pi \int_0^2 \left(4y - y^3\right)\, dy = 8\pi.$$

4. $y = x^2$ and $y = 2x$, around the $x-$axis.
   $y = x^2$ is a parabola open upward with vertex $(0, 0)$ and $y = 2x$ is a straight line passing through the origin.

$x^2 = 2x \iff x = 0\ ,\ x = 2$. The points of intersection between $y = x^2$ and $y = 2x$ are $(0,0)$ and $(2,4)$.
$y = x^2 \iff x = \pm\sqrt{y}$, where $x = \sqrt{y}$ is the right half of the parabola $y = x^2$ and $x = -\sqrt{y}$ is the left half of the parabola. Using Cylindrical shells method, we get

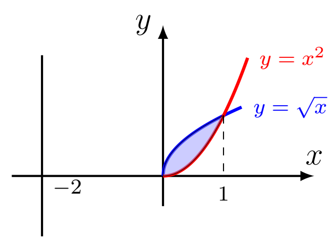


$$V = 2\pi \int_0^4 y\left(\sqrt{y} - \frac{y}{2}\right)\ dy = 2\pi \int_0^4 \left(y^{\frac{3}{2}} - \frac{y^2}{2}\right)\ dy$$
$$= 2\pi \left[\frac{2y^{\frac{5}{2}}}{5} - \frac{y^3}{6}\right]_0^4 = \frac{64}{15}\pi.$$

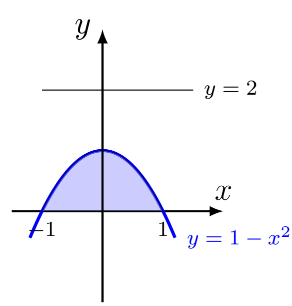
5. $y = \sqrt{x}$ and $y = x^2$, around the line $x = -2$.
   $y = x^2$ is a parabola opens upward with vertex $(0,0)$, and $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$. The points of intersection between $y = x^2$ and $y = \sqrt{x}$.

   $x^2 = \sqrt{x} \iff x = 0\ ,\ x = 1$. Using Cylindrical shells method, we get

   $$V = 2\pi \int_0^1 (x+2)(\sqrt{x} - x^2)\ dx = \frac{49}{30}\pi.$$



6. $y = 1 - x^2$ and $y = 0$, around the line $y = 2$.
   $y = 1 - x^2$ is a parabola opens downward with vertex $(0,1)$ and $y = 0$ is the $x-$axis. $y = 1 - x^2$ intersects $y = 0$ at $x = \pm 1$.
   $y = 1 - x^2 \iff x = \pm\sqrt{1-y}$, where $y = \sqrt{1-y}$ represents the right half of the parabola and $y = -\sqrt{1-y}$ represents the left half.
   Note that the region is symmetric with respect to the $y-$axis. Using Cylindrical shells method, we get



$$\begin{aligned} V &= 2\left(2\pi\int_0^1 (2-y)\sqrt{1-y}\,dy\right) \\ &\overset{u^2=1-y}{=} 8\pi\int_0^1 (u^2+1)u^2 du = \frac{64}{15}\pi. \end{aligned}$$

2.4 Exercises

4-2-1 Find the volume of a ball of radius R.

4-2-2 Find the volume between the sphere of center $(0,0,0)$ and radius R and the sphere of center $(0,0,0)$ and radius $R+r$.

4-2-3 Find the volume of the solid obtained by rotating the region bounded by $y=\sqrt{36-x^2}$, $y=0, x=2, x=4$, about the x-axis.

4-2-4 Sketch the region bounded by the curves and find the volume of the solid generated by revolving the region about the x- or y- axis, as specified below.

1) $y=1-|x|$, $y=0$, revolved around the x-axis,
2) $y=x^2$, $y=2-x$, revolved around the x-axis,
3) $y=|x|$, $y=2-x^2$, revolved around the x-axis,
4) $f(x)=\cos(\frac{\pi}{2}x)$, $y=0$, $x\in[0,1]$, revolved around the x-axis,
5) $x=\sqrt{9-y^2}$, $x=0$, revolved around the y-axis.

4-2-5 Find the volume of the solid obtained by rotating the region bounded by $y=1+\sec(x)$, $y=3$, about the line $y=1$.

4-2-6 Set up an integration to find the volume and draw an illustration each of the solid obtained by rotating the region bounded by $y=0, y=\cos^2 x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$

1) About the x-axis,
2) About the line $y=1$.

4-2-7 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specific axis

1) $y = \cos(x^2)$, $y = 0$, $x = 0$, $x = \sqrt{\frac{\pi}{2}}$ about the y-axis.
2) $y = x^2$, $y = 4 - x^2$ about the y-axis.
3) $x = y^2 + 1$, $x = 2$, about the line $y = -2$.
4) $x^2 - y^2 = 1$, $x = 2$, about the line $y = 3$.
5) $y = x^2 + 1$ and $y = 4 - x^2$ about the line $y = -2$
6) $x^2 - y^2 = 5$, $x = 4$ about the y-axis,
7) $y = \sqrt{x}$, $y = x^2$ about the line $y = 2$,
8) $y = \sqrt{x}$, the x-axis for $0 \leq x \leq 4$ about the line $y = 2$.

4-2-8 Let $R = \{(x, y) \in \mathbb{R}^2;\ \frac{(x-3)^2}{4} + \frac{y^2}{9} \leq 1;\ y > 0\}$. This region is bounded by $x = 1$, $x = 5$, $y = 0$ and the graph of the function $f(x) = 3\sqrt{1 - \dfrac{(x-3)^2}{4}}$. ($R$ is also the region included in the upper half of the ellipse with center $(3, 0)$ and its left vertex is $(1, 0)$, right vertex is $(5, 0)$ and upper vertex is $(3, 3)$).

1) Find the volume of the solid of revolution of R around the x-axis.
2) Find the volume of the solid of revolution of R around the y-axis.
3) Find the volume of the solid of revolution of R around the $x = 1$.
4) Find the volume of the solid of revolution of R around the $x = 6$.
5) Find the volume of the solid of revolution of R around the $y = 4$.
6) Find the volume of the solid of revolution of R around the $y = -2$.

3 Arc Length and Surfaces of Revolution

3.1 Arc Length

Definition 3.1

Let $f\colon I \longrightarrow \mathbb{R}$ be a function. We say that f is continuously differentiable if f is differentiable and f' is itself continuous on I.

Definition 3.2

Let $f\colon [a,b] \longrightarrow \mathbb{R}^+$ be a continuously differentiable function. The length of the curve $(x, f(x))$, for $x \in [a,b]$ is defined by:

$$L_a^b = \int_a^b \sqrt{1 + (f'(x))^2}dx. \tag{3.7}$$

Example 13 :
Let $f\colon [0, \frac{\pi}{4}] \longrightarrow \mathbb{R}$ defined by: $f(x) = \ln(\cos(x))$. The length of the curve defined by f is given by:

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)}dx = \int_0^{\frac{\pi}{4}} \sec(x)dx = \ln(\sqrt{2} + 1).$$

Definition 3.3

Let $f\colon [a,b] \longrightarrow \mathbb{R}^+$ be a continuously differentiable function. Then the arc length function "s" for the graph of f on $[a,b]$ is defined by:

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}dt. \tag{3.8}$$

We have $ds = \sqrt{1 + (f'(x))^2}dx$.

Examples 14 :

1. The arc length of the curve defined by the function $f(x) = \dfrac{x^3}{12} + \dfrac{1}{x}$ on the interval $[1, 2]$ is given by:

$$\begin{aligned} L &= \int_1^2 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\, dx = \int_1^2 \sqrt{\frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}}\, dx \\ &= \int_1^2 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\, dx = \int_1^2 \left(\frac{x^2}{4} + \frac{1}{x^2}\right)\, dx = \frac{13}{12} \end{aligned}$$

2. The arc length of the curve defined by the function $f(x) = \cosh(x)$ on the interval $[0, 2]$ is given by:

$$L = \int_0^2 \sqrt{1 + \sinh^2(x)}\, dx = \int_0^2 \cosh(x)\, dx = \sinh(2).$$

3. Let g be the function defined by: $g(y) = \sqrt{25 - y^2}$ on the interval $[-5, 5]$. The arc length of the curve defined by the function g is equal to half of the perimeter of the circle $x^2 + y^2 = 25$, the arc length is equal to 5π.
$g'(y) = \dfrac{-y}{\sqrt{25 - y^2}}$. Then the arc length of the curve defined by the function g on the interval $[-5, 5]$ is given by:

$$\begin{aligned} L &= \int_{-5}^5 \sqrt{1 + \frac{y^2}{25 - y^2}}\, dy = 5 \int_{-5}^5 \frac{dy}{\sqrt{25 - y^2}} \\ &= 5 \left[\sin^{-1}\left(\frac{y}{5}\right)\right]_{-5}^5 = 5\pi. \end{aligned}$$

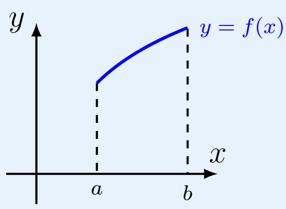
4. The arc length of the curve defined by the function $f(x) = 1 + \dfrac{2}{3} x^{\frac{3}{2}}$ on the interval $[0, 3]$ is:

$$L = \int_0^3 \sqrt{1 + \left(x^{\frac{1}{2}}\right)^2}\ dx = \int_0^3 \sqrt{1+x}\ dx = \int_0^3 (1+x)^{\frac{1}{2}}\ dx$$
$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^3 = \frac{14}{3}$$

3.2 Surfaces of Revolution

Theorem 3.4

Let $f\colon [a,b] \longrightarrow \mathbb{R}^+$ be a continuously differentiable function. The area of the surface generated by revolving the curve $y = f(x)$ around the x-axis denoted by S is given by



$$S = \int_a^b 2\pi|f(x)|\sqrt{1+(f'(x))^2}\,dx. \tag{3.9}$$

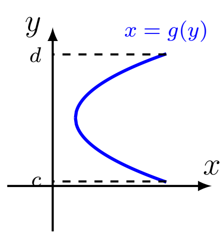
Example 15 :

Let f be the function defined on the interval $[0,1]$ by: $f(x) = \dfrac{x^3}{3}$. The surface of revolution of the graph of f around the $x-$axis is

$$S = 2\pi\int_0^1 \frac{x^3}{3}\sqrt{1+x^4}dx \overset{t^2=1+x^4}{=} \frac{\pi}{3}\int_1^{\sqrt{2}} t^2dt = \frac{\pi}{9}(2\sqrt{2}-1).$$

Remark 15 :

If $x = g(y)$, $y \in [c,d]$ and g continuously differentiable, the surface area generated by revolving the curve of g around the y-axis is given by

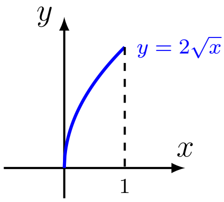


$$S = \int_c^d 2\pi|x|ds = \int_c^d 2\pi|g(y)|ds = \int_c^d 2\pi|g(y)|\sqrt{1+(g'(y))^2}dy.$$

Examples 16 :

1. Consider the function $f(x) = 2\sqrt{x}$ defined on the interval $[0, 1]$.

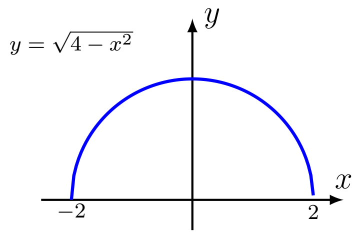
   The surface area generated by revolving the curve defined by the graph of the function f around the $x-$axis is:



$$\begin{aligned} S &= 2\pi \int_0^1 2\sqrt{x}\sqrt{1 + \left[\frac{1}{\sqrt{x}}\right]^2}\, dx = 4\pi \int_0^1 \sqrt{x+1}\, dx \\ &= 4\pi \left[2\frac{(x+1)^{\frac{3}{2}}}{3}\right]_0^1 = \frac{8\pi}{3}\left(2\sqrt{2} - 1\right). \end{aligned}$$

2. Consider the function $f(x) = \sqrt{4 - x^2}$ defined on the interval $[-2, 2]$.

   The surface area generated by revolving the curve defined by the graph of the function f around the $x-$axis is:



$$\begin{aligned} S &= 2\pi \int_{-2}^2 \sqrt{4 - x^2}\,\sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2}\, dx \\ &= 2\pi \int_{-2}^2 \sqrt{4 - x^2}\sqrt{\frac{(4 - x^2) + x^2}{4 - x^2}}\, dx \\ &= 2\pi \int_{-2}^2 \sqrt{4 - x^2}\frac{2}{\sqrt{4 - x^2}}\, dx \\ &= 4\pi \int_{-2}^2 dx = 4\pi\, [x]_{-2}^2 = 16\pi. \end{aligned}$$

Note: It is the surface area of the sphere with radius 2, and it is equal to $4\pi(2)^2 = 16\pi$

3. Let the curve defined on the interval $[1,8]$ by : $y = 2\sqrt[3]{x}$

$y = 2\sqrt[3]{x} \iff \sqrt[3]{x} = \dfrac{y}{2} \iff x = \dfrac{y^3}{8}$.

Let $g(y) = \dfrac{y^3}{8}$, $g'(y) = \dfrac{3}{8}y^2$ and $y \in [2,4]$.

The surface area generated by revolving the curve around the $y-$axis is:

$$\begin{aligned} S &= 2\pi \int_2^4 \frac{y^3}{8}\sqrt{1+\left(\frac{3}{8}y^2\right)^2}\, dy \\ &\overset{t^2=1+\frac{9}{64}y^4}{=} \frac{8\pi}{9}\int_{\frac{\sqrt{13}}{2}}^{\sqrt{37}} t^2 dt = \frac{8\pi}{27}((37)^{\frac{3}{2}} - (\frac{13}{2})^{\frac{3}{2}}). \end{aligned}$$

4. Let the curve defined on the interval $[0,2]$ by: $y = x^2$.

$y = x^2 \iff x = \sqrt{y}$, since $0 \le x \le 2$.

Let $g(y) = \sqrt{y}$ for $y \in [0,4]$, $g'(y) = \dfrac{1}{2\sqrt{y}}$. Then the surface area generated by revolving the curve defined by $y = x^2$, for $x \in [0,2]$ around the $y-$axis is given by:

$$\begin{aligned} S &= 2\pi \int_0^4 \sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\, dy = 2\pi \int_0^4 \sqrt{y}\sqrt{1+\frac{1}{4y}}\, dy \\ &= 2\pi \int_0^4 \sqrt{y+\frac{1}{4}}\, dy = 2\pi \left[\frac{2\left(y+\frac{1}{4}\right)^{\frac{3}{2}}}{3}\right]_0^4 = \frac{17\sqrt{17}-1}{6}\pi. \end{aligned}$$

3.3 Exercises

4-3-1 Find the arc length of the following graphs.

1) $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $x \in [1,5]$

2) $f(x) = \ln(\sin x)$, $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$

3) $f(x) = \cosh x$ on the interval $[0, \ln 2]$

4) $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ on the interval $[1, 2]$

5) $f(x) = \pi + \frac{2}{3}x\sqrt{x}$ on the interval $[0, 8]$

6) $f(x) = \ln|\sec x|$ on the interval $\left[0, \frac{\pi}{4}\right]$

7) $f(x) = \frac{1}{3}\ \left(x^2 + 2\right)^{\frac{3}{2}}$ on the interval $[0, 1]$

8) $f(x) = \dfrac{e^{2x} + e^{-2x}}{4}$ on the interval $[0, 1]$

9) $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $[1, 3]$

10) $f(x) = \frac{1}{3}x^{\frac{3}{2}} - \sqrt{x}$ on the interval $[1, 4]$

4-3-2 Find the length of the following curves

1) $f(x) = e^x$ from $x = 0$ to $x = \dfrac{\ln 3}{2}$.

2) $x = \ln(\cos(y))$, $0 \le y \le \frac{\pi}{3}$,

3) $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$,

4) $x = 1 - e^{-y}$, $0 \le y \le 2$.

4-3-3 Find the area of the surface obtained by revolving the following curves about the x-axis:

1) $y = \sqrt{1 + e^x}$, $0 \le x \le 1$,

2) $y = \dfrac{1}{x}$, $1 \le x \le 2$,

3) $y = \dfrac{x^3}{3}$, $0 \le x \le 1$

4) $y = \sqrt{x}$, $0 \le x \le 4$

5) $y = \sqrt{9 - x^2}$, $0 \le x \le 4$

6) $y = \frac{1}{3}\left(3\sqrt{x} - x^{\frac{3}{2}}\right)$, $1 \leq x \leq 3$

7) $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \leq x \leq 2$

8) $y = \frac{x^4}{4} + \frac{1}{8x^2}$, $1 \leq x \leq 3$

4-3-4 Find the area of the surface obtained by revolving the following curves about the y-axis:

1) $y = 1 - x^2$, $0 \leq x \leq 1$,

2) $y = \frac{x^2}{4} - \frac{\ln(x)}{2}$, $1 \leq x \leq 2$.

CHAPTER 5

PARAMETRIC EQUATIONS AND POLAR COORDINATES

1 Parametric Equations of Plane Curves

1.1 Introduction

The graph of a function $f\colon I \longrightarrow \mathbb{R}$ (I an interval) is an example of plane curve but it is not general enough to represent all types of plane curves, for example a circle or a vertical line segment are not the graph of functions because two distinct points of a graph have different abscissa. In this section we study the trajectory of a point in the plane whose coordinates $(x(t), y(t))$ depend on a parameter t, these are the parametric curves, or curves verifying a Cartesian equation.

1.2 Parametric Equations

Definition 1.1: [Plane Curve]

If f and g are continuous functions on an interval I, the set of ordered pairs $(f(t), g(t))$, $t \in I$ is called a plane curve $\mathcal{C}$.

The equations $x = f(t)$ and $y = g(t)$ are called parametric equations of the curve $\mathcal{C}$ and t is called the parameter.
We can also interpret the curve as the vectorial function
$\gamma\colon I \longrightarrow \mathbb{R}^2$ defined by $\gamma(t) = (f(t), g(t))$, $t \in I$. In this case $\mathcal{C}$ is called the support of the curve γ.

Definition 1.2

1. The curve $\gamma\colon I \longrightarrow \mathbb{R}^2$ is called respectively continuous, differentiable, k-times differentiable, of class $\mathcal{C}^k$, if f and g are continuous, differentiable, k-times differentiable, of class $\mathcal{C}^k$.

2. The Orientation of the curve of the parametric equations $\gamma = (f, g)$ is the direction of movement of the vector γ, for $t \in I$.

Remark 16 :

1. If $\mathcal{C} = \{(x = f(t), y = g(t));\ t \in I\}$ is a curve and the function $f\colon I \longrightarrow J$ is bijective, then $t = f^{-1}(x)$ and the curve is represented by the equation $y = g(t) = g \circ f^{-1}(x)$ and the curve is the graph of the function $y = g \circ f^{-1}(x)$, for $x \in J$.

2. If $\mathcal{C} = \{(x = f(t), y = g(t));\ t \in I\}$ is a curve an the function $g\colon I \longrightarrow J$ is bijective, then $t = g^{-1}(y)$ and the curve is represented by the equation $x = f(t) = f \circ g^{-1}(x)$ and the curve is the graph of the function $x = f \circ g^{-1}(y)$, for $y \in J$.

Examples 1 :

1. The graph of a function $y = f(x)$ is a parametric curve of equation $\gamma(t) = (x(t), y(t)) = (t, f(t))$.

2. A line of equation $y = ax + b$ is the geometric curve of the mapping $\gamma(t) = (t, at + b)$, $t \in \mathbb{R}$, therefore it is parameterizable as in 1).
The parametrization $(x(t), y(t)) = (a, t)$, $t \in \mathbb{R}$ is a parametrization of the vertical line $x = a$.

3. The circle in $\mathbb{R}^2$ of center (a, b) and of radius $r > 0$ is the curve defined by $\{(x, y) \in \mathbb{R}^2;\ (x-a)^2+(y-b)^2 = r^2\}$ and it is parameterized by: $\gamma\colon [0, 2\pi] \longrightarrow \mathbb{R}^2$, where $\gamma(t) = (a + r\cos(t), b + r\sin(t))$.

Remark 17 :

There are infinitely many ways to parametrize a curve.

1. $(x(t), y(t)) = (t, f(t))$ is a parametrization of the graph of the function $y = f(x)$. But $(x(t), y(t)) = (t - a, f(t - a))$ is also a parametrization of this curve.

2. $(a + r\cos(t), b + r\sin(t),\ t \in [0, 2\pi]$ is also a parametrization of the circle of center (a, b) and radius r.

Examples 2 :

1. $x(t) = t + 1$, $y(t) = 2t + 3$, $t \in [-1, 2]$. Then $y = 2x + 1$, $x \in [0, 3]$. The parametric equation represents a straight line.

2. $x(t) = t - 1$, $y(t) = t^2$, $t \in [-1, 3]$. Then $y = (x + 1)^2$, $x \in [-2, 1]$. The parametric equation represents a parabola opens upwards with vertex $(-1, 0)$.

3. $x(t) = 2 + 2\cos t$, $y(t) = -1 + 2\sin(t)$, $t \in [0, 2\pi]$. Then $(x - 2)^2 + (y + 1)^2 = 4$. The parametric equation represents a circle with center $(2, -1)$ and radius 2. It is a closed curve and its direction is counter-clockwise.

4. $x(t) = 1 + 3\cos t$, $y(t) = -1 + 2\sin(t)$, $t \in [0, 2\pi]$. Then $\dfrac{(x-1)^2}{9} + \dfrac{(y+1)^2}{4} = 1$. The parametric equation represents an ellipse with center $(1, -1)$, the endpoints of the major axis are $(4, -1)$, $(-2, -1)$ (its length is 6) and the endpoints of the minor axis are $(1, -3)$, $(1, 1)$ (its length is 4). It is a closed curve and its direction is counter-clockwise.

1.3 Tangent to Parametric Curve

Definition 1.3

Let $\gamma = (f, g) \colon I \longrightarrow \mathbb{R}^2$ be a parametric curve and let $a \in I$ (I an open interval). We assume that $\gamma(t) \neq \gamma(a)$ for t close to a. We say that this curve has tangent at the point $M_0 = (f(a), g(a))$ if the direction of the vector $M_0M_t = \gamma(t) - \gamma(a)$, $(M_t = \gamma(t))$ has a limit when t tends to a. This means that for $t \in I$ close to a $(t \neq a)$, there exists a vector $V(t)$ collinear to the vector M_0M_t such that $\lim\limits_{t \longrightarrow a} V(t) = V \neq 0$. The tangent at $M_0 = \gamma(a)$ to the curve is the line passing through M_0 and parallel to the vector V.

Example 3 :

If $\gamma(t) = (t^2, t^3)$ for $t \in \mathbb{R}$. The tangent to the curve $t \longmapsto \gamma(t)$ at $(0, 0) = \gamma(0)$ is the real axis. Indeed, $\gamma(t) - \gamma(0) = t^2(1, t)$ which is parallel to the vector $V(t) = (1, t)$ and has the limit $(1, 0)$ when t tends to 0.

Theorem 1.4

1. Let $\gamma \colon I \longrightarrow \mathbb{R}^2$ be a plane curve. If γ is differentiable at a and $\gamma'(a) \neq 0$, the curve has a tangent at $M_0 = \gamma(a)$ parallel to the vector $\gamma'(a)$.

2. In general if γ is k-times differentiable at a and $\gamma'(a) = \gamma''(a) = \ldots = \gamma^{(k-1)}(a) = 0$ and $\gamma^{(k)}(a) \neq 0$, then the curve has a tangent at $M_0 = \gamma(a)$ parallel to the vector $\gamma^{(k)}(a)$.

Remark 18 :

1. The slope of the tangent line to a parametric curve if it exists is

$$m = \lim_{t \to t_0} \frac{y'(t)}{x'(t)}. \tag{1.1}$$

2. The tangent line to the parametric curve is horizontal if the slope is equal to zero. In particular if $\dfrac{dy}{dt} = 0$ and $\dfrac{dx}{dt} \neq 0$.

3. The tangent line to the parametric curve is vertical if the slope is equal to ∞. In particular if $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$.

Definition 1.5

Let $\gamma = (f(t), g(t))$ be a parametric curve defined on the interval $I = [a, b]$.

1. If γ is injective, the parametric curve is called simple.

2. If $\gamma(a) = \gamma(b)$, the parametric curve is called closed.

Examples 4 :

1. $x(t) = 1 + 3\cos t$, $y(t) = -1 + 3\sin(t)$, $t \in [0, 2\pi]$. Then $(x-1)^2 + (y+1)^2 = 9$. Since $x'(t) = -3\sin(t)$ and $y'(t) = 3\cos(t)$, the tangent to the curve is parallel to the $x-$axis at the point $(1, 2)$ for $t = \frac{\pi}{2}$ and at the point $(1, -4)$ for $t = \frac{3\pi}{2}$.

   The tangent to the curve is parallel to the $y-$axis at the point $(4, -1)$ for $t = 0$ and at the point $(-2, -1)$ for $t = \pi$.

2. $x(t) = 3 + 3\cos t$, $y(t) = 2 + 2\sin(t)$, $t \in [0, 2\pi]$. Then $\dfrac{(x-3)^2}{9} + \dfrac{(y-2)^2}{4} = 1$. Since $x'(t) = -3\sin(t)$ and $y'(t) = 2\cos(t)$, the tangent to the curve is parallel to the $x-$axis at the point $(3, 5)$ for $t = \frac{\pi}{2}$ and at the point $(3, 0)$ for $t = \frac{3\pi}{2}$.

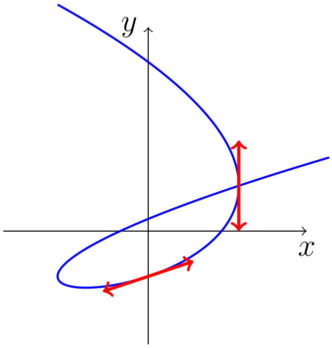
   The tangent to the curve is parallel to the $y-$axis at the point $(6, 2)$ for $t = 0$ and at the point $(0, 2)$ for $t = \pi$.

Examples 5 :

1. The slope of the tangent line to the curve $(x(t) = t^3+1, y(t) = t^4-1)$ at $t = 1$ is $m = \dfrac{y'(1)}{x'(1)} = \dfrac{4}{3}$.

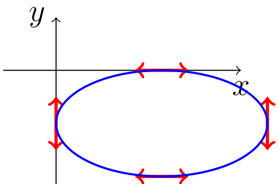
2. Let $x(t) = t^3 - 3t$, $y(t) = t^2 - t - 1$, $t \in \mathbb{R}$.

   The slope of the curve at $t = -1$ is ∞. The tangent line to the curve at $(2, 1)$ is parallel to the $y-$ axis. The slope of the curve at $(0, -1)$ for $t = 0$ is $\frac{1}{3}$. The equation of the tangent line to the curve at $(0, -1)$ is $y = \frac{1}{3}x - 1$.



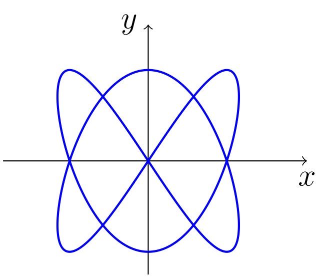
$(t^3 - 3t, t^2 - t - 1)$

3. Let $x(t) = 2 + 2\cos t$, $y(t) = -1 + \sin(t)$, $t \in [0, 2\pi]$.
   The slope of the curve is $m = \dfrac{\cos(t)}{-2\sin(t)}$. The points of the curve at which the tangent line is vertical are $(4, -1)$ and $(0, -1)$. The points of the curve at which the tangent line is horizontal are $(2, 0)$ and $(2, -2)$.



$(2 + 2\cos(t), -1 + \sin(t))$

Example 6 :
$(x(t), y(t)) = (\sin(2t), \cos(3t))$, for $t \in \mathbb{R}$. The curve is periodic of period 2π.
$x(-t) = -x(t)$, $y(-t) = -y(t)$, thus we study the curve on the interval $[0, \frac{\pi}{2}]$ and we take a symmetry with respect to the origin.
$x(\pi - t) = x(-t) = -x(t)$,
$y(\pi - t) = y(t)$, thus we study the curve on $[0, \frac{\pi}{2}]$ and we take a symmetry with respect to the axis (oy) and a symmetry with respect to the origin. $M_0 = (0, 0)$, $f'(0) = (2, 3)$, $f''(0) = (0, 0)$ and
$f^{(3)}(0) = (-8, -27)$. $(0, 0)$ is an inflection point.



$(x(t), y(t)) = (\sin(2t), \cos(3t))$

1.4 Arc Length of Parametric Curve

Definition 1.6

Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^2$ be a smooth curve. The arc length of the curve γ is defined by:

$$L(\gamma) = \int_a^b ||\gamma'(t)||dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt. \qquad (1.2)$$

Remark 19 :
The expression of $L(\gamma)$ is invariant by change of parametrization of class $\mathcal{C}^1$ of the curve. Indeed if $\varphi\colon [\alpha,\beta] \longrightarrow [a,b]$ is a strictly increasing function of class $\mathcal{C}^1$. Set $\psi(s) = \gamma(\varphi(s))$, $\psi'(s) = \gamma'(\varphi(s)).\varphi'(s)$, $||\psi'(s)|| = ||\gamma'(\varphi(s))||\varphi'(s)$. $(\varphi'(s) \geq 0)$. Thus from the change of variables formula $(\varphi(\alpha) = a,\ \varphi(\beta) = b)$ we have:

$$\int_\alpha^\beta ||\psi'(s)||ds = \int_a^b ||\gamma'(t)||dt.$$

The same result if φ is strictly decreasing.

Examples 7 :

1. If the curve is defined in Cartesian coordinates $\gamma\colon [a,b] \longrightarrow \mathbb{R}^2$, with $\gamma(t) = (t, y(t))$, $t \in [a,b]$ and y of class $\mathcal{C}^1$.

$$L(\gamma) = \int_a^b \sqrt{1 + (y'(t))^2}dt.$$

   For example, if $y = \tan(t)$, $t \in [0, \frac{\pi}{4}]$, then $L(\gamma) = \ln(1+\sqrt{2})$.

2. If $\gamma(t) = (\cos(t), \sin(t))$, $t \in [0, 4\pi]$. $L(\gamma) = 4\pi$.

Example 8 :
Find the arc length of the following parametric curves:

1. Consider the parametric curve $x(t) = \frac{1}{3}t^3 + 1,\ y(t) = \frac{1}{2}t^2 + 2$, $t \in [0, 2]$. The arc length of this curve is

$$\begin{aligned} L &= \int_0^2 \sqrt{(t^2)^2 + (t)^2}dt = \frac{1}{2}\int_0^2 (t^2+1)^{\frac{1}{2}}(2t)\ dt \\ &= \frac{1}{2}\left[\frac{2}{3}(t^2+1)^{\frac{3}{2}}\right]_0^2 = \frac{1}{3}\left(5\sqrt{5}-1\right). \end{aligned}$$

2. Consider the parametric curve $x(t) = e^t\cos(t),\ y(t) = e^t\sin(t)$, $t \in [0, \pi]$. The arc length of this curve is

$$\begin{aligned} L &= \int_0^\pi \sqrt{\left[e^t(\cos t - \sin(t))\right]^2 + \left[e^t(\cos(t)+\sin(t))\right]^2}\ dt \\ &= \int_0^\pi e^t\sqrt{(\cos(t)-\sin(t))^2 + (\cos(t)+\sin(t))^2}dt \\ &= \int_0^\pi \sqrt{2}e^t\ dt = \sqrt{2}(e^\pi - 1). \end{aligned}$$

1.5 Surface Area Generated by Revolving a Parametric Curves

Theorem 1.7

If $\gamma(t) = (x(t), y(t))$, $t \in [a, b]$ is a smooth parametric curve:

1. The surface area generated by revolving the curve γ around the $x-$axis is

$$S = 2\pi\int_a^b |y(t)|\sqrt{(x'(t))^2 + (y'(t))^2}dt. \tag{1.3}$$

2. The surface area generated by revolving γ around the $y-$axis is

$$S = 2\pi\int_a^b |x(t)|\sqrt{(x'(t))^2 + (y'(t))^2}dt. \tag{1.4}$$

Examples 9 :

The surface area generated by revolving the following parametric curves:

1. $x(t) = t$, $y(t) = \dfrac{t^3}{3} + \dfrac{1}{4t}$, $t \in [1,2]$, around the $x-$axis.

$$\begin{aligned} S &= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{1}{4t}\right)\sqrt{1 + \left(t^2 - \frac{1}{4t^2}\right)^2}\, dt \\ &= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{1}{4t}\right)\sqrt{t^4 + \frac{1}{2} + \frac{1}{16t^4}}\, dt \\ &= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{1}{4t}\right)\sqrt{\left(t^2 + \frac{1}{4t^2}\right)^2}\, dt \\ &= 2\pi \int_1^2 \left(\frac{t^3}{3} + \frac{1}{4t}\right)\left(t^2 + \frac{1}{4t^2}\right)\, dt \\ &= 2\pi \int_1^2 \left(\frac{t^5}{3} + \frac{t}{3} + \frac{1}{16t^3}\right)\, dt = \frac{509\pi}{64} \end{aligned}$$

2. $x(t) = 4\sqrt{t}$, $y(t) = \frac{1}{2}t^2 + \frac{1}{t}$, $t \in [1,4]$, around the $y-$axis.

$$\begin{aligned} S &= 2\pi \int_1^4 4\sqrt{t}\sqrt{\left(\frac{2}{\sqrt{t}}\right)^2 + \left(t - \frac{1}{t^2}\right)^2}\, dt \\ &= 2\pi \int_1^4 4\sqrt{t}\sqrt{\left(t + \frac{1}{t^2}\right)^2}\, dt \\ &= 2\pi \int_1^4 4\sqrt{t}\left(t + \frac{1}{t^2}\right)\, dt = \frac{288\pi}{5} \end{aligned}$$

Example 10 :

Find the surface area generated by revolving the following parametric curves:

1. $x(t) = 3t$, $y = 4t$, $t \in [0,2]$, around the $x-$axis.
2. $x(t) = t$, $y = 2t$, $t \in [0,4]$, around the $y-$axis.

1.6 Exercises

5-1-1 Explain when a differentiable parametric curve $\gamma(t) = (x(t), y(t))$ has a

1) horizontal tangent at $\gamma(a)$,
2) vertical tangent at $\gamma(a)$.

5-1-2 Find the length of the following curves:

1) $x = 2 + 3t, y = \cosh(3t)$, $0 \leq t \leq 1$,
2) $x = e^t + e^{-t}$, $y = 5 - 2t$ for $0 \leq t \leq 3$,

5-1-3 Find the area of the surface obtained by rotating the curve $x = t^3, y = t^2$ for $0 \leq t \leq 1$ around the $x-$axis.

5-1-4 Find the area of the surface obtained by rotating the curve about the x-axis:

1) $E = \{(x, y) \in \mathbb{R}^2;\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ y \geq 0\}$,
2) $\mathcal{C} = \{(x, y) \in \mathbb{R}^2;\ x^2 + (y - b)^2 = a^2\}$, $0 < a < b$.

2 Polar Coordinates

In the rectangular coordinates system the ordered pair (a, b) represents a point, where "a" is the $x-$coordinate and "b" is the $y-$coordinate.

The polar coordinates system can be used also to represents points in the plane. The **pole** in the polar coordinates system is the origin in the rectangular coordinates system, and the **polar axis** is the directed half-line (the non-negative part of the $x-$axis).

If P is any point in the plane different from the origin, then its polar coordinates consists of two components r and θ, where r is the algebraic distance between P and the pole O, and θ is the measure of an angle determined by the polar axis and OP.

Note: The polar coordinates of a point is not unique, if $P = (r, \theta)$ then other representations are:

1. $P = (r, \theta + 2n\pi)$, where $n \in \mathbb{Z}$.

2. $P = (-r, \theta + \pi + 2n\pi)$, where $n \in \mathbb{Z}$.

Remark 20 :
The polar coordinates (r, θ) and the rectangular coordinates (x, y) of a point P are related by:

$$x = r\cos\theta, \quad y = r\sin(\theta)$$

Examples 11 :

1. If $(r, \theta) = \left(2, \frac{\pi}{2}\right)$, then its other polar coordinates are $\left(2, \frac{\pi}{2} + 2k\pi\right)$ or $\left(-2, \frac{3\pi}{2} + 2n\pi\right)$, $k, n \in \mathbb{Z}$.

2. If $(r, \theta) = \left(-3, \frac{5\pi}{4}\right)$ then its other polar coordinates are $\left(-3, \frac{5\pi}{4} + 2k\pi\right)$ and $\left(3, \frac{\pi}{4} + 2n\pi\right)$, $k, n \in \mathbb{Z}$.

3. The rectangular coordinates (x, y) of the point $(r, \theta) = (-5, \pi)$ are $(x, y) = (5, 0)$.

4. The polar coordinates of the point $\left(2\sqrt{3}, -2\right)$ are $(4, -\frac{\pi}{6} + 2k\pi)$, $k \in \mathbb{Z}$ or $(-4, \frac{5\pi}{6} + 2k\pi)$, $k \in \mathbb{Z}$

5. The rectangular coordinates of the point $(r, \theta) = \left(2, \frac{\pi}{2}\right)$ are $(x, y) = (0, 2)$.

6. The polar coordinates of the point $\left(\sqrt{2}, \sqrt{2}\right)$ are $\left(2, \frac{\pi}{4} + 2k\pi\right)$, $k \in \mathbb{Z}$ or $\left(-2, \frac{5\pi}{4} + 2k\pi\right)$, $k \in \mathbb{Z}$.

2.1 Exercises

5-2-1 Find the rectangular coordinates of the following points

1) $(3, \frac{3\pi}{4})$, 3) $(2, \frac{7\pi}{6})$, 5) $(-2, \frac{8\pi}{3})$.
2) $(-3, \frac{3\pi}{4})$, 4) $(-2, \frac{7\pi}{6})$,

5-2-2 Find the polar coordinates of the following points with $0 \leq \theta < 2\pi$ and $r > 0$

1) $(-3, -3)$, 2) $(1, -\sqrt{3})$, 3) $(3, 3)$, 4) $(-\sqrt{3}, 1)$.

5-2-3 Find the polar coordinates of the following points with $0 \leq \theta < 2\pi$ and $r < 0$

1) $(-3, -3)$, 2) $(1, -\sqrt{3})$, 3) $(3, 3)$, 4) $(-\sqrt{3}, 1)$.

3 Polar Curves

Definition 3.1

A parametric curve $t \longmapsto \gamma(t)$, $(t \in I)$ is called a polar curve if for any $t \in I$, $\gamma(t)$ is determined by a polar coordinates $(r(t), \theta(t))$. In which follows, we study the polar curves with equation $r = f(\theta)$.

A curve in polar coordinates can be studied in Cartesian coordinates by the change of coordinates $x(t) = r(t)\cos(\theta(t))$, $y(t) = r(t)\sin(\theta(t))$.

Examples 12 :

1. The straight lines:

   • Lines passing through the pole:

   Any straight line passing through the pole has the form $\theta = \theta_0$, where θ_0 is the angle between the straight line and the polar axis.
   $\theta = \theta_0 \Rightarrow \tan(\theta) = \tan(\theta_0) \Rightarrow \frac{y}{x} = \tan(\theta_0) \Rightarrow y = \tan(\theta_0)\ x.$

The straight line $\theta = \theta_0$ is passing through the pole with a slope equals to $\tan(\theta_0)$.

For example the equation $\theta = \dfrac{\pi}{4}$ is the equation of a straight line passing through the pole with a slope equals to $\tan\left(\frac{\pi}{4}\right) = 1$. Therefore its equation in $xy-$form is $y = x$.

- Lines perpendicular to the polar axis:

Any straight line perpendicular to the polar axis has the form $r = a\ \sec(\theta)$, where $a \in \mathbb{R}^*$ and $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.
$r = a\ \sec(\theta) \Rightarrow r = \dfrac{a}{\cos(\theta)} \Rightarrow r\cos(\theta) = a \Rightarrow x = a.$
The straight line $r = a\ \sec(\theta)$ is perpendicular to the polar axis at the point $(r, \theta) = (a, 0)$

For example the equation $r = 3\sec(\theta)$ is a straight line perpendicular to the polar axis and passing through the point $(r, \theta) = (3, 0)$. Therefore its equation in $xy-$form is $x = 3$.

The equation $r = -2\csc(\theta)$ is a straight line parallel to the polar axis and passing through the point $(r, \theta) = \left(-2, \frac{\pi}{2}\right)$. Therefore its equation in the $xy-$form is $y = -2$.

- Lines parallel to the polar axis:

Any straight line parallel to the polar axis has the form $r = a\ \csc(\theta)$, where $a \in \mathbb{R}^*$ and $\theta \in (0, \pi)$.
$r = a\ \csc(\theta) \Rightarrow r = \dfrac{a}{\sin(\theta)} \Rightarrow r\sin(\theta) = a \Rightarrow y = a.$
The straight line $r = a\ \sec(\theta)$ is parallel to the polar axis and passing through the point $(r, \theta) = \left(a, \dfrac{\pi}{2}\right)$.

2. Circles:

   - Circles of the form $r = a$, where $a \in \mathbb{R}^*$.
     The equation $r = a$ represents a circle with center $(0, 0)$ and radius equals $|a|$.
   - Circles of the form $r = a\sin(\theta)$, where $a \in \mathbb{R}^*$ and $0 \leq \theta \leq \pi$.

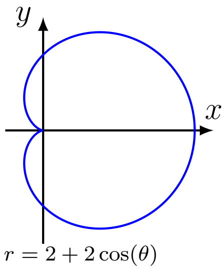
$x = a\sin(\theta)\cos(\theta) = \dfrac{a}{2}\sin(2\theta)$, $y = a\sin^2(\theta) = \dfrac{a}{2} - \dfrac{a}{2}\cos(2\theta)$.
Then the equation $r = a\sin(\theta)$, where $a \in \mathbb{R}^*$ and $0 \le \theta \le \pi$ represents a circle with center $\left(0, \dfrac{a}{2}\right)$ and radius equals to $\dfrac{|a|}{2}$.
$r = 2\sin(\theta)$ represents a circle with center $(0, 1)$ and radius equals to 1

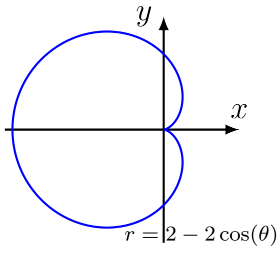
- Circles of the form $r = a\cos(\theta)$, where $a \in \mathbb{R}^*$ and $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$.
  $x = a\cos^2(\theta) = \dfrac{a}{2} + \dfrac{a}{2}\cos(2\theta)$, $y = a\sin(\theta)\cos(\theta) = \dfrac{a}{2}\sin(2\theta)$.
  Then the equation $r = a\cos(\theta)$, where $a \in \mathbb{R}^*$ and $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$ represents a circle with center $\left(\dfrac{a}{2}, 0\right)$ and radius equals to $\dfrac{|a|}{2}$.
  $r = 2\cos(\theta)$ represents a circle with center $(1, 0)$ and radius equals to 1

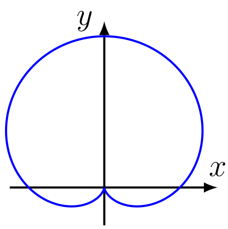
3. The Limaçon curves:

   The general form of a Limaçon curve is $r(\theta) = a + b\sin(\theta)$ or $r(\theta) = a + b\cos(\theta)$, where $a, b \in \mathbb{R}^*$ and $0 \le \theta \le 2\pi$

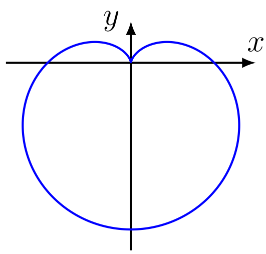
   - Cardioid (Heart-shaped). It has the form $r(\theta) = a \pm a\sin(\theta)$ or $r(\theta) = a \pm a\cos(\theta)$, where $a \in \mathbb{R}^*$ and $0 \le \theta \le 2\pi$







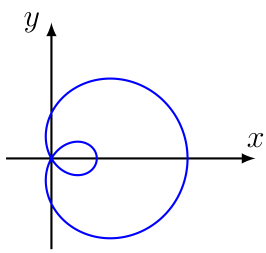
$r = 2 + 2\cos(\theta)$



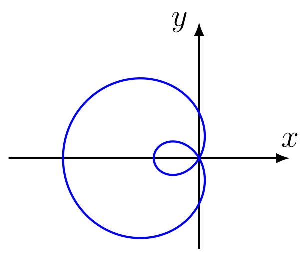
$r = 2 - 2\cos(\theta)$

- Limaçon with inner loop:
  It has the form $r(\theta) = a + b\sin(\theta)$ or $r(\theta) = a + b\cos(\theta)$, where $a, b \in \mathbb{R}^*$, $|a| < |b|$ and $0 \leq \theta \leq 2\pi$

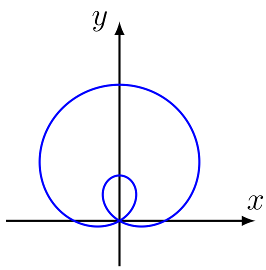
  Note: Note that $|a| < |b|$ in this case.



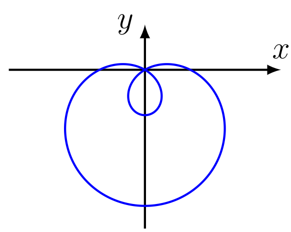
$r = 2 + 2\cos(\theta)$



$r = 2 - 2\cos(\theta)$

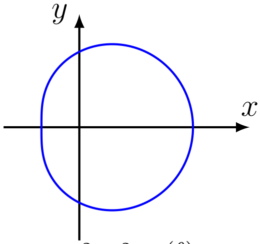


$r = 2 + 2\cos(\theta)$

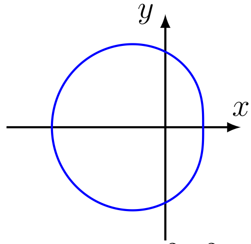


$r = 2 - 2\cos(\theta)$

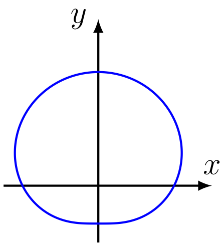
- Dimpled Limaçon:
  It has the form $r(\theta) = a + b\sin(\theta)$ or $r(\theta) = a + b\cos(\theta)$, where $a, b \in \mathbb{R}^*$, $|a| > |b|$ and $0 \leq \theta \leq 2\pi$



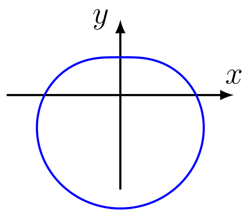
$r = 2 + 2\cos(\theta)$



$r = 2 - 2\cos(\theta)$



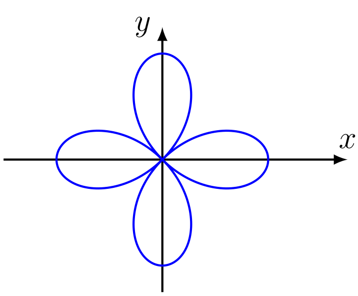
$r = 2 + 2\cos(\theta)$



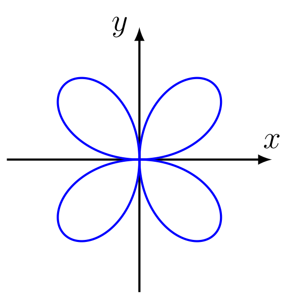
$r = 2 - 2\cos(\theta)$

4. Rose curves:
   It has the form $r(\theta) = a\cos(n\theta)$ or $r(\theta) = a\sin(n\theta)$, where $a \in \mathbb{R}^*$, $n \in \mathbb{N}$ and $n \geq 2$

   - **n is even**: In this case the number of loops (or leaves) is $2n$. For example: $r(\theta) = 2\cos(2\theta)$ or $r(\theta) = 2\sin(2\theta)$, $0 \leq \theta \leq 2\pi$. The number of loops (or leaves) equals 4.

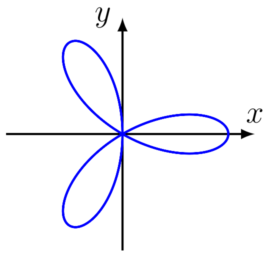


$r = 2 + 2\cos(\theta)$

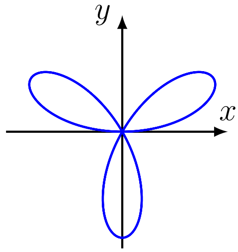


$r = 2 - 2\cos(\theta)$

   - **n is odd**: In this case the number of loops (or leaves) is n. For example: $r(\theta) = 2\cos(3\theta)$ or $r(\theta) = 2\sin(3\theta)$, $0 \leq \theta \leq \pi$ The number of loops (or leaves) equals 3.



$r = 2 + 2\cos(\theta)$



$r = 2 - 2\cos(\theta)$

3.1 Tests of Symmetry

1. If $r(\theta) = r(-\theta)$, the curve is symmetric with respect to the polar axis (the $x-$axis).

   For example, the circle $r = 4\cos(\theta)$ and the cardioid $r = 2+2\cos(\theta)$ are both symmetric with respect to the polar axis.

2. If $r(\theta) = -r(-\theta)$ or $r(\theta) = r(\pi - \theta)$, the curve is symmetric with respect to the $y-$ axis.

   For example the circle $r = 4\sin(\theta)$ and the cardioid $r = 2+2\sin(\theta)$ are both symmetric with respect to the $y-$ axis.

3. If $r(\theta) = r(\pi + \theta)$, the curve is symmetric with respect to the pole.

   For example the rose curve $r = \sin(2\theta)$ is symmetric with respect to the pole.

3.2 Slope of the Tangent Line for Polar Curves

Definition 3.2

If $r = r(\theta)$ is a smooth polar curve, then the slope of the tangent line to the curve $r(\theta)$ at the point $r(\alpha)$ (if it exists) is

$$m = \lim_{\theta\to\alpha} \frac{dy}{dx} = \lim_{\theta\to\alpha} \frac{r(\theta)\cos(\theta) + r'(\theta)\sin(\theta)}{-r(\theta)\sin(\theta) + r'(\theta)\cos(\theta)}. \quad (3.5)$$

Notes:

1. If $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$, the tangent line to $r = r(\theta)$ is horizontal,
2. If $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$, the tangent line to $r = r(\theta)$ is vertical.

Example 13 :

1. Let $r(\theta) = 2\sin(\theta)$, $\theta \in [0, \pi]$.
   $x(\theta) = \sin(2\theta)$ and $\frac{dx}{d\theta} = 2\cos(2\theta)$, $y(\theta) = 2\sin^2(\theta)$ and $\frac{dy}{d\theta} = 2\sin(2\theta)$.

   The tangent line to the curve is vertical if and only if $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} \neq 0$. Thus $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$.
   The points of the curve $r(\theta) = 2\sin(\theta)$, $0 \leq \theta \leq \pi$ at which the tangent line to r is vertical are $(\sqrt{2}, \frac{\pi}{4})$ and $(\sqrt{2}, \frac{3\pi}{4})$.

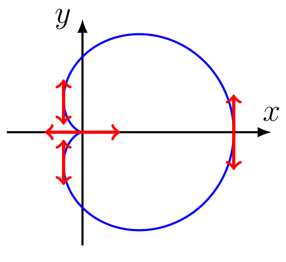
   The tangent line to $r = r(\theta)$ is horizontal if $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} \neq 0$. Thus $\theta = 0$, $\theta = \frac{\pi}{2}$ or $\theta = \pi$.
   The points of the curve $r(\theta) = 2\sin(\theta)$, $0 \leq \theta \leq \pi$ at which the tangent line to r is horizontal are $(0, 0)$, and $(0, 2)$.

2. Consider the polar curve $r(\theta) = 1 + \cos(\theta)$, $\theta \in [0, 2\pi]$.

   $x(\theta) = (1 + \cos\theta)\cos(\theta)$, $\dfrac{dx}{d\theta} = -\sin(\theta)(1 + 2\cos(\theta))$,
   $y(\theta) = (1 + \cos(\theta))\sin(\theta)$ and
   $\dfrac{dy}{d\theta} = \cos(\theta) + \cos(2\theta) = (2\cos(\theta) - 1)(\cos(\theta) + 1)$.
   $\dfrac{dx}{d\theta} = 0 \iff \theta = 0, \pi, 2\pi, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$.
   $\dfrac{dy}{d\theta} = 0 \iff \theta = \pi, \dfrac{\pi}{3}, \dfrac{5\pi}{3}$.
   The slope at the point $r(\pi)$ is

$$\begin{aligned} m &= \lim_{\theta\to\pi} \frac{y'(\theta)}{x'(\theta)} = \lim_{\theta\to\pi} \frac{(2\cos(\theta)-1)(\cos(\theta)+1)}{-\sin(\theta)(1+2\cos(\theta))} \\ &= \lim_{\theta\to\pi} \frac{(2\cos(\theta)-1)(\cos(\theta)+1)}{-\sin(\theta)(1+2\cos(\theta))} = 0. \end{aligned}$$

   The tangent line to the curve $r = r(\theta)$ is horizontal at the points $(0, 0)$, $(\frac{3}{4}, \frac{\sqrt{3}}{4})$ and $(\frac{3}{4}, -\frac{\sqrt{3}}{4})$
   The tangent line to the curve $r = r(\theta)$ is vertical at the points $(2, 0)$, $(-\frac{1}{4}, \frac{\sqrt{3}}{4})$ and $(-\frac{1}{4}, -\frac{\sqrt{3}}{4})$



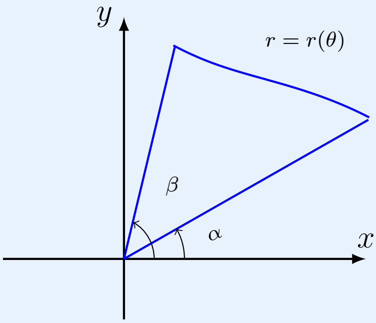
$r = 1 + \cos\theta$

3.3 Area Between Polar Curves

Theorem 3.3

Let $r\colon [\alpha, \beta] \to \mathbb{R}^+$ be a continuous function, where $0 \le \alpha < \beta \le 2\pi$ (generally $0 < \beta - \alpha \le 2\pi$). Then the area of the region bounded by the curve $r(\theta)$, where $\theta \in [\alpha, \beta]$, is equal to

$$A = \int_\alpha^\beta \frac{1}{2} r^2(\theta) d\theta.$$



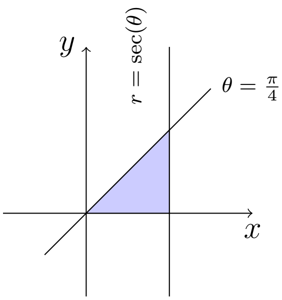
Examples 14 :

1. Let $r = \sec(\theta)$. The area of the region bounded by the curve and the straight lines $\theta = 0$ and $\theta = \frac{\pi}{4}$ is

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sec(\theta))^2 d\theta = \frac{1}{2}\left[\tan(\theta)\right]_0^{\frac{\pi}{4}} = \frac{1}{2}.$$

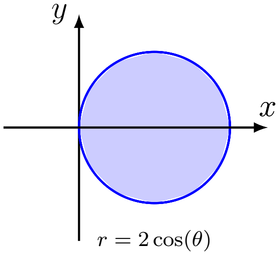
(The area is the area of the triangle of base 1 and height 1).

Note that $r = \sec(\theta)$ is a straight line perpendicular to the polar axis at the point $(r, \theta) = (1, 0)$, $\theta = 0$ is the polar axis and $\theta = \frac{\pi}{4}$ is a straight line passing the pole with a slope equals 1 (in fact it is the line $y = x$).



2. Let $r = 2\cos(\theta)$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$:

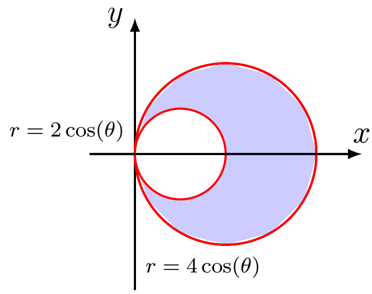
The polar curve $r = 2\cos(\theta)$ is a circle with center $(1, 0)$ and radius 1. The area inside the curve is:



$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^2(\theta)\ d\theta \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[1+\cos(2\theta)\right]\ d\theta \\ &= \left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi. \end{aligned}$$

3. Let $r = 4\cos(\theta)$ and $r = 2\cos(\theta)$.

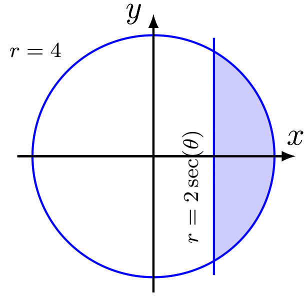
   Note that $r = 4\cos(\theta)$ is a circle with center $(2, 0)$ and radius 2 and the curve $r = 2\cos(\theta)$ is a the circle with center $(1, 0)$ and radius 1. The area inside the curve $r = 4\cos(\theta)$ and outside the curve $r = 2\cos(\theta)$ is:



$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos(\theta))^2 d\theta - \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos(\theta))^2 d\theta \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 12\cos^2(\theta)\ d\theta = 6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[1+\cos(2\theta)\right]\ d\theta \\ &= 3\left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 3\pi \end{aligned}$$

4. Let $r = 4$ and $r = 2\sec(\theta)$. Note that $r = 4$ is a circle with center $(0, 0)$ and radius equals 4 and $r = 2\sec(\theta)$ is a straight line perpendicular to the polar axis (it is the line of equation $x = 2$).

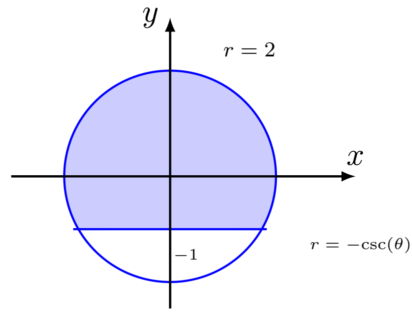
$2\sec(\theta) = 4 \iff \cos(\theta) = \frac{1}{2}$, then $\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$. The area A inside the curve $r = 4$ and at the right of the curve $r = 2\sec(\theta)$ is symmetric with respect to the polar axis, then



$$\begin{aligned} A &= \int_0^{\frac{\pi}{3}} 4^2 d\theta - \int_0^{\frac{\pi}{3}} (2\sec(\theta))^2 d\theta \\ &= \frac{16\pi}{3} - 4\left[\tan(\theta)\right]_0^{\frac{\pi}{3}} = \frac{16\pi}{3} - 4\sqrt{3}. \end{aligned}$$

5. Consider the polar curves $r = 2$ and $r = -\csc(\theta)$.
The curve $r = 2$ is the circle with center $(0, 0)$ and radius 2, $r = -\csc(\theta)$ is the straight line parallel to the polar axis and of equation $y = -1$.

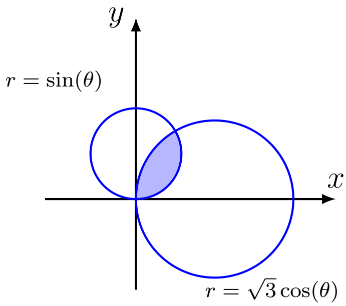
$-\csc(\theta) = 2 \iff \sin(\theta) = -\frac{1}{2}$, then $\theta = -\frac{\pi}{6}$. The angle of intersection between $r = 2$ and $r = -\csc(\theta)$ is $\theta = -\frac{5\pi}{6}$. The area A inside the polar curve $r = 2$ and above the curve $r = -\csc(\theta)$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$ and



$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} (-\csc(\theta))^2 d\theta + \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (2)^2 d\theta\right) \\ &= \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} \csc^2(\theta)\ d\theta + 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta = \sqrt{3} + \frac{2\pi}{3}. \end{aligned}$$

6. Let $r = \sqrt{3}\cos(\theta)$ and $r = \sin(\theta)$. The curve $r = \sqrt{3}\cos(\theta)$ is a circle with center $\left(\frac{\sqrt{3}}{2}, 0\right)$ and radius $\frac{\sqrt{3}}{2}$. The curve $r = \sin(\theta)$ is a circle with center $\left(0, \frac{1}{2}\right)$ and radius $\frac{1}{2}$.

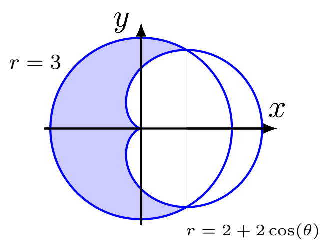
$\sqrt{3}\cos(\theta) = \sin(\theta) \iff \tan(\theta) = \frac{1}{\sqrt{3}}$, then $\theta = \frac{\pi}{3}$, which is the angle of intersection between $r = \sqrt{3}\cos(\theta)$ and $r = \sin(\theta)$. The area of the common region between $r = \sqrt{3}\cos(\theta)$ and $r = \sin(\theta)$ is



$$\begin{aligned} A &= \frac{1}{2}\int_0^{\frac{\pi}{3}} (\sin(\theta))^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos(\theta))^2 d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1}{2}[1 - \cos(2\theta)]\ d\theta + \frac{3}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}[1 + \cos(2\theta)]\ d\theta \\ &= \frac{1}{4}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{3}} + \frac{3}{4}\left[\theta + \frac{\sin(2\theta)}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{5\pi}{24} - \frac{\sqrt{3}}{4}. \end{aligned}$$

7. Let $r = 3$ and $r = 2 + 2\cos(\theta)$. ($r = 2 + 2\cos(\theta)$ is a cardioid.)

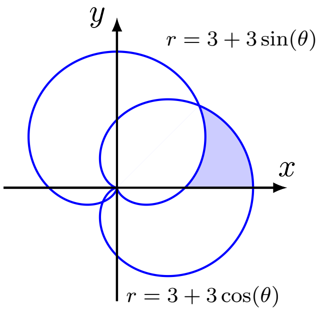
   $2 + 2\cos(\theta) = 3 \iff \cos(\theta) = \dfrac{1}{2}$, then $\theta = \frac{\pi}{3}$. The angles of intersection between $r = 3$ and $r = 2 + 2\cos(\theta)$ are $\theta = \frac{5\pi}{3}$ or $-\frac{\pi}{3}$. Since the desired the area inside $r = 3$ and outside $r = 2 + 2\cos(\theta)$ is symmetric with respect to the polar axis, then this area is equal to:



$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (3)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (2 + 2\cos(\theta))^2 d\theta\right) \\ &= \int_{\frac{\pi}{3}}^{\pi} [9 - (4 + 8\cos(\theta) + 4\cos^2(\theta))]\ d\theta \\ &= \int_{\frac{\pi}{3}}^{\pi} [3 - 8\cos(\theta) - 2\cos(2\theta)]\ d\theta = 2\pi + \frac{9\sqrt{3}}{2}. \end{aligned}$$

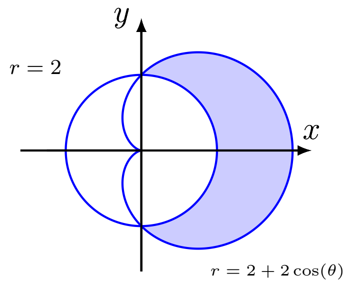
8. Let $r = 3 + 3\cos(\theta)$ and $r = 3 + 3\sin(\theta)$.

$\theta = \dfrac{\pi}{4}$ is the solution of the quation $3 + 3\cos(\theta) = 3 + 3\sin(\theta)$. The other angle of intersection between $r = 3 + 3\cos(\theta)$ and $r = 3 + 3\sin(\theta)$ is $\theta = \frac{5\pi}{4}$. The area inside $r = 3+3\cos(\theta)$, outside $r = 3+3\sin(\theta)$ and at the first quadrant is:



$$\begin{aligned} A &= \frac{1}{2}\int_0^{\frac{\pi}{4}} (3+3\cos(\theta))^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} (3+3\sin(\theta))^2 d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[18\cos(\theta) - 18\sin(\theta) + 9\cos^2(\theta) - 9\sin^2(\theta)\right] d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} [18\cos(\theta) - 18\sin(\theta) + 9\cos(2\theta)]\, d\theta = \frac{18}{\sqrt{2}} - \frac{27}{4}. \end{aligned}$$

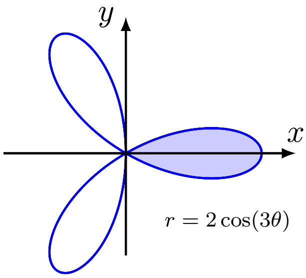
9. Let $r = 2 + 2\cos(\theta)$ and $r = 2$.
Note that $r = 2$ is a circle with center $(0,0)$ and radius 2 and $r = 2 + 2\cos(\theta)$ is a cardioid.
$2 + 2\cos(\theta) = 2 \iff \cos(\theta) = 0$, then $\theta = \dfrac{\pi}{2}$. The other angle of intersection between $r = 2$ and $r = 2+2\cos(\theta)$ is $\theta = \dfrac{3\pi}{2}$. The area inside $r = 2 + 2\cos(\theta)$ and outside $r = 2$ is symmetric with respect to the polar axis and equal to



$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2+2\cos(\theta))^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} (2)^2 d\theta\right) \\ &= \int_0^{\frac{\pi}{2}} \left(4 + 8\cos(\theta) + 4\cos^2(\theta) - 4\right)\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(8\cos(\theta) + 2(1+\cos(2\theta))\right)\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(2 + 8\cos(\theta) + 2\cos(2\theta)\right)\, d\theta) = \pi + 8. \end{aligned}$$

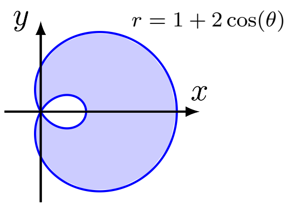
10. Consider the curve $r = 2\cos(3\theta)$.
The rose curve $r = 2\cos(3\theta)$, $0 \le \theta \le \pi$ starts at $(r, \theta) = (2, 0)$ and reaches the pole when $r = 0$.

$r = 0 \iff 2\cos(3\theta) = 0$, then $\theta = \frac{\pi}{6}$. The area inside one leaf of the rose curve $r = 2\cos(3\theta)$ is symmetric with respect to the polar axis, then



$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}} (2\cos(3\theta))^2 d\theta\right) = 4\int_0^{\frac{\pi}{6}} \cos^2(3\theta)\ d\theta \\ &= 4\int_0^{\frac{\pi}{6}} \frac{1}{2}(1+\cos(6\theta))\ d\theta = 2\int_0^{\frac{\pi}{6}} (1+\cos(6\theta))\ d\theta = \frac{\pi}{3}. \end{aligned}$$

11. Consider the curve $r = 1 + 2\cos(\theta)$.
$r = 0 \iff 1 + 2\cos(\theta) = 0$, then $\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$. The interior loop starts at $\theta = \dfrac{2\pi}{3}$ and ends at $\theta = \dfrac{4\pi}{3}$. The area between the loops of the curve $r = 1 + 2\cos(\theta)$ is:



$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_0^{\frac{2\pi}{3}} (1+2\cos(\theta))^2 d\theta - \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} (1+2\cos(\theta))^2 d\theta\right) \\ &= \int_0^{\frac{2\pi}{3}} (1+4\cos(\theta)+4\cos^2(\theta))\ d\theta - \int_{\frac{2\pi}{3}}^{\pi} (1+4\cos(\theta)+4\cos^2(\theta))\ d\theta \\ &= \int_0^{\frac{2\pi}{3}} (3+4\cos(\theta)+2\cos(2\theta))\ d\theta - \int_{\frac{2\pi}{3}}^{\pi} (3+4\cos(\theta)+2\cos(2\theta))\ d\theta \\ &= [3\theta + 4\sin(\theta) + \sin(2\theta)]_0^{\frac{2\pi}{3}} - [3\theta + 4\sin(\theta) + \sin(2\theta)]_{\frac{2\pi}{3}}^{\pi} = \pi + 3\sqrt{3}. \end{aligned}$$

3.4 Arc Length for Polar Curves

Definition 3.4

The arc length of a smooth polar curve $r = r(\theta)$ from θ_1 to θ_2 is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3.6}$$

Examples 15 :
Find the arc length of the following polar curves :

1. $r = 1 + \cos(\theta)$, $0 \leq \theta \leq 2\pi$.
   The curve is symmetric with respect to the polar axis then the arc length of the curve is

$$\begin{aligned} L &= 2\int_0^{\pi} \sqrt{(1+\cos(\theta))^2 + (-\sin(\theta))^2}\, d\theta \\ &= 2\int_0^{\pi} \sqrt{(1+2\cos(\theta)+\cos^2(\theta)) + \sin^2(\theta)}\, d\theta \\ &= 2\int_0^{\pi} \sqrt{2+2\cos(\theta)}\, d\theta = 2\int_0^{\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta \\ &= 4\int_0^{\pi} \cos\left(\frac{\theta}{2}\right)\, d\theta = 8. \end{aligned}$$

2. $r = 2\cos(\theta)$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
   The arc length of the curve is

$$\begin{aligned} L &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(2\cos(\theta))^2 + (-2\sin(\theta))^2}\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4\cos^2(\theta) + 4\sin^2(\theta)}\, d\theta = 2\pi. \end{aligned}$$

3. $r = e^{-\theta}$, $0 \leq \theta \leq \pi$.
   The arc length of the curve is

$$\begin{aligned} L &= \int_0^{\pi} \sqrt{\left(e^{-\theta}\right)^2 + \left(-e^{-\theta}\right)^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{e^{-2\theta} + e^{-2\theta}}\, d\theta = \sqrt{2} \int_0^{\pi} e^{-\theta} d\theta = \sqrt{2}\left(1 - e^{-\pi}\right). \end{aligned}$$

3.5 Surface Area Generated by Revolving Polar Curves

Definition 3.5

The surface area generated by revolving the smooth polar curve $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$ around the polar axis is

$$S = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\sin(\theta)| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{3.7}$$

The surface area generated by revolving the smooth polar curve $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$ around the line $\theta = \dfrac{\pi}{2}$ is

$$A = 2\pi \int_{\theta_1}^{\theta_2} |r(\theta)\cos(\theta)| \sqrt{(r(\theta))^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{3.8}$$

Examples 16 :

1. Let $r = e^{\frac{\theta}{2}}$, $0 \leq \theta \leq \pi$. The surface area generated by revolving the smooth polar curve around the polar axis is

$$\begin{aligned} S &= 2\pi \int_0^{\pi} \left| e^{\frac{\theta}{2}} \sin(\theta) \right| \sqrt{\left(e^{\frac{\theta}{2}}\right)^2 + \left(\frac{1}{2} e^{\frac{\theta}{2}}\right)^2}\, d\theta \\ &= \pi\sqrt{5} \int_0^{\pi} e^{\theta} \sin(\theta)\, d\theta = \sqrt{5}\pi \left[\frac{1}{2} e^{\theta}(\sin(\theta) - \cos(\theta))\right]_0^{\pi} \\ &= \frac{\sqrt{5}\pi}{2}\left(e^{\pi} + 1\right). \end{aligned}$$

(We use integration by parts).

2. Let $r = 2 + 2\cos(\theta)$, $0 \le \theta \le \dfrac{\pi}{2}$, The surface area generated by revolving the smooth polar curve around the polar axis is

$$\begin{aligned} S &= 2\pi \int_0^{\frac{\pi}{2}} |(2+2\cos(\theta))\sin(\theta)| \sqrt{(2+2\cos(\theta))^2 + (-2\sin(\theta))^2}\, d\theta \\ &= 4\pi \int_0^{\frac{\pi}{2}} (2+2\cos(\theta))\sin(\theta)\sqrt{2+2\cos(\theta)}\, d\theta \\ &= 4\pi \int_0^{\frac{\pi}{2}} (2+2\cos(\theta))^{\frac{3}{2}} \sin(\theta)\, d\theta \\ &= -2\pi \left[\frac{2}{5}(2+2\cos(\theta))^{\frac{5}{2}}\right]_0^{\frac{\pi}{2}} = \frac{16\pi}{5}(8-\sqrt{2}). \end{aligned}$$

3. Let $r = \cos(\theta)$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, The surface area generated by revolving the smooth polar curve around the line $\theta = \frac{\pi}{2}$ is

$$\begin{aligned} S &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos(\theta)\ \cos(\theta)| \sqrt{(\cos(\theta))^2 + (-\sin(\theta))^2}\, d\theta \\ &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta)\, d\theta = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos(2\theta))\, d\theta \\ &= \pi \left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi^2. \end{aligned}$$

4. Let $r = 2\sin(\theta)$, $0 \le \theta \le \dfrac{\pi}{2}$, The surface area generated by revolving the smooth polar curve around the line $\theta = \dfrac{\pi}{2}$ is

$$\begin{aligned} S &= 2\pi \int_0^{\frac{\pi}{2}} |2\sin(\theta)\ \cos(\theta)| \sqrt{(2\sin(\theta))^2 + (2\cos(\theta))^2}\, d\theta \\ &= 4\pi \int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta = 4\pi. \end{aligned}$$

(The surface area of a sphere of radius 1.)

3.6 Exercises

5-3-1 Sketch the curve with the given polar equation:

1) $r = 1 + 2\cos(\theta)$ 2) $r = 3 + \sin(\theta)$ 3) $r = 2\cos(4\theta)$

5-3-2 Find the polar equations of the following Cartesian equations:

1) $y = x$,
2) $4y^2 = x$,
3) $xy = 4$.
4) $(x^2+y^2)^2 = 2xy$

5-3-3 Find the Cartesian equations of the following polar equations:

1) $r = 2$;
2) $r = -3$;
3) $r = 2\cos(2\theta)$;
4) $r = 2\sin(2\theta)$;
5) $r = 2\sec(\theta)$;
6) $r = 2\csc(2\theta)$;
7) $r = 4\sec(\theta)$;
8) $\theta = \dfrac{\pi}{3}$;
9) $r = \tan(\theta)\sec(\theta)$.

5-3-4 Find the area of the region bounded by the curve

1) $r = \tan(\theta)$, $\theta \in [\frac{\pi}{6}, \frac{\pi}{3}]$;
2) $r = 1 - \sin(\theta)$, $\theta \in [0, \pi]$;
3) $r = 1 - \sin(\theta)$, $\theta \in [0, 2\pi]$;
4) $r^2 = 9\sin(2\theta)$, $\theta \in [0, \frac{\pi}{2}]$;
5) $r = 2\tan\theta$, $\theta \in [0, \frac{\pi}{8}]$;
6) $r = 2(4\cos\theta - \sec\theta)$, $\theta \in [0, \frac{\pi}{4}]$.

5-3-5 Find the area of the region that lies inside both curves: $r = 3 - 2\cos(\theta)$, $r = 3 - 2\sin(\theta)$.

5-3-6 Find the area of the region that lies inside the curve $r = 2 + \sin\theta$ and outside the curve $r = 3\sin\theta$.

5-3-7

1) Sketch the region inside the curve $r = 3$ and outside the curve $r = 2$ and find its area;

2) Sketch the region inside the curve $r = 2$ and over the straight line $r = -\csc\theta$ and find its area;

3) Sketch the region inside the curve $r = 4$ and outside the curve $r = 4\sin\theta$ and find its area;

4) Sketch the region inside the curve $r = 4\cos\theta$ and outside the curve $r = 2\cos\theta$ and find its area;

5) Sketch the region inside the curve $r = 1$ and outside the curve $r = 1 - \cos\theta$ and find its area;

6) Sketch the region inside the curve $r = 2 + 2\cos\theta$ and outside the curve $r = 3$ and find its area;

7) Sketch the region inside the curve $r = 3\sin\theta$ and outside the curve $r = 1 + \sin\theta$ and find its area:

8) Sketch the region inside the curve $r = 1 + \cos\theta$ and outside the curve $r = 1 - \cos\theta$ and find its area;

9) Sketch the region inside the curve $r = 1 + \cos\theta$ and outside the curve $r = 3\cos\theta$ and find its area;

10) Sketch the region inside the curve $r = \cos(\theta)$ and outside the curve $r = 1 - \cos(\theta)$ and find its area;

11) Sketch the region inside the curve $r = 1$ and outside the curve $r = 1 - \cos(\theta)$ and find its area;

12) Sketch the common region between the curves $r = 2\sin(\theta)$ and $r = 2\cos(\theta)$ and find its area;

5-3-8 Find the area enclosed the curve $r = 1 - 2\sin(\theta)$.

5-3-9 Set up the integral that gives the area of the region that lies inside the curve $r = 2 + \cos(2\theta)$ and outside the curve $r = 2 + \sin(\theta)$.

5-3-10 Find the length of the polar curves

1) $r = 5^{\theta}$, $\theta \in [0, 2\pi]$;

2) $r = \sin^3(\dfrac{\theta}{3})$, $\theta \in [0, \pi]$;

3) $r = 2(1 + \cos(\theta))$, $\theta \in [0, 2\pi]$.

5-3-11 Find the surface area of the curve $r = e^{5\theta}$ rotated around the line $\theta = \frac{\pi}{2}$ from 0 to $\frac{\pi}{4}$.

5-3-12 Find the surface area of the revolution of the curve $r = 5\cos\theta$ revolved around the polar axis (the $x-$axis) from 0 to $\frac{\pi}{3}$.

5-3-13 Find the surface area of the curve $r = -3 - 3\sin\theta$ revolved around the line $\theta = \frac{\pi}{2}$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

5-3-14 Find the surface area of the curve $r = \sin\theta$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

5-3-15 Find the surface area of the curve $r = \cos\theta$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \pi$.

5-3-16 Find the surface area of the curve $r = e^{\theta}$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

5-3-17 Find the surface area of the curve $r = 1 + \cos\theta$ rotated about the $x-$axis from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

5-3-18 Set up, but do not evaluate, an integral that gives the surface area of the curve rotated about the given axis.

1) $r = 2 + 2\sin\theta$ rotated about the $x-$axis from $\theta = 0$ to $\theta = \pi$;
2) $r = \cos\theta\sin\theta$ rotated about the $x-$axis from $\theta = 0$ to $\theta = \frac{\pi}{2}$;
3) $r = \sin(2\theta)$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{3}$;
4) $r = \cos\theta\sin(2\theta)$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{3}$;
5) $r = 5 - 4\cos\theta$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$;
6) $r = \theta\cos\theta$ rotated about the $x-$axis from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$.

Appendices

APPENDIX A

CONSTRUCTION OF GRAPHS OF FUNCTIONS

Let $f\colon I \longrightarrow \mathbb{R}$ be a smooth function (continuously differentiable). The equation of the tangent at a point $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a).$$

Definition 0.1

Let $c \in \mathbb{R} \cup \{-\infty, +\infty\}$ be a cluster point of the interval I. (i.e there is a sequence in I with limit c). We say that the function f has an infinite branch when x tends to c if $\lim\limits_{t \longrightarrow c} ||(x, f(x))|| = +\infty$, where $||(x, f(x))|| = \sqrt{x^2 + f^2(x)}$.

Definition 0.2

Assume that the function f has an infinite branch when x tends to c.

1. If $c \in \mathbb{R}$, the line of equation $x = c$ is the equation of the asymptote to the curve of f.

   If $c = +\infty$ and $\lim\limits_{x \to +\infty} f(x) = \ell$, the line of equation $y = \ell$

is the equation of the asymptote to the curve of f when $t \to +\infty$.

2. If $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = \pm\infty$, we say that f presents a parabolic branch parallel to the $y-$axis.

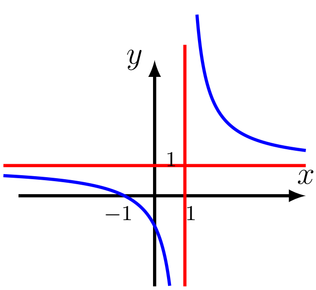
3. If $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = a$ and $\lim\limits_{x\to+\infty} f(x) - ax = \pm\infty$, we say that f presents a parabolic branch parallel to the line of equation $y = ax$.

4. If $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = a$ and $\lim\limits_{x\to+\infty} f(x) - ax = b$, we say that the line of equation $y = ax + b$ is the asymptote to the graph of f when $x \to +\infty$.

Examples 17 :

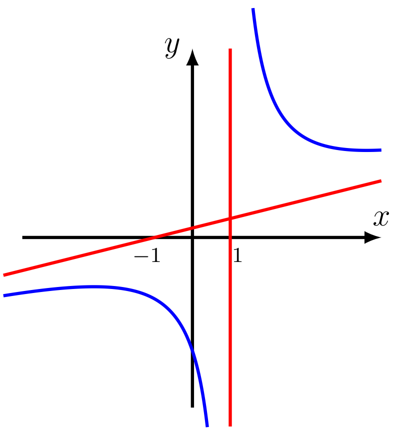
1. Let $f(x) = \dfrac{x+1}{x-1}$ defined for $x \neq 1$. f has an infinite branch when x tends to 1. The line of equation $x = 1$ is an asymptote to the graph of f. $\lim\limits_{x\to\pm\infty} f(x) = 1$, the $y = 1$ is the equation of the asymptote to the curve of f. $f'(x) = \dfrac{-2}{(x-1)^2} < 0$.

x	$-\infty$ 1	$+\infty$
$f'(x)$	$-$	$-$
$f(x)$	$1 \searrow -\infty$	$+\infty \searrow 1$



2. Let $f(x) = \dfrac{x^2+3}{4(x-1)}$ defined for $x \neq 1$. f has an infinite branch when x tends to 1. The line of equation $x = 1$ is an asymptote to the graph of f. $\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = \dfrac{1}{4}$ and $\lim\limits_{x\to\pm\infty} f(x) - \dfrac{1}{4}x = \dfrac{1}{4}$. Then $y = \frac{1}{4}(x+1)$ is the equation of the asymptote to the curve of f. $f'(x) = \dfrac{(x+1)(x-3)}{4(x-1)^2} < 0$.

x	$-\infty$		-1		1		3		$+\infty$
$f'(x)$		$+$	0	$-$	‖	$-$	0	$+$	
$f(x)$	$-\infty$	↗	$-\frac{1}{2}$	↘	$-\infty$ ‖ $+\infty$	↘	$\frac{3}{2}$	↗	$+\infty$



APPENDIX B

CONSTRUCTION OF CURVES IN CARTESIAN COORDINATES

1 Tangent to Parametric Curve

Definition 1.1

Let $f\colon I \longrightarrow \mathbb{R}^2$ be a parametric curve and let $a \in I$ (I an open interval). We assume that for t close to a we have: $f(t) \neq f(a)$. We say that this curve has a tangent at $A = f(a)$ if the direction of the vector $AM = f(t) - f(a)$, ($M = f(t)$) has a limit when t tends to a. This means that for $t \in I$ close to a ($t \neq a$), there exists a vector $V(t)$ collinear to the vector AM such that $\lim\limits_{t \longrightarrow a} V(t) = V \neq 0$. The tangent to the curve at $A = f(a)$ is the line passing through A and parallel to V.
If $f(t) = (x(t), y(t))$, the slop of the tangent to the curve at A is

$$p = \lim_{t \to a} \frac{y(t) - y(a)}{x(t) - x(a)}.$$

Examples 18 :

1. Let $f(t) = (t^2, t^3)$ for $t \in \mathbb{R}$. The tangent to the curve $t \longmapsto f(t)$ at

$(0,0) = f(0)$ is the real axis. Indeed, $f(t) - f(0) = t^2(1,t)$ which is parallel to the vector $V(t) = (1,t)$ and $\lim_{t \to 0} V(t) = (1,0)$. The slope is $p = \lim_{t \to 0} \frac{t^3}{t^2} = 0$.

2. Let $f(t) = (t\ln(t), \sin(\pi t))$ for $t \in]0, +\infty[$. The tangent to the curve $t \longmapsto f(t)$ at $f(1) = (0,0)$ is the line passing through the point $y-$axis. Indeed $f(t) - f(1) = t(\ln(t), \frac{\sin(\pi t)}{t})$.
If $V(t) = (\ln(t), \frac{\sin(\pi t)}{t})$, $\lim_{t \to 1} V(t) = (0, -\pi)$.

Theorem 1.2

1. If f is differentiable at a and $f'(a) \neq 0$, the curve has a tangent at $A = f(a)$ parallel to the vector $f'(a)$.
2. In general if f is k-times differentiable at a and $f'(a) = f''(a) = \ldots = f^{(k-1)}(a) = 0$ and $f^{(k)}(a) \neq 0$, then the curve has a tangent at $A = f(a)$ parallel to the vector $f^{(k)}(a)$. (We use the L'Hôpital rule)

2 Local Study of Curves

2.1 Taylor Formulas

Theorem 2.1: (Taylor-Young Formula)

Let $f\colon I \longrightarrow \mathbb{R}^2$ be a curve n-times differentiable at $a \in I$, we have:
$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + |x-a|^n \varepsilon(x)$; with $\lim_{x \longrightarrow a} \|\varepsilon(x)\| = 0$.

Theorem 2.2: (Taylor Formula with integral remainder)

Let $f\colon I \longrightarrow \mathbb{R}^2$ be a curve of class $\mathcal{C}^{n+1}$ on a neighborhood I of a. If the interval $[a,x] \subset I$ or $[x,a] \subset I$, we have:

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + (x-a)^{n+1} \int_0^1 \frac{(1-t)^n}{n!} f^{(n+1)}(a+t(x-a))dt.$$

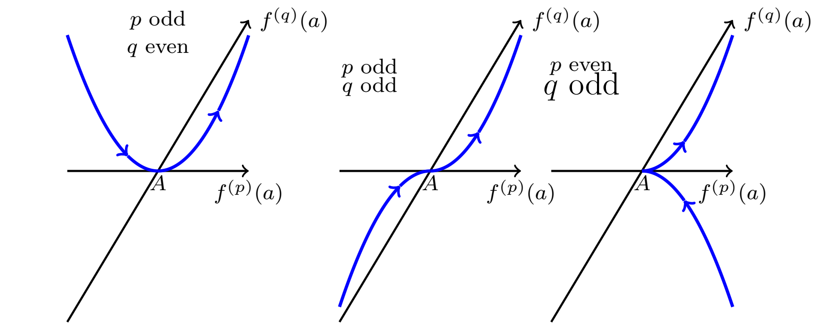
Moreover if $||f^{(n+1)}(a+t(x-a))|| \leq M \ \forall \ t \in [0,1]$, we have

$$||f(x) - \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a)|| \leq \frac{M|x-a|^{n+1}}{(n+1)!}.$$

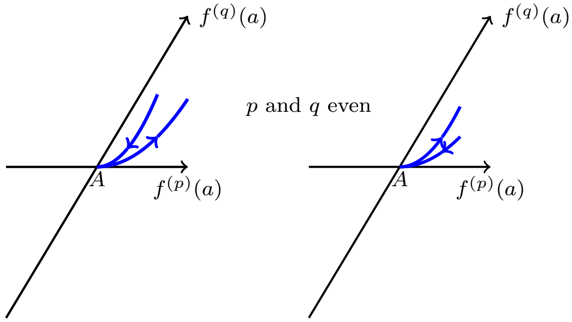
Remark 21 :
Let $f\colon I \longrightarrow \mathbb{R}^2$ be a curve of class $\mathcal{C}^n$ on a neighborhood I of a. If $f^{(k)}(a) = 0$, for $1 \leq k \leq m-1 \leq n-1$ and $f^{(m)}(a) \neq 0$, then the tangent to the curve at $M_a = f(a)$ is $f^{(m)}(a)$.

2.2 Local Study of Curves

Let $a \in I$. Assume that there exists two integers p and q, $1 \leq p < q$ and f is q times differentiable at a and we have the following:
$f^{(k)}(a) = 0$, for any $1 \leq k \leq p-1$, $f^{(p)}(a) \neq 0$, $f^{(m)}(t_0)$ is collinear to $f^{(p)}(t_0)$ for $p \leq m \leq q-1$ and $f^{(q)}(a)$ is not collinear to $f^{(q)}(a)$.
It results that the vectors $(f^{(p)}(a), f^{(q)}(a))$ form a basis of $\mathbb{R}^2$. The curve in a neighborhood of $M_a = f(a)$ has the following aspect form which depends to the parity of p and q. (The arrows indicate the direction when t increases).



ordinary point   inflection point   cusp of first kind (or a simple cusp)



cusp of second kind

Indeed by Taylor-Young formula in a neighborhood of a:

$$\begin{aligned} f(t)-f(a) &= \frac{(t-a)^p}{p!}f^{(p)}(a)+\sum_{m=p+1}^{q-1}\frac{(t-a)^m}{m!}f^{(m)}(a) \\ &\quad +\frac{(t-a)^q}{q!}f^{(q)}(a)+(t-a)^q\varepsilon(t), \end{aligned}$$

$\lim\limits_{t\longrightarrow a}\varepsilon(t)=0$. The vector $\sum_{m=p+1}^{q-1}\frac{(t-a)^m}{m!}f^{(m)}(a)$ is collinear to the vector $f^{(p)}(a)$. Therefore for $(t-a)$ small enough the sign of the two components of M_aM_t in the basis $(f^{(p)}(a), f^{(q)}(a))$ have the same sign that those of $(t-a)^p$ and $(t-a)^q$ respectively.

3 Infinite Branches

Definition 3.1

Let $a \in \mathbb{R} \cup \{-\infty, +\infty\}$ be a cluster point of the interval I. (i.e there is a sequence in I with limit a). We say that the parametric curve f has an infinite branch when t tends to a if $\lim_{t \longrightarrow a} ||f(t)|| = +\infty$, where $f(t) = (x(t), y(t))$ and $\|f(t)\| = \sqrt{x^2(t) + y^2(t)}$.

Definition 3.2

Assume that the parametric curve f has an infinite branch when t tends to t_0. If the direction of the vector $f(t)$ has a limit when $t \longrightarrow t_0$, this direction is called the asymptotic direction of the curve.

Examples 19 :

1. Let $f(t) = (\frac{1}{t}, \frac{1}{t^2})$ defined for $t \neq 0$. f has an infinite branch when t tends to 0. The asymptotic direction is parallel to the vector $(0, 1)$ because $f(t) = \frac{1}{t^2}(t, 1)$.

2. $f(t) = (\frac{1}{1-t}, \frac{1}{1-t^2})$. The curve has an infinite branch when t tends to 1 or -1. $\lim_{t \longrightarrow 1}(1-t)f(t) = (1, \frac{1}{2})$. The asymptotic direction (when $t \to 1$) is the line passing through the origin and parallel to the vector $(1, \frac{1}{2})$. The slope is $\frac{1}{2}$.
   If $t \to -1$, $\lim_{t \longrightarrow -1}(1-t^2)f(t) = (0, 1)$, thus the asymptotic direction is the $y-$axis.

3.1 Asymptotes

Definition 3.3

1. Assume that the parametric curve $t \longmapsto f(t)$ has an infinite branch when t tends a and has an asymptotic direction. We say that this branch has an asymptote if there exists a line D parallel to the asymptotic direction such that the distance of $f(t)$ to this line D tends to 0 when $t \longrightarrow a$. The line D is called the asymptote of the curve when t tends to a.

2. We say that the infinite branch is parabolic if the distance of $f(t)$ to any parallel line to the asymptotic direction tends to ∞ when $t \longrightarrow a$.

3.2 Determination of Asymptotes

Let p be the slope of the asymptotic direction of the curve $f(t) = (x(t), y(t))$ at a.

First case: $p = 0$:
The curve has an asymptote if and only if $y(t)$ has a limit λ when t tends to a. In this case the asymptote is the line of equation $y = \lambda$. The sign of $y(t) - \lambda$ allows to place the curve with respect to the asymptote.

Second case $p = +\infty$.
The parametric curve has an asymptote if and only if $x(t)$ has a limit λ when $t \longrightarrow a$. The equation of the asymptote is $x = \lambda$ and the sign of $x(t) - \lambda$ allows to place the curve with respect to the asymptote.

Third case $p \in \mathbb{R}$.
We have an asymptote if and only if $\lim_{t \longrightarrow t_0} y(t) - px(t) = \lambda$, $\lambda \in \mathbb{R}$. The equation of the asymptote is $y = px + \lambda$. The sign of $y(t) - px(t) - \lambda$ allows to place the curve with respect to the asymptote.

4 Construction of Curves in Cartesian Coordinates

Example 20 :

$f(t) = (x(t), y(t)) = (\dfrac{1}{1-t}, \dfrac{1}{1-t^2})$. The functions $x(t)$ and $y(t)$ are defined on $\mathbb{R} \setminus \{-1, 1\}$, $x'(t) = \dfrac{1}{(1-t)^2}$ and $y'(t) = \dfrac{2t}{(1-t^2)^2}$.

Infinite branches:

In a neighborhood of $t = -1$, $\lim\limits_{t \longrightarrow -1^+} y(t) = -\infty$ and $\lim\limits_{t \longrightarrow -1^-} y(t) = \infty$. Also $\lim\limits_{t \longrightarrow -1} x(t) = \dfrac{1}{2}$. Then the line of equation $x = \dfrac{1}{2}$ is an asymptote to the curve.

In a neighborhood of $t = 1$, $\lim\limits_{t \longrightarrow 1} \dfrac{y(t)}{x(t)} = \dfrac{1}{2}$, thus the curve has an asymptotic direction parallel to the vector $(2, 1)$. In this case the slope is $p = \frac{1}{2}$.

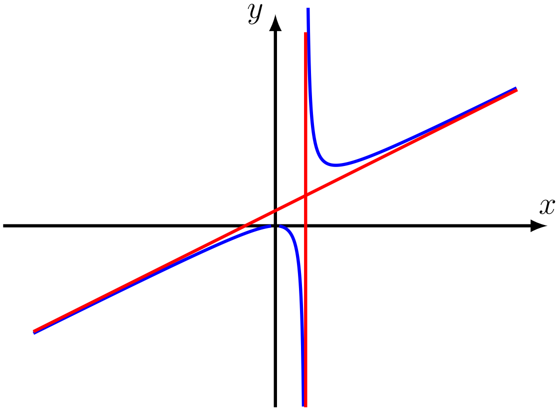
$y(t) - \dfrac{x(t)}{2} = \dfrac{1}{1-t^2} - \dfrac{1}{2(1-t)} = \dfrac{1}{2(1+t)} \overset{t \longrightarrow 1}{\longrightarrow} \dfrac{1}{4}$. The equation of the asymptote is $y = \frac{1}{2}x + \frac{1}{4}$.

If $t < 1$ the curve is above the asymptote. If $t > 1$ the curve is under the asymptote.

Tangents:

$M_{-\infty}M_t = (\frac{1}{1-t}, \frac{1}{1-t^2})$, $\lim\limits_{t \longrightarrow -\infty} (-tM_{-\infty}M_t) = (1, 0)$. The tangent at the point $(0, 0)$ is parallel to the $x-$axis. The same result when $t \longrightarrow +\infty$.

t	$-\infty$		-1		0		1		$+\infty$
$x'(t)$		$+$					‖	$+$	
x	0			↗			$+\infty$ ‖ $-\infty$	↗	0
$y'(t)$		$-$	‖	$-$	0	$+$	‖	$+$	
y	0	↘	$-\infty$ ‖ $+\infty$	↘	1	↗	$+\infty$ ‖ $-\infty$	↗	0



Example 21 :

$f(t) = (x(t), y(t)) = (\sin(2t), \sin(3t))$. f is $2\pi-$periodic.

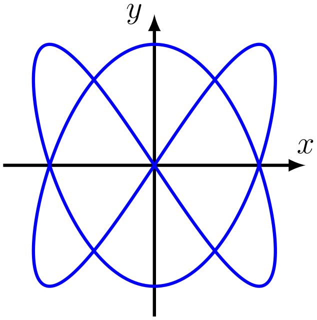
$x(-t) = -x(t)$, $y(-t) = -y(t)$, thus we study the curve on $[0, \pi]$ and we take a symmetry with respect to the origin.

$x(\pi - t) = x(-t) = -x(t)$, $y(\pi - t) = y(t)$, thus we study the curve on $[0, \frac{\pi}{2}]$ and we take a symmetry with respect to the $y-$axis and a symmetry with respect to the origin.

$M_0 = (0, 0)$, $f'(0) = (2, 3)$, $f''(0) = (0, 0)$ and $f^{(3)}(0) = (-8, -27)$. $(0, 0)$ is an inflexion point.

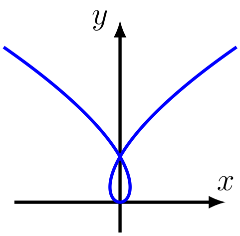
$x'(t) = 2\cos(2t)$, $y'(t) = 3\cos(3t)$

t	0		$\frac{\pi}{6}$		$\frac{\pi}{4}$		$\frac{\pi}{2}$
$x'(t)$		$+$			0	$-$	
$x(t)$	0	$\nearrow$			1	$\searrow$	0
$y'(t)$		$+$	0	$-$			
$y(t)$	0	$\nearrow$	1	$\searrow$			0



Example 22 :
$f(t) = (x(t), y(t)) = (t - \frac{1}{3}t^3, t^2)$.
$x(-t) = -x(t)$, $y(-t) = y(t)$, thus we study the curve on the interval $[0, +\infty[$ and one take a symmetry with respect to the axis (oy).
The double points: $t - \frac{t^3}{3} = s - \frac{s^3}{3}$ and $s^2 = t^2 \Rightarrow t = \sqrt{3}$. The double point is $(0, 3)$.
$x'(t) = 1 - t^2$, $y'(t) = 2t$. $\lim_{t \longrightarrow +\infty} \dfrac{y}{x} = 0$, thus the axis (ox) is an asymptotic direction. $x'(t) = 1 - t^2$, $y'(t) = 2t$

t	0		1		$+\infty$
$x'(t)$		$+$	0	$-$	
$x(t)$	0	$\nearrow$	$\frac{2}{3}$	$\searrow$	$-\infty$
$y'(t)$	0		$+$		
$y(t)$	0		$\nearrow$		$+\infty$



APPENDIX C

CONSTRUCTION OF CURVES IN POLAR COORDINATES

1 Local Study of Curves in Polar Coordinates

Consider the Euclidean plane $\mathbb{R}^2$ equipped with an orthonormal basis (e_1, e_2) and the system of axes (ox, oy).

Definition 1.1

Let M be a point of the plane of coordinates (x, y). A polar coordinate system of M is any pair of real numbers (r, θ) such that $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Remark 22 :
A point M have an infinite polar coordinates. For example, the origin 0 has $(0, \theta)$ as polar coordinates for all $\theta \in \mathbb{R}$.
If $M \neq (0,0)$ and (r, θ) is a polar coordinates of M, then (r', θ') is a polar coordinates of M if and only if $r = r'$ and $\theta = \theta' + 2k\pi$, $k \in \mathbb{Z}$ or $r = -r'$ and $\theta = \theta' + (2k+1)\pi$, $k \in \mathbb{Z}$.

Definition 1.2

A parametric curve $t \longmapsto f(t)$, $(t \in I)$ is called in polar coordinates if for any $t \in I$, $f(t)$ is determined by a polar coordinates $(r(t), \theta(t))$.

A curve in polar coordinates can be studied in Cartesian coordinates by the change of coordinates $x(t) = r(t)\cos(\theta(t))$, $y(t) = r(t)\sin(\theta(t))$.

Notations

For any $\theta \in \mathbb{R}$ we define the vectors

$$u_\theta = \cos(\theta)e_1 + \sin(\theta)e_2 \quad \text{and} \quad v_\theta = -\sin(\theta)e_1 + \cos(\theta)e_2.$$

Remark 23 :

$v_\theta = u_{\theta+\frac{\pi}{2}}$, this which yields that (u_θ, v_θ) is a basis of $\mathbb{R}^2$ positively oriented. Moreover $\dfrac{du_\theta}{d\theta} = v_\theta$ and $\dfrac{dv_\theta}{d\theta} = -u_\theta$.

In which follows we shall limit our study to the case $\theta(t) = t$.

$f(\theta) = (r(\theta)\cos(\theta), r(\theta)\sin(\theta)) = r(\theta)u_\theta$, where $r(\theta)$ is a real function.

Examples 23 :

1. If $r(\theta) = 1$, $(\theta \in \mathbb{R})$. The support of this parametric curve is the unit circle.

2. Let $I = [-\frac{\pi}{2}, 0[\cup]0, \frac{\pi}{2}]$ and $r(\theta) = \frac{1}{\sin(\theta)}$. The support of the parametric curve is the line of equation $y = 1$.

Let $f(\theta) = r(\theta)u_\theta$ be a parametric curve given in polar coordinates defined on an interval I. If r is differentiable, f is differentiable and

$$f'(\theta) = r'(\theta)u_\theta + r(\theta)v_\theta.$$

The coordinates of f' in the basis (u_θ, v_θ) are $(r'(\theta), r(\theta))$.

If $f(\alpha) \neq (0,0)$, the vector $f'(\alpha)$ is non zero. The curve has a tangent at the point $f(\alpha)$. If $r'(\alpha) = 0$, $f'(\alpha) = r(\alpha)v_\alpha$. Thus the tangent at the point $f(\alpha)$ is orthogonal to the line which passes through 0 and $f(\alpha)$.

If $f(\alpha) = 0$, we assume that $r(\theta) \neq 0$ for θ close to α. The slope of the line which passes through 0 and $f(\theta)$ is $\tan(\theta)$. The curve has a tangent at the origin of slope equal to $\lim\limits_{\theta \longrightarrow \alpha} \tan(\theta)$.

2 Infinite Branches

We have an infinite branch in a neighborhood of α if $\lim\limits_{\theta\longrightarrow\alpha}|r(\theta)| = +\infty$. The asymptotic direction is the direction of the line $\theta = \alpha$. (if α is finite.) Let H be the projection of M on the line parallel to the vector v_α and passing through 0. $OH = r(\beta)\sin(\beta-\alpha)$. If $r(\beta)\sin(\beta-\alpha)$ has a limit when β tends to α, then the curve has the asymptote, the line parallel to u_α and passing through the point K in the system of coordinates $(0_\alpha, u_\alpha, v_\alpha)$. $OK = \lim\limits_{\beta\longrightarrow\alpha} r(\alpha)\sin(\beta-\alpha) = d$, the equation of this asymptote is $r = \frac{d}{\sin(\theta-\alpha)}$.

3 Asymptotic Point , Asymptotic Circle, Spiral

If $\lim\limits_{\theta\longrightarrow\infty} r(\theta) = 0$. 0 is an asymptotic point. (Example $r = \frac{1}{\theta}$).
If $\lim\limits_{\theta\longrightarrow\infty} r(\theta) = t > 0$, the circle of equation $r = t$ is an asymptotic circle. (Example $r = 1 - \frac{1}{\theta}$).
If $\lim\limits_{\theta\longrightarrow\infty} r(\theta) = +\infty$, we have a spiral. (Example $r = e^\theta$).

4 Symmetries and Reduction of Interval

Periodical Curves
Let f be a $T-$periodic function. We consider the parametric curve $r = f(\theta)$, with $\theta \in \mathbb{R}$. (We assume that T is the smallest positive period of f).

1. If $T = 2k\pi$, with $k \in \mathbb{Z}$, then we study the curve on an interval of length T.
2. If $T = \frac{2k\pi}{n}$, with $k, n \in \mathbb{N}$ and $\frac{k}{n}$ irreducible. We study the curve on an interval of length T and we do n rotations of angle T. $(nT = 2k\pi)$.
3. If $T = 2\alpha\pi$, with $\alpha \notin \mathbb{Q}$. We construct the curve on an interval of length T and we do an infinite rotations of angle T.

1. If $r(\alpha-\theta)=r(\theta)$, we study the curve on the interval $[\frac{\alpha}{2},+\infty[$ and we do symmetry with respect to the polar axis $(\frac{\alpha}{2})$.

2. If $r(\alpha-\theta)=-r(\theta)$, we study the curve on the interval $[\frac{\alpha}{2},+\infty[$ and we do a symmetry with respect to the polar axis $(\frac{\alpha}{2}+\frac{\pi}{2})$.

5 Construction of curves in polar coordinates

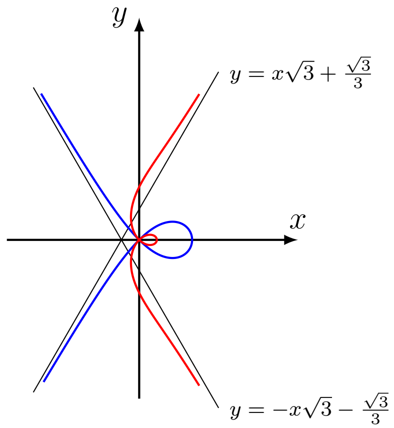
Examples 24 :

1. $r(\theta)=\dfrac{\cos(2\theta)}{2\cos(\theta)-1}$. The curve is defined on $\mathbb{R}\backslash\{\pm\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}\}$.

   $r(-\theta)=r(\theta)$ and r is $2\pi-$periodic. It suffices to study the curve on $[0,\frac{\pi}{3}[\cup]\frac{\pi}{3},\pi]$ and we do a symmetry with respect to the real axis. $\lim\limits_{\theta\longrightarrow\frac{\pi}{3}^-} r(\theta)=-\infty$, $\lim\limits_{\theta\longrightarrow\frac{\pi}{3}^+} r(\theta)=+\infty$, $\lim\limits_{\theta\longrightarrow\frac{\pi}{3}} r(\theta)\sin(\theta-\dfrac{\pi}{3})=-\dfrac{1}{2\sqrt{3}}$. Thus the line of equation $y=x\sqrt{3}+\frac{\sqrt{3}}{3}$ is an asymptote in a neighborhood of $\frac{\pi}{3}$.

   $$r'(\theta)=\frac{-2\sin(\theta)(1-2\cos(\theta)+2\cos^2(\theta))}{(2\cos(\theta)-1)^2}$$

θ	0			$\frac{\pi}{3}$			π
$r'(\theta)$	0	$-$		‖		$-$	0
$r(\theta)$	1	$\searrow$	$-\infty$	‖	$+\infty$	$\searrow$	$-\frac{1}{3}$

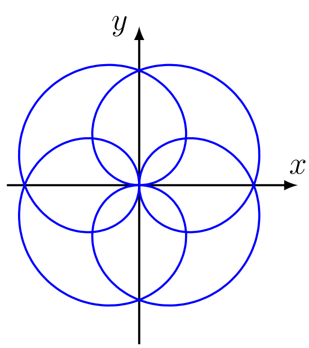


2. $r(\theta) = \sin(\frac{2\theta}{3})$. r is $3\pi-$periodic and $r(\theta + 3\pi) = r(\theta)$, thus it suffices to study the curve on an interval of length 3π and make a symmetry with respect to O.

   Moreover $r(-\theta) = r(\theta)$, we study the curve on $[0, 3\frac{\pi}{2}]$ and we make in order a symmetry with respect to the real axis and a symmetry with respect the origin 0.

   $r(\frac{3\pi}{2} - \theta) = r(\theta)$. We study the curve on $[0, \frac{3\pi}{4}]$ and we do in order a symmetry with respect to the polar axis $\theta = \frac{\pi}{4}$, a symmetry with respect to the real axis and a symmetry with respect to the origin 0. $r'(\theta) = \frac{2}{3}\cos(\frac{2\theta}{3})$, $\frac{r(\theta)}{r'(\theta)} = \frac{2}{3}\tan(\frac{2\theta}{3})$.

θ	0		$\frac{3\pi}{4}$
$r'(\theta)$	1	$+$	0
$r(\theta)$	0	$\nearrow$	1



APPENDIX D

DIFFERENTIAL GEOMETRY OF PLANE CURVES

1 Length of Plane Curves

Let $\gamma\colon [a,b] \longrightarrow \mathbb{R}^2$ be a curve piecewise continuously differentiable. The length of γ is defined by:

$$L(\gamma) = \int_a^b ||\gamma'(t)||dt.$$

Remark 24 :
The expression of $L(\gamma)$ is invariant by change of parametrization of class $\mathcal{C}^1$ of the curve. indeed let $\varphi\colon [\alpha,\beta] \longrightarrow [a,b]$ is a strictly increasing function of class $\mathcal{C}^1$. Set $\psi(s) = \gamma(\varphi(s))$, $\psi'(s) = \gamma'(\varphi(s)).\varphi'(s)$, $||\psi'(s)|| = ||\gamma'(\varphi(s))||\varphi'(s)$. $(\varphi'(s) \geq 0)$. Thus from the change of variables formula , $(\varphi(\alpha) = a,\, \varphi(\beta) = b)$ and we have:

$$\int_\alpha^\beta ||\psi'(s)||ds = \int_a^b ||\gamma'(t)||dt.$$

The same result if φ is strictly decreasing.

Examples 25 :

1. If the curve is defined in Cartesian coordinates by a mapping $f\colon [a,b] \longrightarrow \mathbb{R}^2$, with $f(t) = (t, y(t))$, $t \in [a,b]$ and y of class $\mathcal{C}^1$.

$$L(f) = \int_a^b \sqrt{1 + y'^2(t)}dt.$$

   For example, if $y = \tan(t)$, $t \in [0, \frac{\pi}{4}]$. $L(f) = \ln(1+\sqrt{2})$.

2. If the curve is defined in Cartesian coordinates by a mapping $f(t) = (x(t), y(t))$, $t \in [a,b]$, with x and y of class $\mathcal{C}^1$.

$$L(f) = \int_a^b \sqrt{x'^2(t) + y'^2(t)}dt.$$

   For example, $x(t) = \cos(t)$ and $y(t) = \sin(t)$, $t \in [0, 4\pi]$. $L(f) = 4\pi$.

3. If the curve is defined in polar coordinates by a mapping $f\colon [a,b] \longrightarrow \mathbb{R}^2$ with $f(\theta) = r(\theta)u_\theta$, then $f'(\theta) = r'(\theta)u_\theta + r(\theta)v_\theta$ and $||f'(\theta)||^2 = r^2(\theta) + r'^2(\theta)$. Thus

$$L(f) = \int_a^b \sqrt{r^2(\theta) + r'^2(\theta)}d\theta.$$

   For example if $r(\theta) = e^{k\theta}$, $k \in \mathbb{R}^*$ and $\theta \in [a,b]$.
   $L(f) = \dfrac{\sqrt{1+k^2}}{k}(e^{kb} - e^{ka})$.

2 Curvilinear Abscissa and Normal Parametrization

Definition 2.1

Let γ be a plane curve of class $\mathcal{C}^1$ defined on the interval $[a,b]$. If $t_0 \in [a,b]$ and $\gamma'(t_0) = 0$, we say that $\gamma(t_0)$ is a singular point. The point $\gamma(t_0)$ is said a regular point if $\gamma'(t_0) \neq 0$.
The curve γ is called regular if $\gamma'(t) \neq 0$ for all $t \in [a,b]$.

Definition 2.2

Let γ be a plane curve of class $\mathcal{C}^1$ defined on the interval $[a, b]$. We assume that γ regular on $[a, b]$. For $t_0 \in [a, b]$, we set

$$\varphi(t) = \int_{t_0}^{t} ||\gamma'(\theta)||d\theta.$$

φ is a function of class $\mathcal{C}^1$ strictly increasing, then φ^{-1} is of class $\mathcal{C}^1$ and $(\varphi^{-1})'\varphi(t) = (\varphi'(t))^{-1}$. By definition $\varphi(t)$ is called the curvilinear abscissa of γ.
If $g(s) = \gamma(\varphi^{-1}(s))$, $||g'(s)|| = 1$. g is a parametrization of the same curve γ, g is called the normal parametrization of the curve $(||\gamma'(s)|| = 1)$.

3 Curvature of Plane Curves

We consider a regular curve γ of class $\mathcal{C}^2$ on an interval I. We denote s the curvilinear abscissa of γ.
$\dfrac{ds}{dt} = ||\gamma'(t)||$. Let T be the unitary tangent vector $T = \dfrac{\gamma'(t)}{||\gamma'(t)||}$ and N the unitary vector defined by the rotation of center 0 and of angle $\frac{\pi}{2}$ of the vector T. (If (a, b) are the components of T in the basis (e_1, e_2), then $(-b, a)$ are those of the vector N.) By definition N is called the normal vector of the curve.

Remark 25 :
T is a unitary vector, thus $< T, T, >= 1$, this which yields that $\langle \frac{dT}{ds}, T\rangle = 0$. Thus the vector N is collinear to the vector $\dfrac{dT}{ds}$. ($\langle , \rangle$ is the Cartesian inner product on $\mathbb{R}^2$.)

Definition 3.1

If $\dfrac{dT}{ds} = CN$, the real number C is called the algebraic curvature of the curve γ.

Remark 26 :
If φ is the polar angle of T. $(T = (\cos(\varphi), \sin(\varphi)), \frac{dT}{ds} = \frac{d\varphi}{ds}(-\sin(\varphi), \cos(\varphi)))$.
Then the algebraic curvature of γ is $\frac{d\varphi}{ds}$, because $N = (-\sin(\varphi), \cos(\varphi))$.
We have also $\frac{dN}{ds} = -CT$.

Definition 3.2

At any point where $C \neq 0$, we define the radius of curvature and denoted by R the real $\frac{1}{C}$.

Definition 3.3

The point K such that $MK = RN$ is called the center of curvature of γ at M.

4 Curvature in Cartesian Coordinates

$T = \frac{\gamma'(t)}{||\gamma'(t)||}$, $\frac{dT}{ds} = \frac{dT}{dt}\frac{1}{\frac{ds}{dt}}$.
$T = (\frac{x'}{\sqrt{x'^2 + y'^2}}, \frac{y'}{\sqrt{x'^2 + y'^2}})$.
$\frac{dT}{ds} = (\frac{-y'}{\sqrt{x'^2 + y'^2}}, \frac{x'}{\sqrt{x'^2 + y'^2}})(\frac{x'y'' - y'x''}{\sqrt{(x'^2 + y'^2)^3}})$. Thus

$$C = \frac{x'y'' - y'x''}{\sqrt{(x'^2 + y'^2)^3}}.$$

In particular if $x(t) = t$. $C = \dfrac{y''}{\sqrt{(1+y'2)^3}}$. We get the notion of convexity of the functions of class $\mathcal{C}^2$. The curve is convex if and only if $x'C$ is positive and concave if and only if $x'C$ is non negative.

5 Curvature in Polar Coordinates

$OM = r(\theta)u_\theta$, $\dfrac{d(OM)}{d\theta} = r'(\theta)u_\theta + r(\theta)v_\theta$.

$T = \dfrac{r'}{\sqrt{r^2+r'^2}}u_\theta + \dfrac{r}{\sqrt{r^2+r'^2}}v_\theta \Rightarrow N = \dfrac{-ru_\theta + r'v_\theta}{\sqrt{r^2+r'^2}}$.

$\dfrac{ds}{dt} = \dfrac{d||(OM)(\theta)||}{d\theta} = \sqrt{r^2+r'^2}$. Then

$\dfrac{dT}{ds} = \dfrac{-ru_\theta + r'v_\theta}{\sqrt{r^2+r'^2}}(\dfrac{r^2+2r'^2-rr''}{\sqrt{(r^2+r'^2)^3}})$ and

$$C = \frac{r^2+2r'^2-rr''}{\sqrt{(r^2+r'^2)^3}}.$$

6 Exercises

6-1-1 Study the following parametric curves:

1) $x(t) = \cos(3t), y(t) = \sin(2t)$,
2) $x(t) = 3\cos(t) - \cos(3t), y(t) = 3\sin(t) - \sin(3t)$,
3) $x(t) = \cos(2t), y(t) = \sin(3t)$,
4) $x(t) = \dfrac{t}{1+t^3}, y(t) = \dfrac{t^2}{1+t^3}$,
5) $x(t) = t - \sin(t), y(t) = 1 - \cos(t)$,
6) $x(t) = \sin(\dfrac{t}{2}), y(t) = \tan(t)$,
7) $x(t) = \dfrac{t^2}{t-1}, y(t) = \dfrac{t}{t^2-1}$,
8) $x(t) = t^2 + \dfrac{1}{t}, y(t) = e^t + 2t^2$,

6-1-2 Study the following curves in polar coordinates:

1) $r(\theta) = 1 + \cos(\theta)$,

2) $r(\theta) = \cos(2\theta)$,

3) $r(\theta) = \sin(\theta)\cos(2\theta)$,

4) $r(\theta) = \cos\left(\frac{\theta}{3}\right) + \frac{\sqrt{2}}{2}$,

5) $r(\theta) = \sin^3(\frac{\theta}{3})$,

6) $r(\theta) = \sec(\theta) + \csc(\theta)$,

7) $r(\theta) = 2 + \sec(\theta)$,

8) $r(\theta) = \sqrt{\cos(2\theta)}$,

9) $r(\theta) = \frac{\sin^2(\theta)}{\cos(\theta)}$,

10) $r(\theta) = \frac{1}{\sin\frac{\theta}{2}} = \csc(\frac{\theta}{2})$.