INTEGRAL CALCULUS

Solutions Manual

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Solutions of Exercises
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Preface

This solutions manual contains the detailed solutions to each exercise in the textbook "Integral Calculus". These solutions are formatted in an appropriate style to aid in its understanding This allows students to easily verify their answers. This manual includes also graphs or figures whenever the exercise requires, or as needed to aid in explanation.

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1 Exercises on Chapter I

Exercises

1-1-1 Evaluate the following indefinite integrals:

1) $\displaystyle\int \sec^2(x)dx,$

2) $\displaystyle\int \csc^2(x)dx,$

3) $\displaystyle\int \tan^2(x)dx,$

4) $\displaystyle\int \cot^2(x)dx,$

5) $\displaystyle\int \sec(x)\tan(x)dx,$

6) $\displaystyle\int \csc(x)\cot(x)dx,$

7) $\displaystyle\int \left(x - \frac{1}{x^{\frac{2}{3}}} + \frac{1}{x^2}\right) dx,$

8) $\displaystyle\int \left(x + 2 + \frac{4}{(x+1)^2}\right) dx,$

9) $\displaystyle\int \left(\frac{1}{\sec(x)} - \frac{1}{\csc(x)}\right) dx.$

Solutions of Exercises

$\boxed{\text{1-1}}$ 1) $\displaystyle\int \sec^2(x)dx = \tan(x) + c,$

2) $\displaystyle\int \csc^2(x)dx = -\cot(x) + c,$

3) $\displaystyle\int \tan^2(x)dx = \int (\sec^2(x) - 1)dx = \tan(x) - x + c,$

4) $\displaystyle\int \cot^2(x)dx = \int (\csc^2(x) - 1)dx = -\cot(x) - x + c,$

5) $\displaystyle\int \sec(x)\tan(x)dx = \sec(x) + c,$

6) $\displaystyle\int \csc(x)\cot(x)dx = -\csc(x) + c,$

7) $\displaystyle\int (x - \frac{1}{x^{\frac{2}{3}}} + \frac{1}{x^2})dx = \frac{x^2}{2} - 3x^{\frac{1}{3}} - \frac{1}{x} + c,$

8) $\displaystyle\int \left(x + 2 + \frac{4}{(x+1)^2}\right) dx = \frac{x^2}{2} + 2x - \frac{4}{x+1} + c,$

9)

$$\begin{aligned}\int \left(\frac{1}{\sec(x)} - \frac{1}{\csc(x)}\right) dx &= \int (\cos(x) - \sin(x))\, dx \\ &= \sin(x) + \cos(x) + c.\end{aligned}$$

Exercises

1-2-1 Evaluate the following integrals

1) $\displaystyle\int \sin(2x+3)dx,$

2) $\displaystyle\int \frac{1}{\cos^2(\pi x)}dx,$

3) $\displaystyle\int x\sqrt{x+1}dx,$

4) $\displaystyle\int \frac{x}{\sqrt{3-4x^2}}dx,$

5) $\displaystyle\int \frac{1}{\sqrt{x}\cos^2\left(\sqrt{x}\right)}dx,$

6) $\displaystyle\int \frac{x^2+3x+6}{\sqrt{x+1}}dx,$

7) $\displaystyle\int x^3\sqrt{x^4+1}\ dx$

8) $\displaystyle\int \frac{3x}{(1+x^2)^{\frac{2}{3}}}\ dx$

9) $\displaystyle\int \frac{3x^5}{\sqrt[3]{x^6+1}}\ dx$

10) $\displaystyle\int x^2\sec^2(x^3-2)\ dx$

11) $\displaystyle\int \frac{\csc^2\left(\sqrt{x}\right)}{\sqrt{x}}\ dx$

12) $\displaystyle\int \frac{\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)}{x^2}\ dx$

13) $\displaystyle\int \frac{(2+\sin x)^4}{\sec x}\ dx$

1-2-2 Evaluate the following integrals with the indicated change of variable:

1) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^3(x)}dx, \quad (t=\cos(x)),$

2) $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^2}, \quad (x=\tan(\theta)),$

3) $\displaystyle\int_0^1 x\sqrt{x^2+1}dx, \quad (t=x^2+1),$

Solutions of Exercises

$\boxed{\text{1-2-1}}$ 1)

$$\begin{aligned}\int \sin(2x+3)dx &\overset{(t=2x+3)}{=} \frac{1}{2}\int \sin(t)dt \\ &= -\frac{1}{2}\cos(t)+c = -\frac{1}{2}\cos(2x+3)+c\end{aligned}$$

2) $\displaystyle\int \frac{1}{\cos^2(\pi x)}dx \overset{(t=\pi x)}{=} \frac{1}{\pi}\int \sec^2(t)dt = \frac{1}{\pi}\tan(\pi x)+c,$

3)

$$\begin{aligned}\int x\sqrt{x+1}dx &\overset{(t=\sqrt{x+1})}{=} 2\int (t^2-1)t^2dt = \frac{2}{5}t^5 - \frac{2}{3}t^3 + c \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c,\end{aligned}$$

4)

$$\begin{aligned}\int \frac{x}{\sqrt{3-4x^2}}dx &\overset{(t=\sqrt{3-4x^2})}{=} -\frac{1}{4}\int dt = -\frac{1}{4}t+c \\ &= -\frac{1}{4}\sqrt{3-4x^2}+c,\end{aligned}$$

5)

$$\begin{aligned}\int \frac{1}{\sqrt{x}\cos^2\left(\sqrt{x}\right)}dx &\overset{(t=\sqrt{x})}{=} 2\int \sec^2(t)dt = 2\tan(t)+c \\ &= 2\tan(\sqrt{x})+c\end{aligned}$$

6)

$$\begin{aligned}\int \frac{x^2+3x+6}{\sqrt{x+1}}dx &\overset{(t=\sqrt{x+1})}{=} 2\int \left((t^2-1)^2+3(t^2-1)+6\right)dt \\ &= 2\int \left(t^4+t^2+4\right)dt \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + 4(x+1)^{\frac{1}{2}} + c,\end{aligned}$$

7) $\displaystyle\int x^3\sqrt{x^4+1}\ dx \overset{t=x^4+1}{=} \frac{2}{9}(x^4+1)^{\frac{3}{2}}+c$

8) $\displaystyle\int \frac{3x}{(1+x^2)^{\frac{2}{3}}}\ dx \overset{t=x^2+1}{=} \frac{3}{2}\int \frac{dt}{t^{\frac{2}{3}}} = \frac{1}{2}(1+x^2)^{\frac{1}{3}}+c$

9) $\displaystyle\int \frac{3x^5}{\sqrt[3]{x^6+1}}\ dx \overset{t=x^6+1}{=} \frac{1}{2}\int \frac{dt}{t^{\frac{1}{3}}} = \frac{3}{4}(x^6+1)^{\frac{2}{3}}+c$

10) $\displaystyle\int x^2\sec^2(x^3-2)\ dx \overset{t=x^3-2}{=} \frac{1}{3}\int \sec^2 t\ dt = \frac{1}{3}\tan(x^3-2)+c$

11) $\displaystyle\int \frac{\csc^2(\sqrt{x})}{\sqrt{x}}\ dx \overset{t=\sqrt{x}}{=} 2\int \csc^2 t dt = -2\cot(\sqrt{x})+c$

12) $\displaystyle\int \frac{\sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)}{x^2}\ dx \overset{t\frac{1}{x}}{=} -\int \sec t\tan t dt = -\sec(\frac{1}{x})+c$

13) $\displaystyle\int \frac{(2+\sin x)^4}{\sec x}\ dx \overset{2+\sin x}{=} \int t^4 dt = \frac{1}{5}(2+\sin x)^5+c$

1-2-2

1) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^3(x)}dx \overset{(t=\cos(x))}{=} -\int_1^{\frac{\sqrt{2}}{2}} \frac{dt}{t^3} = \left[\frac{1}{2t^2}\right]_1^{\frac{\sqrt{2}}{2}} = \frac{1}{2},$

2)

$$\begin{aligned}\int_0^1 \frac{dx}{(1+x^2)^2} &\overset{(x=\tan(\theta))}{=} \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sec^2(\theta)} = \int_0^{\frac{\pi}{4}} \cos^2(\theta)d\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} 1+\cos(2\theta)d\theta = \frac{\pi}{8}+\frac{1}{8},\end{aligned}$$

3) $\displaystyle\int_0^1 x\sqrt{x^2+1}dx \overset{(t=1+x^2)}{=} \frac{1}{2}\int_1^2 \sqrt{t}dt = \frac{1}{3}\left[t^{\frac{3}{2}}\right]_1^2 = \frac{2^{\frac{3}{2}}-1}{3},$

Exercises

Recall that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$,
$\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$.

1-3-1 1) Find the value of n such that $\displaystyle\sum_{k=1}^{n}(2k^2 - k + 1) = 147$.

2) Find the value of α such that $\displaystyle\sum_{k=1}^{6}\left(k^2 + 3k + 2\alpha\right) = 130$,

1-3-2 Express the sum $\displaystyle\sum_{k=1}^{n} k(k+1)$ in terms of n.

1-3-3 Find the value of a satisfying the following identities

1) $\displaystyle\sum_{k=1}^{10}(ak - 10) = 120$

2) $\displaystyle\sum_{k=1}^{5}(ak^2 + 2) = 120$

3) $\displaystyle\sum_{k=5}^{15}(ak + 5) = 275$

1-3-4 Find the following limits.

1) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n}(3k - 2)$

2) $\displaystyle\lim_{n\to+\infty} \sum_{k=1}^{n}(2\frac{k}{n^2} - \frac{3}{n})$

3) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^3}\sum_{k=1}^{n}(3k^2 - 2k + 1)$

4) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^4}\sum_{k=1}^{n}(k^3 - 3k^2 + 2)$

5) $\lim\limits_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}(\frac{k^2}{n^2} - 3\frac{k}{n} + 1)$

6) $\lim\limits_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n}(k^2 - k + 1)$

7) $\lim\limits_{n\to\infty} \frac{1}{n^4}\sum_{k=1}^{n}(2k^3 + 4)$

1-3-5 Find the Riemann sum $R(f, P, w)$ for the function f defined by $f(x) = 3x - 2$ on the interval $[-2, 2]$ with respect to the partition $P = \{-2, 0, 1, 1.5, 2\}$ by choosing on each sub-interval of the partition

1) The left-hand end point $w_k = x_{k-1}$
2) The right-hand end point $w_k = x_k$
3) The mid-point $w_k = \frac{x_{k-1}+x_k}{2}$

1-3-6 Use the Riemann sums to find the following integrals:

1) $\int_0^1 (3x + 7)dx,$
2) $\int_1^4 (x^2 + x + 2)dx,$
3) $\int_0^2 (6x^3 + 1)dx,$
4) $\int_0^2 (3x - 2)\ dx$
5) $\int_1^3 (5x - 6)\ dx$
6) $\int_{-1}^4 (2x + 1)\ dx$
7) $\int_0^4 (x^2 + 1)\ dx$
8) $\int_2^4 (x^2 - x)\ dx$
9) $\int_0^3 (x^3 - 1)\ dx$
10) $\int_1^4 (x^3 + x)\ dx$

1-3-7 Find the following limits:

1) $\lim\limits_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}\sec^2\left(\frac{k}{n}\right).$

2) $\lim\limits_{n\to+\infty} \frac{1}{n^3}\sum_{k=1}^{n}(k-1)(k+2).$

1-3-8 Evaluate the following integrals:

1) $\displaystyle\int_{-\frac{\pi}{2}}^{\pi} f(t)dt$, where $f(t) = \begin{cases} \cos(t), & \text{for } \ t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \sin(t), & \text{for } \ t \in [\frac{\pi}{2}, \pi]. \end{cases}$,

2) $\displaystyle\int_0^2 |x-1|\, dx.$

1-3-9 Express the following limits as an indefinite integrals:

1) $\displaystyle\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} \frac{1}{n+k}$,

2) $\displaystyle\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{n}{(2n+k)^2}$,

3) $\displaystyle\lim_{n \longrightarrow +\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2}$,

4) $\displaystyle\lim_{n \longrightarrow +\infty} \frac{1}{n} \sum_{k=1}^{n} \sin(\frac{kx}{n}), \quad x \in \mathbb{R}$,

5) $\displaystyle\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2}$,

6) $\displaystyle\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$,

7) $\displaystyle\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^2}}$,

8) $\displaystyle\lim_{n \longrightarrow +\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 \sin(\frac{k\pi}{n})$,

9) $\displaystyle\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{n} \cos\left(\frac{k\pi}{n}\right)$,

10) $\displaystyle\lim_{n \to +\infty} \sum_{k=1}^{2^n} \frac{k^3}{2^{4n}}$.

Solutions of Exercises

$\boxed{\text{1-3-1}}$ 1) $\displaystyle\sum_{k=1}^{n}(2k^2-5k+1)=\frac{n}{6}\left(4n^2-9n-7\right).$

The value of n such that $\displaystyle\sum_{k=1}^{n}(2k^2-5k+1)=147$ is 7.

2) $\displaystyle\sum_{k=1}^{6}\left(k^2+3k+2\alpha\right)=\frac{6(7)(13)}{6}+3\frac{6(7)}{2}+12\alpha=154+12\alpha.$

Then the value of α such that $\displaystyle\sum_{k=1}^{6}\left(k^2+3k+2\alpha\right)=130$ is -2.

$\boxed{\text{1-3-2}}$ $\displaystyle\sum_{k=1}^{n}k(k+1)=\sum_{k=1}^{n}k^2+k=(n+1)\left(\frac{n(2n+1)}{6}+\frac{n}{2}\right)=\frac{n(n+1)(n+2)}{3}.$

$\boxed{\text{1-3-3}}$ 1) $\displaystyle\sum_{k=1}^{10}(ak-10)=55a-100=120.$ Then $a=4$.

2) $\displaystyle\sum_{k=1}^{5}(ak^2+2)=55a+10=120.$ Then $a=2$.

3) $\displaystyle\sum_{k=5}^{15}(ak+5)=110a+55=275.$ Then $a=2$.

$\boxed{\text{1-3-4}}$ 1) $\displaystyle\frac{1}{n^2}\sum_{k=1}^{n}(3k-2)=\frac{1}{n^2}\left(\frac{3n(n+1)}{2}-2n\right).$ Then

$\displaystyle\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}(3k-2)=\frac{3}{2}.$

2) $\displaystyle\sum_{k=1}^{n}(2\frac{k}{n^2}-\frac{3}{n})=\frac{n+1}{n}-3.$ Then $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n}(2\frac{k}{n^2}-\frac{3}{n})=-2$

3)

$$\begin{aligned}\sum_{k=1}^{n}(3k^2-2k+1) &= \frac{n(n+1)(2n+1)}{2}-n(n+1)+n\\ &= \frac{n(n+1)(2n+1)}{2}-n^2.\end{aligned}$$

Then $\lim\limits_{n\to+\infty}\dfrac{1}{n^3}\sum\limits_{k=1}^{n}(3k^2-2k+1)=1.$

4)

$$\begin{aligned}\frac{1}{n^4}\sum_{k=1}^{n}(k^3-3k^2+2) &= \frac{1}{n^4}\left(\frac{n^2(n+1)^2}{4}-\frac{n(n+1)(2n+1)}{2}+2n\right)\\ &= \frac{1}{4}(1+\frac{1}{n})^2-\frac{1}{2n}(1+\frac{1}{n})(2+\frac{1}{n})+\frac{2}{n^3}.\end{aligned}$$

Then $\lim\limits_{n\to+\infty}\dfrac{1}{n^4}\sum\limits_{k=1}^{n}(k^3-3k^2+2)=\dfrac{1}{4}.$

5)

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}(\frac{k^2}{n^2}-\frac{3k}{n}+1) &= \frac{1}{n}\left(\frac{n(n+1)(2n+1)}{2n^2}-3\frac{n(n+1)}{2n}+n\right)\\ &= \frac{1}{2}(1+\frac{1}{n})(2+\frac{1}{n})-\frac{3}{2}(1+\frac{1}{n})+1.\end{aligned}$$

Then $\lim\limits_{n\to+\infty}\dfrac{1}{n}\sum\limits_{k=1}^{n}(\dfrac{k^2}{n^2}-3\dfrac{k}{n}+1)=1-\dfrac{3}{2}+1=\dfrac{1}{2}.$

6)

$$\begin{aligned}\frac{1}{n^3}\sum_{k=1}^{n}(k^2-k+1) &= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}+n\right)\\ &= \frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n})-\frac{1}{2n}(1+\frac{1}{n})+\frac{1}{n^2}.\end{aligned}$$

Then $\lim\limits_{n\to\infty}\dfrac{1}{n^3}\sum\limits_{k=1}^{n}(k^2-k+1)=\dfrac{1}{3}.$

7)

$$\begin{aligned}\frac{1}{n^4}\sum_{k=1}^{n}(2k^3+4) &= \frac{1}{n^4}\left(\frac{n^2(n+1)^2}{2}+4n\right)\\ &= \frac{1}{2}(1+\frac{1}{n})^2+\frac{4}{n^3}.\end{aligned}$$

Then $\lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n}(2k^3+4)=\frac{1}{2}$.

1-3-5

1) $R(f,P,w)=((-8)(2)-2+(0.5)+(2.5)(0.5))=-16.25$
2) $R(f,P,w)=((2)(-2)+1+(0.5)(2.5)+(4)(0.5))=0.25$
3) $R(f,P,w)=((-5)(2)+1.5+(0.5)(1.75)+(0.5)(3.25))=-7$

1-3-6

1) $\int_0^1(3x+7)dx=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}(3\frac{k}{n}+7)=\lim_{n\to+\infty}(\frac{3}{n^2}\frac{n(n+1)}{2}+7)=\frac{3}{2}+7=\frac{17}{2}$.

2)

$$\begin{aligned}\int_1^4(x^2+x+2)dx &= \lim_{n\to+\infty}\frac{3}{n}\sum_{k=1}^{n}\left(\left(1+3\frac{k}{n}\right)^2+(1+3\frac{k}{n})+2\right)\\ &= \lim_{n\to+\infty}\frac{3}{n}\sum_{k=1}^{n}\left(1+6\frac{k}{n}+9\frac{k^2}{n^2}+1+3\frac{k}{n}+2\right)\\ &= \frac{69}{2}.\end{aligned}$$

3)

$$\begin{aligned}\int_0^2(6x^3+1)dx &= \lim_{n\to+\infty}\frac{2}{n}\sum_{k=1}^{n}6\left(2\frac{k}{n}\right)^3+1\\ &= \lim_{n\to+\infty}\frac{2}{n}\left(12\frac{(n+1)^2}{n}+n\right)=26.\end{aligned}$$

4)

$$\begin{aligned}\int_0^2 (3x-2)\ dx &= \lim_{n\to+\infty} \frac{2}{n}\left(\sum_{k=1}^{n} 3\frac{2k}{n} - 2\right)\\ &= \lim_{n\to+\infty} 6(1+\frac{1}{n}) - 4 = 2.\end{aligned}$$

Then $\int_0^2 (3x-2)\ dx = 2.$

5)

$$\begin{aligned}\int_1^3 (5x-6)\ dx &= \lim_{n\to+\infty} \frac{2}{n}\left(\sum_{k=1}^{n} 5(1+\frac{2k}{n}) - 6\right)\\ &= \lim_{n\to+\infty} 10 + 10(1+\frac{1}{n}) - 12 = 8.\end{aligned}$$

Then $\int_1^3 (5x-6)\ dx = 8.$

6)

$$\begin{aligned}\int_{-1}^4 (2x+1)\ dx &= \lim_{n\to+\infty} \frac{5}{n}\left(\sum_{k=1}^{n} 2(-1+\frac{5k}{n} + 1\right)\\ &= \lim_{n\to+\infty} -10 + \frac{25}{2}(1+\frac{1}{n}) + 5 = \frac{15}{2}.\end{aligned}$$

Then $\int_{-1}^4 (2x+1)\ dx = \frac{15}{2}$

7)

$$\begin{aligned}\int_0^4 (x^2+1)\ dx &= \lim_{n\to+\infty} \frac{4}{n}\left(\sum_{k=1}^{n} \frac{16k^2}{n^2}) + 1\right)\\ &= \lim_{n\to+\infty} \frac{64}{6}(1+\frac{1}{n})(2+\frac{1}{n}) + 4 = \frac{76}{3}.\end{aligned}$$

Then $\int_0^4 (x^2+1)\ dx = \frac{76}{3}.$

8)

$$\begin{aligned}\int_2^4 (x^2 - x)\ dx &= \lim_{n\to+\infty} \frac{2}{n}\left(\sum_{k=1}^{n}(2+\frac{4k^2}{n^2}) - \frac{2k}{n}\right)\\ &= \lim_{n\to+\infty} 4 + \frac{4}{3}(1+\frac{1}{n})(2+\frac{1}{n}) - 2(1+\frac{1}{n}) = \frac{14}{3}.\end{aligned}$$

Then $\int_2^4 (x^2 - x)\ dx = \frac{14}{3}$.

9)

$$\begin{aligned}\int_0^3 (x^3 - 1)\ dx &= \lim_{n\to+\infty} \frac{3}{n}\left(\sum_{k=1}^{n}(\frac{27k^3}{n^3}) - 1\right)\\ &= \lim_{n\to+\infty} \frac{81}{4}(1+\frac{1}{n}) - 3 = \frac{69}{4}.\end{aligned}$$

Then $\int_0^3 (x^3 - 1)\ dx = \frac{69}{4}$.

10)

$$\begin{aligned}\int_1^4 (x^3 + x)\ dx &= \lim_{n\to+\infty} \frac{3}{n}\left(\sum_{k=1}^{n}(1+\frac{3k}{n})^3 + (1+\frac{3k}{n})\right)\\ &= \lim_{n\to+\infty} \frac{3}{n}\left(\sum_{k=1}^{n} 2 + 12\frac{k}{n} + 27\frac{k^2}{n^2} + 27\frac{k^3}{n^3}\right)\\ &= \frac{285}{4}.\end{aligned}$$

Then $\int_1^4 (x^3 + x)\ dx = \frac{285}{4}$.

1-3-7 1) $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \sec^2\left(\frac{k}{n}\right) = \int_0^1 \sec^2(x)dx = \tan 1.$

2)

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{n^3}\sum_{k=1}^{n}(k-1)(k+2) &= \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k^2}{n^2} + \frac{k}{n^2} - \frac{2}{n^2}\right)\\ &= \int_0^1 x^2 dx = \frac{1}{3}\end{aligned}$$

1-3-8 1) $\displaystyle\int_{-\frac{\pi}{2}}^{\pi} f(t)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t)dt + \int_{\frac{\pi}{2}}^{\pi} \sin(t)dt = 3.$

2)

$$\begin{aligned}\int_0^2 |x-1|\, dx &= \int_0^1 (1-x)dx + \int_1^2 (x-1)dx \\ &= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2 = 1\end{aligned}$$

1-3-9 1) Let $f(x) = \dfrac{1}{1+x}$ on the interval $[0,1]$. The Riemann sum of f is

$$\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) = \sum_{k=1}^{n} \frac{1}{n+k},$$

then

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \int_0^1 \frac{dx}{1+x}.$$

2) Let $f(x) = \dfrac{1}{(2+x)^2}$ on the interval $[0,1]$. The Riemann sum of f is
$\displaystyle\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) = \sum_{k=1}^{n} \frac{n}{(2n+k)^2},$
then

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{n}{(2n+k)^2} = \int_0^1 \frac{dx}{(2+x)^2} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

3) Let $f(x) = x^2$ on the interval $[0,1]$. The Riemann sum of f is $\displaystyle\frac{1}{n}\sum_{k=1}^{n} \frac{k^2}{n^2}$, then

$$\lim_{n\longrightarrow+\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k^2}{n^2} = \frac{1}{3}.$$

4) Let $f(t) = \sin(xt)$ on the interval $[0, 1]$. The Riemann sum of f is $\frac{1}{n}\sum_{k=1}^{n} \sin(\frac{kx}{n})$, then

$$\lim_{n \longrightarrow +\infty} \frac{1}{n}\sum_{k=1}^{n} \sin(\frac{kx}{n}) = \frac{1 - \cos(x)}{x}.$$

5) $\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{k}{n^2 + k^2} = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\frac{k}{n}}{1 + \frac{k^2}{n^2}} = \int_0^1 \frac{x}{1 + x^2} dx.$

6) Let $f(t) = \frac{1}{1 + x^2}$ on the interval $[0, 1]$. The Riemann sum of f is $\frac{1}{n}\sum_{k=1}^{n} \frac{1}{1 + \frac{k^2}{n^2}} = \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$, then

$$\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2} = \int_0^1 \frac{dx}{1 + x^2}.$$

7) Let $f(t) = \frac{1}{\sqrt{1 + x^2}}$ on the interval $[0, 1]$. The Riemann sum of f is $\sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k^2}}$, then

$$\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k^2}} = \int_0^1 \frac{dx}{\sqrt{1 + x^2}}.$$

8) Let $f(t) = x^2 \sin(\pi x)$ on the interval $[0, 1]$. The Riemann sum of f is $\frac{1}{n^3}\sum_{k=1}^{n} k^2 \sin(\frac{k\pi}{n})$, then

$$\lim_{n \longrightarrow +\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 \sin(\frac{k\pi}{n}) = \int_0^1 x^2 \sin(\pi x) dx.$$

9) $\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{n} \cos\left(\frac{k\pi}{n}\right) = \int_0^1 \cos(\pi x) dx = 0.$

10) $\lim_{n \to +\infty} \sum_{k=1}^{2^n} \frac{k^3}{2^{4n}} = \lim_{n \to +\infty} \frac{1}{2^n} \sum_{k=1}^{2^n} \frac{k^3}{2^{3n}} = \int_0^1 x^3 dx = \frac{1}{4}.$

Exercises

1-4-1 Let I and J be the integrals defined by:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sin(x)+\cos(x)}dx \text{ and } J = \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)+\cos(x)}dx.$$

1) Prove that $I = J$, (Hint: use the substitution $t = \frac{\pi}{2} - x$).
2) Evaluate $I + J$.
3) Deduce the values of I and J.

1-4-2 Differentiate the following functions:

1) $f(x) = x\displaystyle\int_{\sqrt{\pi}}^{x} \cos(t^2)dt,$
2) $f(x) = \displaystyle\int_1^x \sin^3(t)dt,$
3) $f(x) = \displaystyle\int_x^{x^2} \cos^5(t)dt,$
4) $f(x) = \displaystyle\int_{\sin(x)}^{\cos(x)} (1-t^2)^{\frac{3}{2}}dt,$
5) $f(x) = \displaystyle\int_{\tan(x)}^{\sec(x)} (1+t^3)^{\frac{1}{3}}dt,$
6) $f(x) = \displaystyle\int_{\frac{1}{x}}^{2} (4+t^2)^{\frac{5}{2}}dt,$
7) $f(x) = \displaystyle\int_{2x}^{x^2} t\ln t dt,\ x > 0,$
8) $f(x) = \displaystyle\int_{-1}^{x} \sqrt{2+\sin t}\ dt$
9) $f(x) = \displaystyle\int_2^x \frac{1}{\sqrt{1+t^2}}\ dt$
10) $f(x) = \displaystyle\int_0^{\sqrt{x}} \frac{1}{1+t^2}\ dt$
11) $f(x) = \displaystyle\int_{-3}^{\sin x} \frac{t}{1+t^4}\ dt$
12) $f(x) = \displaystyle\int_{-1}^{x^2} \sqrt{3+\cos t}\ dt$
13) $f(x) = \displaystyle\int_{x^2}^{3} \sqrt{3+\cos t}\ dt$
14) $f(x) = \displaystyle\int_{\sin x}^{x^3} \frac{t}{2+t^2}\ dt$
15) $f(x) = \displaystyle\int_{\sqrt{x}}^{x^2+1} \frac{t^2}{1+t^2}\ dt$

1-4-3 Find $F'(0)$ if $F(x) = \displaystyle\int_{3x}^{3x^2+1} \frac{t}{4+t^2}\ dt$

1-4-4 Find $F'\left(\dfrac{\pi}{2}\right)$ if $F(x) = \displaystyle\int_{\cos x}^{\sin x} \frac{1}{\sqrt{t^2+1}}\ dt$

1-4-5 Find the number c that satisfies the conclusion of the Mean Value Theorem for the following functions

1) $f(x) = 3x + 7$ on $[0, 1]$.
2) $f(x) = x^2 + x + 2$ on $[1, 4]$.
3) $f(x) = 6x^3 + 1$ on $[0, 2]$.
4) $f(x) = ax + b, \quad a \neq 0$, on $[\alpha, \beta]$.

Solutions of Exercises

$\boxed{\text{1-4-1}}$ 1) $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{\sin(x)+\cos(x)}dx \overset{(t=\frac{\pi}{2}-x)}{=} \int_0^{\frac{\pi}{2}} \frac{\cos(t)}{\sin(t)+\cos(t)}dt = J.$

2) $I + J = \displaystyle\int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$

3) $2I = I + J = \dfrac{\pi}{2}$, then $I = \dfrac{\pi}{4} = J.$

$\boxed{\text{1-4-2}}$ 1) $f'(x) = \displaystyle\int_{\sqrt{\pi}}^{x} \cos(t^2)dt + x\cos(x^2),$

2) $f'(x) = \sin^3(x),$

3) $f'(x) = 2x\cos^5(x^2) - \cos^5(x),$

4)

$$\begin{aligned} f'(x) &= -\sin(x)(1-\cos^2(x))^{\frac{3}{2}} - \cos(x)(1-\sin^2(x))^{\frac{3}{2}} \\ &= -\sin(x)|\sin^3(x)| - \cos(x)|\cos^3(x)|, \end{aligned}$$

5) $f'(x) = \sec(x)\tan(x)(1+\sec^3(x))^{\frac{1}{3}} - \sec^2(x)(1+\tan^3(x))^{\frac{1}{3}},$

6) $f'(x) = \dfrac{1}{x^2}(4 + \dfrac{1}{x^2})^{\frac{5}{2}}$

7) $f'(x) = 4x^3\ln(x) - 4x\ln(2x),$

8) $f'(x) = \sqrt{2+\sin x}$

9) $f'(x) = \dfrac{1}{\sqrt{1+x^2}}$

10) $f'(x) = \dfrac{1}{2\sqrt{x}(1+x)}$

11) $f'(x) = \dfrac{\sin x\cos x}{1+\sin^4 x}$

12) $f'(x) = 2x\sqrt{3+\cos(x^2)}$

13) $f'(x) = -2x\sqrt{3+\cos(x^2)}$

14) $f'(x) = \dfrac{3x^5}{2+x^6} - \dfrac{\cos x\sin x}{2+\sin^2 x}$

15) $f'(x) = \dfrac{2x(x^2+1)^2}{1+(x^2+1)^2} - \dfrac{1}{2\sqrt{x}}\dfrac{x}{1+x}.$

$\boxed{\text{1-4-3}}$ $F'(x) = \dfrac{6x(3x^2+1)}{4-(3x^2+1)^2} - \dfrac{9x}{4+9x^2}$, then $F'(0) = 0$.

$\boxed{\text{1-4-4}}$ $F'(x) = \dfrac{\cos x}{\sqrt{1+\sin^2 x}} + \dfrac{\sin x}{\sqrt{1+\cos^2 x}}$, then $F'\left(\dfrac{\pi}{2}\right) = 1$.

$\boxed{\text{1-4-5}}$ 1) Since $\displaystyle\int_0^1 3x+7dx = \frac{3}{2}+7$, then the point c where f reached its average value verifies $3c+7 = \frac{3}{2}+7 \Rightarrow c = \dfrac{1}{2}$.

2) Since $\displaystyle\int_0^1 (x^2+x+2)dx = \frac{17}{6}$, then the point c where f reached its average value verifies $c^2+c+2 = \frac{17}{6} \Rightarrow c = -\dfrac{1}{2}(1-\dfrac{\sqrt{13}}{\sqrt{3}})$.

3) Since $\displaystyle\int_0^1 (6x^3+1)dx = \frac{5}{2}$, then the point c where f reached its average value verifies $6c^3+1 = \frac{5}{2} \Rightarrow c = 2^{-\frac{2}{3}}$.

4) $\displaystyle\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)dx = \frac{a}{2}(\beta+\alpha)+b$. Then $c = \dfrac{\beta+\alpha}{2}$.

Exercises

1-5-1 1) Approximate the integral $\int_0^{\pi} \sqrt{1+\sin(x)}dx$ using trapezoidal rule with $n = 4$ and the regular partition. Give an approximation of the error.

2) Approximate $\int_0^5 \frac{dx}{\sqrt{1+x^4}}$ using trapezoidal rule with $n = 5$.

3) Approximate the integral $\int_0^2 \frac{x}{\sqrt{x+1}}dx$ using Simpson's rule for $n = 4$ and $n = 8$. Give an approximate of the reminder in each case.

1-5-2 Let $f(x) = 2x-1$ and $g(x) = x^2+3x-1$ defined on the interval $[1, 3]$. Use trapezoidal method for $n = 5$ to approximate the integrals $\int_1^3 f(x)dx$ and $\int_1^3 g(x)dx$.

1-5-3 Let $g(x) = x^2 + 3x - 1$ and $h(x) = x^3$ defined on the interval $[1, 3]$. Use Simpson method for $n = 8$ to approximate the integrals $\int_1^3 (x^2 + 3x - 1)dx$ and $\int_1^3 x^3 dx$.

Solutions of Exercises

1-5-1 1)

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	0	1	1	1
1	$\pi/4$	$\sqrt{1+\frac{\sqrt{2}}{2}}$	2	$2\sqrt{1+\frac{\sqrt{2}}{2}}$
2	$\pi/2$	$\sqrt{2}$	2	$2\sqrt{2}$
3	$3\pi/4$	$\sqrt{1+\frac{\sqrt{2}}{2}}$	2	$2\sqrt{1+\frac{\sqrt{2}}{2}}$
4	π	1	1	1
				10.0546

$\displaystyle\int_0^{\pi} \sqrt{1+\sin(x)}dx \approx \frac{\pi}{8}(10.0546) \approx 3.9484632.$

If $f(x) = \sqrt{1+\sin(x)}$, $f''(x) = -\dfrac{\sqrt{1+\sin(x)}}{4}$. The reminder R_2 fulfills
$|R_2| \leq 610^{-2}$.
The exact value of the integral can be calculated by the change of variable
$t = \tan(\dfrac{x}{2})$ as follows:

$$\begin{aligned}
\int_0^{\pi} \sqrt{1+\sin(x)}dx &= 2\int_0^{\frac{\pi}{2}} \sqrt{1+\sin(x)}dx \\
&\overset{(t=\tan(\frac{x}{2}))}{=} 4\int_0^{1} \frac{1+t}{(1+t^2)^{\frac{3}{2}}}dt \\
&\overset{(t=\tan(\theta))}{=} 4\int_0^{\frac{\pi}{4}} (\cos(\theta)+\sin(\theta))d\theta \\
&= 4.
\end{aligned}$$

2) $\displaystyle\int_0^{5} \frac{dx}{\sqrt{1+x^4}} \approx \frac{1}{2}(3.284871) \approx 1.642435$

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	0	1	1	1
1	1	$1/\sqrt{2}$	2	$\sqrt{2} \approx 1.414213$
2	2	$1/\sqrt{17}$	2	$2/\sqrt{17} \approx 0.485071$
3	3	$1/\sqrt{82}$	2	$2/\sqrt{82} \approx 0.220863$
4	4	$1/\sqrt{257}$	2	$2/\sqrt{257} \approx 0.124756$
5	5	$1/\sqrt{626}$	1	$1/\sqrt{626} \approx 0.039968$
				3.284871

3)

$$\int_0^2 \frac{x}{\sqrt{x+1}}dx \overset{(t=\sqrt{x+1})}{=} 2\int_1^{\sqrt{3}} (t^2-1)dt = \frac{4}{3}$$

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	0	0	1	0
1	$\frac{1}{2}$	$\sqrt{2}/2\sqrt{3}$	4	$2\sqrt{2}/\sqrt{3}$
2	1	$1/\sqrt{2}$	2	$\sqrt{2}$
3	$\frac{3}{2}$	$3/\sqrt{10}$	4	$12/\sqrt{10}$
4	2	$2/\sqrt{3}$	1	$2/\sqrt{3}$
				7.9966404

$$\int_0^2 \frac{x}{\sqrt{x+1}}dx \approx 1.332773.$$

$f''(x) = -4(4+x)(x+1)^{-\frac{5}{2}}$. It is easy to prove that $f^{(5)} \geq 0$, then

$$\sup_{x\in[0,2]} |f''(x)| = 8.3^{-\frac{3}{2}}.$$

The reminder R_4 verifies $|R_4| \leq 3^{-\frac{5}{2}} \leq 7.10^{-2}$.

For $n = 8$

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	0	0	1	0
1	$\frac{1}{4}$	$\frac{1}{2\sqrt{5}}$	4	$\frac{2}{\sqrt{5}} \approx 0.894427$
2	$\frac{1}{2}$	$\frac{1}{\sqrt{6}}$	2	$\frac{2}{\sqrt{6}} \approx 0.8164965$
3	$\frac{3}{4}$	$\frac{3}{2\sqrt{7}}$	4	$\frac{6}{\sqrt{7}} \approx 2.2677868$
4	1	$\frac{1}{\sqrt{2}}$	2	$\sqrt{2} \approx 1.41421356$
5	$\frac{5}{4}$	$\frac{5}{6}$	4	$\frac{10}{3} \approx 3.33333333$
6	$\frac{3}{2}$	$\frac{3}{\sqrt{10}}$	2	$\frac{6}{\sqrt{10}} \approx 1.89736659$
7	$\frac{7}{4}$	$\frac{7}{2\sqrt{11}}$	4	$\frac{14}{\sqrt{11}} \approx 4.2211588$
8	2	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}} \approx 1.154701$
				15.99948

$\displaystyle\int_0^2 \frac{x}{\sqrt{x+1}}dx \approx 1.33329.$

$f^{(4)}(x) = -\dfrac{15}{16}(8+x)(x+1)^{-\frac{9}{2}}$. It is easy to prove that $f^{(5)} \geq 0$, then $\displaystyle\sup_{x\in[0,2]} |f^{(4)}(x)| = \frac{15}{16}(12)(3)^{-\frac{9}{2}} = \frac{5}{36\sqrt{3}} \leq 0.081.$

The reminder R_8 verifies $|R_8| \leq \frac{5}{2880.9.8^3} \leq 3.10^{-4}$.

1-5-2 $x_k = 1+\dfrac{2k}{5}$, $f(x_k) = 1+\dfrac{4k}{5}$ and $g(x_k) = 3+2k+\dfrac{4k^2}{25}$, $k = 0, \dots, 5$.

$$\int_1^3 (2x-1)dx \approx \frac{1}{5}\left(1+5+2\sum_{k=1}^{4}(1+\frac{4k}{5})\right) = 6.$$

The exact value of the integral $\displaystyle\int_1^3 (2x-1)dx$ is 6.

$$\begin{aligned}\int_1^3 (x^2+3x-1)dx &\approx \frac{1}{5}\left(3+17+2\sum_{k=1}^{4}(1+\frac{4k}{5})^2+3(1+\frac{4k}{5})-1\right)\\ &= \frac{1}{5}\left(20+2\sum_{k=1}^{4}(\frac{4k^2}{25}+2k+3)\right)\\ &= \frac{1}{5}(93+\frac{3}{5}) = 18.72.\end{aligned}$$

The exact value of the integral $\int_1^3 (x^2+3x-1)dx$ is $19-\frac{1}{3}$.

1-5-3 $x_k = 1+\frac{k}{4}$, $g(x_k) = 3+\frac{5k}{4}+\frac{k^2}{16}$ and $h(x_k) = 1+\frac{3k}{4}+\frac{3k^2}{16}+\frac{k^3}{64}$, $k = 0,\ldots,8$.

k	x_k	$g(x_k)$	m_k	$m_k g(x_k)$
0	1	3	1	3
1	$\frac{5}{4}$	$\frac{69}{16}$	4	$\frac{69}{4} = 17.25$
2	$\frac{3}{2}$	$\frac{23}{4}$	2	$\frac{23}{2} = 11.5$
3	$\frac{7}{4}$	7.3125	4	29.25
4	2	9	2	18
5	$\frac{9}{4}$	10.8125	4	43.25
6	$\frac{5}{2}$	12.75	2	25.5
7	$\frac{11}{4}$	14.8125	4	59.25
8	3	17	1	17
				233

$$\int_1^3 (x^2+3x-1)dx \approx \frac{224}{12} = 18.66.$$

k	x_k	$h(x_k)$	m_k	$m_k h(x_k)$
0	1	1	1	1
1	$\frac{5}{4}$	1.953125	4	7.8125
2	$\frac{3}{2}$	$\frac{27}{8}$	2	$\frac{27}{4} = 6.75$
3	$\frac{7}{4}$	5.359375	4	21.4375
4	2	8	2	16
5	$\frac{9}{4}$	11.390625	4	45.5625
6	$\frac{5}{2}$	15.625	2	31.25
7	$\frac{11}{4}$	20.796875	4	83.1875
8	3	27	1	27
				240

$$\int_1^3 x^3 dx \approx \frac{240}{12} = 20$$

2 Exercises on Chapter II

Exercises

2-1-1 Solve the following equations:

1) $\ln|x-1| = 7$ 2) $\ln|x^3-1| = 0$

2-1-2 Differentiate the following functions:

1) $f(x) = \ln(x^2+2x+4)$,
2) $f(x) = \ln(|2-3x|^5)$,
3) $f(x) = \ln\left(\dfrac{1-x}{1+x}\right)$, $-1 < x < 1$,
4) $f(x) = \ln\left|x^4+x^3+1\right|$
5) $f(x) = \ln\left|x^2+\cos(2x)\right|$
6) $f(x) = \sin x \ \ln|5x|$
7) $f(x) = \tan\left(\ln|3x|\right)$
8) $f(x) = [3x + \ln|\sin x|]^8$
9) $f(x) = \ln\left|\dfrac{\sqrt{x^2+1}\ \sin^5 x}{(x^3+4)^2}\right|$
10) $f(x) = \dfrac{(3x+1)^{\frac{3}{2}}\ (x^2-1)^{\frac{2}{3}}}{\sqrt[3]{x^2+2}}$

2-1-3 Find the derivative of the following functions:

1) $f(x) = \mathrm{Log}(x^2+4)$,
2) $f(x) = \ln(x+\sqrt{x^2-4})$,
3) $f(x) = \ln(x+\sqrt{4+x^2})$,

2-1-4 Differentiate the following functions:

1) $f(x) = \dfrac{(x^2+1)^3(x^2+4)^{10}}{(x^2+2)^5(x^2+3)^4}$, 2) $f(x) = \dfrac{(x+1)^3(2x-3)^{\frac{3}{4}}}{(1+7x)^{\frac{1}{3}}(2x+3)^{\frac{3}{2}}}$,

3) $f(x) = \sqrt{(3x^2+2)\sqrt{6x-7}}$,   4) $f(x)=(x+1)^2(x+2)^3(x-5)^7$.

$\boxed{\text{2-1-5}}$ Use implicit differentiation to find y' if

1) $y^2 + \ln(\frac{x}{y}) - 4x = -3$,   2) $xe^y + 2x - \ln(y+1) = 3$.

Solutions of Exercises

$\boxed{\text{2-1-1}}$ 1) $\ln|x-1| = 7 \iff x = 1 + e^7$

2) $\ln|x^3 - 1| = 0 \iff x^3 - 1 = 1 \iff x = \sqrt[3]{2}$

$\boxed{\text{2-1-2}}$ 1) $f'(x) = \dfrac{2x+2}{x^2+2x+4}$;

2) $f'(x) = \dfrac{15}{3x-2}$;

3) $f'(x) = -\dfrac{1}{1-x} - \dfrac{1}{1+x} = \dfrac{2}{x^2-1}$;

4) $f'(x) = \dfrac{4x^3+3x^2}{x^4+x^3+1}$;

5) $f'(x) = \dfrac{2x - 2\sin(2x)}{x^2+\cos(2x)}$;

6) $f'(x) = \cos x - \dfrac{1}{x}$;

7) $f'(x) = \dfrac{1}{x}\sec^2(\ln|3x|)$;

8) $f'(x) = 8(3+\cot x)\left[3x + \ln|\sin x|\right]^7$;

9) $f'(x) = \dfrac{1}{x^2+1} + 5\cot x + \dfrac{6x^2}{x^3+4}$;

10) $f'(x) = f(x)\left(\dfrac{9}{2(3x+1)} + \dfrac{4}{3(x^2-1)} - \dfrac{2x}{3(x^2+2)}\right)$;

11) $f'(x) = \dfrac{2x}{(x^2+4)\ln(10)}$;

12) $f'(x) = \dfrac{1}{\sqrt{x^2-4}}$;

13) $f'(x) = \dfrac{1}{\sqrt{4+x^2}}$.

14) $f'(x) = f(x)\left(\dfrac{6x}{x^2+1} + \dfrac{20x}{x^2+4} - \dfrac{10x}{x^2+2} - \dfrac{8x}{x^2+3}\right)$,

15) $f'(x) = f(x)\left(\dfrac{3}{x+1} + \dfrac{3}{2(2x-3)} - \dfrac{7}{3(1+7x)} - \dfrac{3}{2x+3}\right)$,

16) $\ln(f(x)) = \dfrac{1}{2}\ln(3x^2+2) + \dfrac{1}{4}\ln(6x-7)$. Then
$\dfrac{f'(x)}{f(x)} = \dfrac{3x}{3x^2+2} + \dfrac{3}{2(6x-7)}$ and

$$f'(x) = \Big(\frac{3x}{3x^2+2} + \frac{3}{2(6x-7)}\Big)\sqrt{(3x^2+2)\sqrt{6x-7}}.$$

17) $\ln|f(x)| = 2\ln|x+1| + 3\ln|x+2| + 7\ln|x-5|$.
Then $\dfrac{f'(x)}{f(x)} = \dfrac{2}{x+1} + \dfrac{3}{x+2} + \dfrac{7}{x-5}$ and

$$f'(x) = \Big(\frac{2}{x+1} + \frac{3}{x+2} + \frac{7}{x-5}\Big)(x+1)^2(x+2)^3(x-5)^7.$$

2-1-3 1) $2yy' + \frac{1}{x} - \frac{y'}{y} - 4 = 0$. Then $(2y - \dfrac{1}{y})y' = 4 - \dfrac{1}{x}$ and
$y' = \dfrac{4 - \frac{1}{x}}{2y - \frac{1}{y}} = \dfrac{4xy - y}{2y^2x - x}$.

2) $e^y + xy'e^y + 2 - \dfrac{y'}{y+1} = 0$. Then $xy'e^y - \dfrac{y'}{y+1} = -(2+e^y)$
and $y' = -\dfrac{2+e^y}{xe^y - \frac{1}{y+1}}$.

Exercises

$\boxed{\text{2-2-1}}$ Solve the following equations:

1) $e^{2x-1} = 5$

2) $e^{x^2-4} = 1$

$\boxed{\text{2-2-2}}$ Differentiate the following functions:

1) $f(x) = e^{1-x^2}$,

2) $f(x) = e^{x\ln(x)}$,

3) $f(x) = x^2e^{-x^3}$,

$\boxed{\text{2-2-3}}$ Find the equation of the tangent line to the graph of the function $f(x)=x-e^{-x}$ that is parallel to the line (D) of equation $6x-2y=7$.

$\boxed{\text{2-2-4}}$ Solve the following equation for x:

$$\frac{e^x}{1+e^x} = \frac{1}{3}.$$

Solutions of Exercises

2-2-1 1) $e^{2x-1} = 5 \iff 2x - 1 = \ln 5 \iff x = \frac{1}{2}(\ln 5 + 1)$

2) $e^{x^2-4} = 1 \iff x^2 = 4 \iff x = \pm 2$

2-2-2 1) $f'(x) = -2xe^{1-x^2}$,

2) $f'(x) = (\ln(x) + 1)e^{x \ln(x)}$,

3) $f'(x) = 2xe^{-x^3} - 3x^4 e^{-x^3}$,

2-2-3 The equation of the tangent line is $y - y_1 = f'(x_1)(x - x_1)$ and the equation of D is $y = 3x - \frac{7}{2}$. Then the tangent line is parallel to D if and only if $f'(x_1) = 3$. Now, it suffices to find the solutions of the equation $f'(x) = 3$.
$f'(x) = 1 + e^{-x} = 3 \iff e^{-x} = 2 \iff x = -\ln(2)$. Therefore the equation of the tangent line is $y - 2 + \ln(2) = 3(x + \ln(2))$.

2-2-4 $\dfrac{e^x}{1 + e^x} = \dfrac{1}{3} \iff e^x = \dfrac{1}{2} \iff x = -\ln 2.$

Exercises

2-3-1 Evaluate the following integrals with the indicate change of variable:

1) $\displaystyle\int xe^{-x^2}dx, \quad (t = x^2),$

2) $\displaystyle\int \frac{\sin(\ln x)}{x}dx, \quad (t = \ln x),$

3) $\displaystyle\int_0^1 \frac{dx}{e^x+1}, \quad (t = e^x),$

2-3-2 Evaluate the following integrals:

1) $\displaystyle\int \frac{x-2}{x^2-4x+9}dx,$

2) $\displaystyle\int \frac{(2+\ln(x))^{10}}{x}dx,$

3) $\displaystyle\int \frac{\tan(e^{-3x})}{e^{3x}}dx,$

Solutions of Exercises

2-3-1 1) $\int xe^{-x^2}dx \overset{t=x^2}{=} \frac{1}{2}\int e^{-t}dt = -\frac{1}{2}e^{-x^2} + c,$

2) $\int \frac{\sin(\ln x)}{x}dx \overset{t=\ln x}{=} \int \sin(t)dt = -\cos(t)+c = -\cos(\ln x)+c,$

3) $\int_0^1 \frac{dx}{e^x+1} \overset{t=e^x}{=} \int_1^e \frac{dt}{t(t+1)} = \int_1^e (\frac{1}{t} - \frac{1}{t+1})dt = \ln\left(\frac{2e}{e+1}\right),$

2-3-2 1) $\frac{1}{2}\int \frac{2(x-2)}{x^2-4x+9}dx = \frac{1}{2}\ln|x^2-4x+9| + c,$

2)

$$\begin{aligned} \int (2+\ln(x))^{10}\frac{1}{x}dx &\overset{u=2+\ln(x)}{=} \int u^{10}du \\ &= \frac{u^{11}}{11} + c = \frac{(2+\ln(x))^{11}}{11} + c, \end{aligned}$$

3)

$$\begin{aligned} \int \frac{\tan(e^{-3x})}{e^{3x}}dx &\overset{u=e^{-3x}}{=} -\frac{1}{3}\int \tan(u)du \\ &= \frac{1}{3}\ln|\cos(u)| + c = \frac{1}{3}\ln|\cos(e^{-3x})| + c. \end{aligned}$$

Exercises

2-4-1 Find the derivative of the following functions:

1) $f(x) = 10^{x^2}$,
2) $f(x) = 2^{(x^3+1)}$,
3) $f(x) = 5^{(x^4+x^2)}$,
4) $f(x) = 6^{\sqrt{x}}$
5) $f(x) = (x^2+1)^{\sin(2x)}$,
6) $f(x) = (x^2+4)^{(x^3+1)}$,
7) $f(x) = (\sin(x)+3)^{(4\cos(x)+7)}$,
8) $f(x) = (e^{x^2}+1)^{(2x+1)}$,
9) $f(x) = x^2(x^2+1)^{(x^3+1)}$,

2-4-2 Evaluate the following integrals:

1) $\displaystyle\int \frac{(2^x+1)^2}{2^x}dx$,
2) $\displaystyle\int e^{3x}\sec^2(2+e^{3x})dx$,
3) $\displaystyle\int 10^{\cos(x)}\sin(x)dx$,
4) $\displaystyle\int x10^{x^2+3}dx$,
5) $\displaystyle\int_1^8 \left(\sqrt[3]{\frac{5}{x}}\right) dx$,
6) $\displaystyle\int x3^{2x^2}(3^{2x^2}+1)^{-4}dx$.

Solutions of Exercises

$\boxed{\text{2-4-1}}$

1) $f'(x) = 2x10^{x^2}\ln(10);$
2) $f'(x) = 3x^2 2^{(x^3+1)}\ln 2;$
3) $f'(x) = (4x^3 + 2x)5^{(x^4+x^2)}\ln 5;$
4) $f'(x) = 6^{\sqrt{x}}\ln(6)\dfrac{1}{2\sqrt{x}};$
5) $f'(x) = (x^2+1)^{\sin(2x)}\left(\dfrac{2x\sin(2x)}{1+x^2} + 2\cos(2x)\ln(1+x^2)\right);$
6) $f'(x) = \left(3x^2\ln(x^2+4) + \dfrac{2x(x^3+1)}{x^2+4}\right) f(x);$
7) $f'(x) = \left(-4\sin(x)\ln(\sin(x)+3) + \dfrac{\cos(x)(4\cos(x)+7)}{\sin(x)+3}\right) f(x);$
8) $f'(x) = \left(2\ln(e^{x^2}+1) + \dfrac{2x(2x+1)e^{x^2}}{e^{x^2}+1}\right) f(x);$
9) $f'(x) = \left(\dfrac{2}{x} + 3x^2\ln(x^2+1) + \dfrac{2x(x^3+1)}{x^2+1}\right) f(x);$
10) $f'(x) = f(x)\left(\ln(x^2+1) + \dfrac{2x^2}{x^2+1}\right);$
11) $f'(x) = f(x)\left(\cos x\ln x + \dfrac{\sin x}{x}\right);$
12) $f'(x) = f(x)\left(3x^2\ln(\cos x) - (x^3-1)\tan x\right).$

$\boxed{\text{2-4-2}}$

1)

$$\begin{aligned}\int \frac{(2^x+1)^2}{2^x}dx &= \int \frac{(2^x)^2 + 2.2^x + 1}{2^x}dx = \int 2^x + 2 + \frac{1}{2^{-x}}dx \\ &= \frac{2^x}{\ln 2} + 2x - \frac{2^{-x}}{\ln(2)} + c,\end{aligned}$$

2) $\displaystyle\int e^{3x}\sec^2(2+e^{3x})dx \overset{t=2+e^{3x}}{=} \frac{1}{3}\int \sec^2(t)dt = \frac{1}{3}\tan(2+e^{3x})+c,$

3) $\displaystyle\int 10^{\cos(x)}\sin(x)dx \overset{t=\cos(x)}{=} -\int 10^t dt = -\frac{10^{\cos(x)}}{\ln 10} + c,$

4) $\displaystyle\int x10^{x^2+3}dx \overset{t=x^2+3}{=} \frac{1}{2}\int 10^t dt = \frac{10^{x^2+3}}{2\ln(10)} + c,$

5) $\displaystyle\int_1^8 5^{\frac{1}{3}}x^{-\frac{1}{3}}dx = \frac{3}{2}.5^{\frac{1}{3}}.(4-1) = \frac{9}{2}5^{\frac{1}{3}}.$

6)

$$\begin{aligned}
\int x3^{2x^2}(3^{2x^2}+1)^{-4}dx \quad &\overset{(u=3^{2x^2}+1)}{=} \quad \frac{1}{4\ln 3}\int u^{-4}du \\
&= \quad -\frac{1}{12\ln 3}u^{-3}+c \\
&= \quad -\frac{1}{12\ln 3}(3^{2x^2}+1)^{-3}+c.
\end{aligned}$$

Exercises

$\boxed{\text{2-5-1}}$ Solve the following equations for x:
$\log_3(x^4) + \log_3(x^3) - 2\log_3(x^{\frac{1}{2}}) = 5.$

$\boxed{\text{2-5-2}}$ Find the derivative of the following functions:

1) $f(x) = \log_5(x^3 + 1)$;
2) $f(x) = \sqrt{1 + \text{Log}(1 + x^2)}\,\log_3(1 + x^4)$.

Solutions of Exercises

2-5-1 $\dfrac{\ln(x^4)+\ln(x^3)-2\ln(x^{\frac{1}{2}})}{\ln 3} = 5 \iff \ln x^6 = 5\ln 3 \iff x = 3^{\frac{5}{6}}.$

2-5-2 1) $f'(x) = \dfrac{3x^2}{(x^3+1)\ln 5}$,

2)

$$\begin{aligned} f'(x) &= \frac{x}{(1+x^2)\sqrt{1+\mathrm{Log}(1+x^2)}}\log_3(1+x^4) \\ &\quad + \frac{4x^3}{(1+x^4)\ln 3}\sqrt{1+\mathrm{Log}(1+x^2)}. \end{aligned}$$

Exercises

2-6-1 Compute $\dfrac{dy}{dx}$ for each of the following:

1) $y = \sin^{-1}(\dfrac{x}{2})$,
2) $y = \cos^{-1}(\dfrac{x}{3})$,
3) $y = \tan^{-1}(\dfrac{x}{5})$,
4) $y = \cot^{-1}(\dfrac{x}{7})$,
5) $y = \sec^{-1}(\dfrac{x}{2})$,
6) $y = \csc^{-1}(\dfrac{x}{3})$,

2-6-2 Find the exact value of y in each of the following

1) $y = 3\sin^{-1}(\dfrac{1}{2})$
2) $y = 2\cos^{-1}(\dfrac{\sqrt{3}}{2})$,
3) $y = 4\tan^{-1}(\dfrac{1}{\sqrt{3}})$
4) $y = 5\cot^{-1}(\dfrac{1}{\sqrt{3}})$,
5) $y = 2\sec^{-1}(-2)$,
6) $y = 3\cos^{-1}(-\dfrac{\sqrt{3}}{2})$,
7) $y = \cos(2\cos^{-1}(x))$,
8) $y = \sin(2\cos^{-1}(x))$.
9) $y = \cos^{-1}(-\frac{1}{2})$,
10) $y = \sin^{-1}(\frac{\sqrt{3}}{2})$,
11) $y = \tan^{-1}(-\sqrt{3})$,
12) $y = \cot^{-1}(-\frac{\sqrt{3}}{3})$,
13) $y = \sec^{-1}(-\sqrt{2})$,
14) $y = \csc^{-1}(-\sqrt{2})$
15) $y = \sec^{-1}(-\frac{2}{\sqrt{3}})$,
16) $y = \csc^{-1}(-\frac{2}{\sqrt{3}})$,
17) $y = \sec^{-1}(-2)$,
18) $y = \csc^{-1}(-2)$,
19) $y = \tan^{-1}(\frac{-1}{\sqrt{3}}$,
20) $y = \cot^{-1}(-\sqrt{3})$.

$\boxed{\text{2-6-3}}$ Evaluate the following integrals.

1) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}dx$,

2) $\displaystyle\int_0^{2^{-\frac{1}{4}}} \frac{x}{\sqrt{1-x^4}}dx$,

3) $\displaystyle\int \frac{e^{\tan^{-1} x}}{1+x^2}dx$,

4) $\displaystyle\int \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}dx$,

5) $\displaystyle\int e^{\sin(2x)} \cos(2x)dx$,

6) $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}dx$,

7) $\displaystyle\int e^x \cos(1+2e^x)dx$,

8) $\displaystyle\int \frac{4^{\sec^{-1}(x)}}{x\sqrt{x^2-1}}dx$.

Solutions of Exercises

$\boxed{\text{2-6-1}}$

1) $y' = \dfrac{1}{\sqrt{4-x^2}}$,

2) $y' = -\dfrac{1}{\sqrt{9-x^2}}$,

3) $y' = \dfrac{5}{25+x^2}$,

4) $y' = -\dfrac{7}{49+x^2}$,

5) $y' = \dfrac{2}{x\sqrt{x^2-4}}$,

6) $y' = -\dfrac{3}{x\sqrt{x^2-9}}$,

$\boxed{\text{2-6-2}}$

1) $y = 3\sin^{-1}(\dfrac{1}{2}) = 3\dfrac{\pi}{6}$,

2) $y = 2\cos^{-1}(\dfrac{\sqrt{3}}{2}) = 2\dfrac{\pi}{6} = 5\dfrac{\pi}{6}$,

3) $y = 4\tan^{-1}(\dfrac{1}{\sqrt{3}}) = 4\dfrac{\pi}{6}$,

4) $y = 5\cot^{-1}(\dfrac{1}{\sqrt{3}}) = 5\dfrac{\pi}{3} = 7\dfrac{\pi}{3}$,

5) $y = 2\sec^{-1}(-2) = 2\dfrac{2\pi}{3}$,

6) $y = 3\cos^{-1}(-\dfrac{\sqrt{3}}{2}) = 3\dfrac{5\pi}{6} = 3\pi$,

7) $y = \cos(2\cos^{-1}(x)) = 2\cos^2(\cos^{-1}(x)) - 1 = 2x^2 - 1$,

8) $y = \sin(2\cos^{-1}(x)) = 2x\sin(\cos^{-1}(x)) = 2x\sqrt{1-x^2}$.

9) $y = \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$,

10) $y = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$,

11) $y = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$,

12) $y = \cot^{-1}(-\frac{\sqrt{3}}{3}) = \frac{2\pi}{3}$,

13) $y = \sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$,

14) $y = \csc^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$

15) $y = \sec^{-1}(-\frac{2}{\sqrt{3}}) = \frac{5\pi}{6}$,

16) $y = \csc^{-1}(-\frac{2}{\sqrt{3}}) = -\frac{\pi}{3}$,

17) $y = \sec^{-1}(-2) = \frac{2\pi}{3}$,

18) $y = \csc^{-1}(-2) = -\frac{\pi}{6}$,

19) $y = \tan^{-1}(\frac{-1}{\sqrt{3}} = \frac{5\pi}{6}$,

20) $y = \cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$.

2-6-3

1) $$\int \frac{x}{\sqrt{1-x^4}}dx \overset{t=x^2}{=} \frac{1}{2}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}\sin^{-1}(x^2) + c,$$

2) $$\int_0^{2^{-\frac{1}{4}}} \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\sin^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{8},$$

3) $$\int \frac{e^{\tan^{-1}x}}{1+x^2}dx \overset{t=\tan^{-1}x}{=} \int e^t dt = e^{\tan^{-1}x} + c,$$

4) $$\int \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}dx \overset{t=\sin^{-1}(x)}{=} \int e^t dt = e^{\sin^{-1}(x)} + c,$$

5) $$\int e^{\sin(2x)}\cos(2x)dx \overset{t=\sin(2x)}{=} \frac{1}{2}\int e^t dt = \frac{1}{2}e^{\sin(2x)} + c,$$

6) $$\int \frac{e^{2x}}{1+e^{2x}}dx \overset{t=1+e^{2x}}{=} \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln(1+e^{2x}) + c,$$

7) $$\int e^x\cos(1+2e^x)dx \overset{t=1+2e^x}{=} \frac{1}{2}\int \cos(t)dt = \frac{1}{2}\sin(1+2e^x) + c,$$

8) $$\int \frac{4^{\sec^{-1}(x)}}{x\sqrt{x^2-1}}dx \overset{t=\sec^{-1}(x)}{=} \int 4^t dt = \frac{4^{\sec^{-1}(x)}}{\ln 4} + c,$$

Exercises

$\boxed{\text{2-7-1}}$ Find the derivative of the following functions:

1) $f(x) = 4\mathrm{csch}^2(2x-1)$,
2) $f(x) = \sinh(2x)\mathrm{csch}(3x)$,
3) $f(x) = \log_2(\sec(x) + \tan(x))$,
4) $f(x) = (3\sinh(x) + \cos(x) + 5)^{(x^3+1)}$,
5) $\mathrm{sech}\,(1+\sqrt{x})$,
6) $\tan^{-1}(\sinh(x))$,
7) $\ln|\sinh(1-x^2)|$,
8) $x^{\cosh(x)}$.

$\boxed{\text{2-7-2}}$ Compute the following integrals:

(a) $\displaystyle\int \frac{1}{\mathrm{sech}(x)\sqrt{4-\sinh^2(x)}}dx$,

(b) $\displaystyle\int \frac{e^x}{1-e^{2x}}dx$,

(c) $\displaystyle\int \frac{e^x}{\sqrt{4e^{2x}+9}}dx$,

(d) $\displaystyle\int \frac{dx}{\sqrt{x}\sqrt{4+x}}$,

(e) $\displaystyle\int \frac{dx}{\sqrt{16-e^{2x}}}$.

(f) $\displaystyle\int \frac{dx}{\sqrt{1+e^{2x}}}$,

(g) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x-8}}$,

(h) $\displaystyle\int \frac{dx}{(x-1)\sqrt{-x^2+2x+3}}$,

Solutions of Exercises

2-7-1 1) $f'(x) = -16\text{csch}^2(2x-1)\text{coth}(2x-1)$,

2) $f'(x) = 2\cosh(2x)\text{csch}(3x) - 3\sinh(2x)\text{csch}(3x)\text{coth}(3x)$,

3) $f'(x) = \dfrac{\sec(x)}{\ln 2}$

4)

$$\begin{aligned} f'(x) &= \Big(3x^2\ln(3\sinh(x)+\cos(x)+5) \\ &\qquad + \frac{(x^3+1)(3\cosh(x)-\sin(x))}{3\sinh(x)+\cos(x)+5}\Big) f(x). \end{aligned}$$

5) $\dfrac{d}{dx}\text{sech}\left(1+\sqrt{x}\right) = -\text{sech}\left(1+\sqrt{x}\right)\tanh\left(1+\sqrt{x}\right)\dfrac{1}{2\sqrt{x}}$,

6) $\dfrac{d}{dx}\tan^{-1}(\sinh(x)) = \dfrac{\cosh(x)}{1+\sinh^2(x)} = \dfrac{\cosh(x)}{\cosh^2(x)} = \dfrac{1}{\cosh(x)} =$ $\text{sech}(x)$,

7) $\dfrac{d}{dx}\ln|\sinh(1-x^2)| = \dfrac{-2x\cosh(1-x^2)}{\sinh(1-x^2)} = -2x\coth(1-x^2)$,

8)

$$\begin{aligned} \frac{d}{dx}x^{\cosh(x)} &= \frac{d}{dx}e^{\cosh(x)\ln x} \\ &= \left(\sinh(x)\ln|x| + \frac{\cosh(x)}{x}\right)x^{\cosh(x)} \end{aligned}$$

2-7-2 (a)

$$\begin{aligned} \int \frac{1}{\text{sech}(x)\sqrt{4-\sinh^2(x)}}dx &= \int \frac{\cosh(x)}{\sqrt{(2)^2-(\sinh(x))^2}}dx \\ &= \sin^{-1}\left(\frac{\sinh(x)}{2}\right) + c, \end{aligned}$$

(b) $\displaystyle\int \frac{e^x}{1-e^{2x}}dx \overset{u=e^x}{=} \int \frac{du}{1-u^2} = \frac{1}{2}\ln\left|\frac{1+e^x}{1-e^x}\right| + c,$

(c) $\displaystyle\int \frac{e^x}{\sqrt{4e^{2x}+9}}dx \overset{u=2e^x}{=} \frac{1}{2}\int \frac{du}{\sqrt{u^2+(3)^2}} = \frac{1}{2}\sinh^{-1}\left(\frac{2e^x}{3}\right)+c,$

(d) $\displaystyle\int \frac{dx}{\sqrt{x}\sqrt{4+x}} \overset{x=t^2}{=} 2\int \frac{dt}{\sqrt{4+t^2}} = 2\sinh^{-1}\left(\frac{\sqrt{x}}{2}\right)+c,$

(e) $\displaystyle\int \frac{dx}{\sqrt{16-e^{2x}}} \overset{u=e^x}{=} \int \frac{du}{u\sqrt{4^2-u^2}} = -\frac{1}{4}\mathrm{sech}^{-1}\left(\frac{e^x}{4}\right)+c.$

(f) $\displaystyle\int \frac{dx}{\sqrt{1+e^{2x}}} \overset{u=e^x}{=} \int \frac{du}{u\sqrt{1+u^2}} = -\mathrm{csch}^{-1}(e^x)+c,$

(g)

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+2x-8}} &= \int \frac{dx}{\sqrt{(x^2+2x+1)-9}}\\ &= \int \frac{dx}{\sqrt{(x+1)^2-(3)^2}}\\ &= \cosh^{-1}\left(\frac{x+1}{3}\right)+c,\end{aligned}$$

(h)

$$\begin{aligned}\int \frac{dx}{(x-1)\sqrt{-x^2+2x+3}} &= \int \frac{dx}{(x-1)\sqrt{-(x^2-2x+1)+4}}\\ &= \int \frac{dx}{(x-1)\sqrt{(2)^2-(x-1)^2}}\\ &= -\frac{1}{2}\mathrm{sech}^{-1}\left(\frac{x-1}{2}\right)+c,\end{aligned}$$

Exercises

2-8-1 Use L'Hospital's rule when appropriate. When not appropriate, say so.

1) $\lim_{x\to 0} \dfrac{x}{\tan^{-1}(4x)}$,

2) $\lim\limits_{x\to+\infty} \sqrt{x}e^{-\frac{x}{2}}$,

3) $\lim\limits_{x\to+\infty} x\sin(\dfrac{1}{x})$,

4) $\lim\limits_{x\to 0}(\cot(x) - \dfrac{1}{x})$,

5) $\lim\limits_{x\to 0}(\csc(x) - \cot(x))$,

6) $\lim\limits_{x\to 0^+} (\tan(2x))^x$,

7) $\lim\limits_{x\to 0} \dfrac{\tan(x) - x}{x^3}$,

8) $\lim\limits_{x\to(\frac{\pi}{2})^-} \dfrac{2 - \sec(x)}{3\tan(x)}$,

9) $\lim\limits_{x\to 0^+} x^x$,

10) $\lim\limits_{x\to 0} \dfrac{2\sinh(x) - \sinh(2x)}{2x(\cos(x) - 1)}$,

11) $\lim\limits_{x\to 0} \dfrac{\sin(3x) - 3\sin(x)}{3x^3}$,

12) $\lim\limits_{x\to 0^+} (2x+1)^{\cot(x)}$,

13) $\lim\limits_{x\to 0} \dfrac{x - \tan(x)}{1 - \cos(x)}$,

14) $\lim\limits_{x\to 0^+} \big(\sec(x) + \tan(x)\big)^{\csc(x)}$,

15) $\lim\limits_{x\to 1^+} (\dfrac{3}{\ln(x)} - \dfrac{2}{x-1})$,

16) $\lim\limits_{x\to+\infty} (e^x + 1)^{\frac{1}{x}}$,

17) $\lim\limits_{x\to\infty} \dfrac{4e^x}{x^2}$,

18) $\lim\limits_{x\to\infty} \dfrac{e^{2x} - 1}{x}$,

19) $\lim\limits_{x\to\infty} e^{-x}\sqrt{x}$,

20) $\lim\limits_{x\to\infty} (1+4x)^{\frac{1}{x^2}}$.

2-8-2 Use L'Hospital's rule to find the sum $\displaystyle\sum_{k=1}^{n} k$.

Solutions of Exercises

2-8-1

1) $\lim_{x\to 0} \dfrac{x}{\tan^{-1}(4x)} = \lim\limits_{x\to 0} \dfrac{1+(4x)^2}{4} = \dfrac{1}{4}$,

2) $\lim\limits_{x\to +\infty} \sqrt{x}e^{-\frac{x}{2}} = \lim\limits_{x\to +\infty} \dfrac{\sqrt{x}}{e^{\frac{x}{2}}} = \lim\limits_{x\to +\infty} \dfrac{1}{\sqrt{x}e^{\frac{x}{2}}} = 0$,

3) $\lim\limits_{x\to +\infty} x\sin(\dfrac{1}{x}) = \lim\limits_{x\to +\infty} \dfrac{\sin(\frac{1}{x})}{\frac{1}{x}} = 1$,

4)

$$\begin{aligned}
\lim_{x\to 0}(\cot(x) - \frac{1}{x}) &= \lim_{x\to 0} \frac{x\cos(x) - \sin(x)}{x\sin(x)} \\
&= \lim_{x\to 0} \frac{-x\sin(x)}{\sin(x) + x\cos(x)} \\
&= \lim_{x\to 0} \frac{-\sin(x) - x\cos(x)}{2\cos(x) - x\sin(x)} = 0,
\end{aligned}$$

5) $\lim\limits_{x\to 0}(\csc(x) - \cot(x)) = \lim\limits_{x\to 0} \dfrac{1-\cos(x)}{\sin(x)} = \lim\limits_{x\to 0} \dfrac{\sin(x)}{\cos(x)} = 0$,

6)

$$\begin{aligned}
\lim_{x\to 0^+} (\tan(2x))^x &= \lim_{x\to 0^+} e^{x\ln(\tan(2x))} = \lim_{x\to 0^+} e^{\frac{\ln(\tan(2x))}{\frac{1}{x}}} \\
&= \lim_{x\to 0^+} e^{-\frac{2x^2}{\sin(4x)}} = 1,
\end{aligned}$$

7)

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan(x) - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{\tan^2 x}{3x^2} \\
&= \frac{1}{3} \lim_{x\to 0} \left(\frac{\tan(x)}{x}\right)^2 = \frac{1}{3}(1)^2 = \frac{1}{3},
\end{aligned}$$

8)

$$\begin{aligned}
\lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{2-\sec(x)}{3\tan(x)} &= \lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{-\sec(x)\tan(x)}{3\sec^2(x)}\\
&= \lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{-\tan(x)}{3\sec(x)}\\
&= \lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{-\sin(x)}{3} = -\frac{1}{3}.
\end{aligned}$$

9) $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{x\ln x} = \lim\limits_{x\to 0^+} e^{\frac{\ln x}{x^{-1}}} = \lim\limits_{x\to 0^+} e^{-x} = 1.$

10) $\dfrac{2\sinh(x)-\sinh(2x)}{2x(\cos(x)-1)} = \dfrac{2\sinh(x)-\sinh(2x)}{x^3}\dfrac{x^2}{2(\cos(x)-1)}.$

$$\begin{aligned}
\lim_{x\to 0}\frac{2\sinh(x)-\sinh(2x)}{x^3} &= \lim_{x\to 0}\frac{2\cosh(x)-2\cosh(2x)}{3x^2}\\
&= \lim_{x\to 0}\frac{2\sinh(x)-4\sinh(2x)}{6x}\\
&= \lim_{x\to 0}\frac{2\cosh(x)-8\cosh(2x)}{6} = -1,
\end{aligned}$$

and $\lim\limits_{x\to 0}\dfrac{x^2}{2(\cos(x)-1)} = \lim\limits_{x\to 0}\dfrac{2x}{-2\sin(x)} = \lim\limits_{x\to 0}\dfrac{2}{-2\cos(x)} = -1,$
then $\lim\limits_{x\to 0}\dfrac{2\sinh(x)-\sinh(2x)}{2x(\cos(x)-1)} = 1.$

11) $\lim\limits_{x\to 0}\dfrac{\sin(3x)}{3x^3} - \dfrac{\sin(x)}{x^3} = \lim\limits_{x\to 0}\dfrac{\sin(3x)-3\sin(x)}{3x^3},$

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(3x)-3\sin(x)}{3x^3} &= \lim_{x\to 0}\frac{3\cos(3x)-3\cos(x)}{9x^2}\\
&= \lim_{x\to 0}\frac{-9\sin(3x)+3\sin(x)}{18x}\\
&= \lim_{x\to 0}\frac{-27\cos(3x)+3\cos(x)}{18} = -\frac{4}{3}.
\end{aligned}$$

12) $\lim\limits_{x\to 0^+}(2x+1)^{\cot(x)} = \lim\limits_{x\to 0^+} e^{\frac{\ln(2x+1)}{\tan(x)}} = \lim\limits_{x\to 0^+} e^{\frac{2}{(2x+1)\sec^2(x)}} = e^2,$

13) $\lim_{x\to 0} \frac{x-\tan(x)}{1-\cos(x)} = \lim_{x\to 0} \frac{-\tan^2(x)}{\sin(x)} = 0,$

14)

$$\begin{aligned}\lim_{x\to 0^+} \big(\sec(x)+\tan(x)\big)^{\csc(x)} &= \lim_{x\to 0^+} e^{\frac{\ln(\sec(x)+\tan(x))}{\sin(x)}}\\ &= \lim_{x\to 0^+} e^{\sec^2(x)} = e,\end{aligned}$$

15)

$$\begin{aligned}\lim_{x\to 1^+} \left(\frac{3}{\ln(x)} - \frac{2}{x-1}\right) &= \lim_{x\to 1^+} \frac{3(x-1)-2\ln x}{(x-1)\ln(x)}\\ &= \lim_{x\to 1^+} \frac{3-\frac{2}{x}}{\ln(x)+\frac{x-1}{x}} = +\infty\end{aligned}$$

16) $\lim_{x\to+\infty} (e^x+1)^{\frac{1}{x}} = \lim_{x\to+\infty} e^{\frac{\ln(e^x+1)}{x}} = \lim_{x\to+\infty} e^{\frac{e^x}{e^x+1}} = e.$

17) $\lim_{x\to\infty} \frac{4e^x}{x^2} = \lim_{x\to\infty} \frac{2e^x}{x} = \lim_{x\to\infty} 2e^x = +\infty,$

18) $\lim_{x\to\infty} \frac{e^{2x}-1}{x} = \lim_{x\to\infty} 2e^{2x} = +\infty,$

19) $\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}e^x} = 0,$

20) $\lim_{x\to\infty} (1+4x)^{\frac{1}{x^2}} = \lim_{x\to\infty} e^{\frac{\ln(1+4x)}{x^2}} = \lim_{x\to\infty} e^{\frac{2}{x(1+4x)}} = 1,$

$\boxed{\text{2-8-2}}$ Let $f(x) = \sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$, for $x \neq 1$.

$$\begin{aligned}f'(1) = \sum_{k=1}^{n} k &= \lim_{x\to 1} f'(x)\\ &= \lim_{x\to 1} \frac{(n+1)x^n(x-1)-(x^{n+1}-1)}{(x-1)^2}\\ &= \lim_{x\to 1} \frac{(n+1)x^n + n(n+1)x^{n-1}(x-1)-(n+1)x^n}{2(x-1)}\\ &= \lim_{x\to 1} \frac{n(n+1)x^{n-1}}{2} = \frac{n(n+1)}{2}.\end{aligned}$$

3 Exercises on Chapter III

Exercises

2-9-1 Evaluate the following integrals:

1) $\displaystyle\int \ln^3(x)dx,$

2) $\displaystyle\int \ln(x^2-1)dx,$

3) $\displaystyle\int \ln\left(x^2+x+1\right)dx,$

4) $\displaystyle\int x^3e^x dx,$

5) $\displaystyle\int x^3\cos(x)dx,$

6) $\displaystyle\int x^4\cos(x)dx,$

7) $\displaystyle\int x^3\sin(x)dx,$

8) $\displaystyle\int x^4\sin(x)dx,$

9) $\displaystyle\int \cos(x)\cosh(x)dx,$

10) $\displaystyle\int e^{ax}\cos(bx)dx,$

11) $\displaystyle\int e^{ax}\sin(bx)dx,$

12) $\displaystyle\int x\tan^{-1}(x)dx,$

13) $\displaystyle\int x\sinh^{-1}(x)dx,$

Solutions of Exercises

3-1-1 1)

$$\begin{aligned}\int \ln^3(x)dx &\overset{u=\ln^3(x),\ v=x}{=} x\ln^3(x) - 3\int \ln^2(x)dx\\ &= x(\ln^3(x) - 3\ln^2(x) + 6\ln(x) - 6) + c\end{aligned}$$

2) Using integration by parts with $u = \ln(x^2-1)$, $v = x$, we get

$$\begin{aligned}\int \ln(x^2-1)dx &= x\ln(x^2-1) - 2\int \frac{x^2-1+1dx}{x^2-1}\\ &= x\ln(x^2-1) + \ln\left|\frac{x+1}{x-1}\right| - 2x + c,\end{aligned}$$

3) Using integration by parts with $u = \ln(x^2+x+1)$, $v' = 1$, we get

$$\begin{aligned}\int \ln(x^2+x+1)dx &= x\ln(x^2+x+1) - \int \frac{2x^2+2x+2-x-2}{x^2+x+1}dx\\ &= x\ln(x^2+x+1) - 2x + \frac{1}{2}\int \frac{2x+1+3}{x^2+x+1}dx\\ &= (x+\frac{1}{2})\ln(x^2+x+1) - 2x\\ &\quad +\frac{3}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}dx\\ &= \sqrt{3}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) - 2x\\ &\quad + \left(\frac{1}{2}+x\right)\ln\left(x^2+x+1\right) + c,\end{aligned}$$

4) $\displaystyle\int x^3e^x dx = (x^3 - 3x^2 + 6x - 6)e^x + c,$

5) $\displaystyle\int x^3\cos(x)dx = (x^3 - 6x)\sin(x) + (3x^2 - 6)\cos(x) + c,$

6) $\displaystyle\int x^4\cos(x)dx = (x^4 - 12x^2 + 24)\sin(x) + (4x^3 - 24x)\cos(x) + c,$

7) $\displaystyle\int x^3\sin(x)dx = (-x^3 + 6x)\cos(x) + (3x^2 - 6)\sin(x) + c,$

8) $\displaystyle\int x^4 \sin(x)dx = (-x^4 + 12x^2 - 24)\cos(x) + (4x^3 - 24x)\sin(x) + c,$

9) Using integration by parts in the first integration with $u = \cos(x)$, $v' = \cosh(x)$ and in the second integration with $u = \sin(x)$, $v' = \sinh(x)$, we get

$$\begin{aligned}\int \cos(x)\cosh(x)dx &= \cos(x)\sinh(x) + \int \sin(x)\sinh(x)dx \\ &= \cos(x)\sinh(x) + \sin(x)\cosh(x) \\ &\quad - \int \cos(x)\cosh(x)dx.\end{aligned}$$

Then

$$\int \cos(x)\cosh(x)dx = \frac{1}{2}\left[\sin(x)\cosh(x) + \cos(x)\sinh(x)\right] + c.$$

10) $\displaystyle\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos(bx) + b\sin(bx)\right) + c,$

11) $\displaystyle\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin(bx) - b\cos(bx)\right) + c,$

12) Using integration by parts in the first integration with $u = \tan^{-1}(x)$ and $v' = x$, we get

$$\begin{aligned}\int x\tan^{-1}(x)dx &= \frac{x^2}{2}\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}dx \\ &= \frac{1+x^2}{2}\tan^{-1}(x) - \frac{x}{2} + c,\end{aligned}$$

13) Using integration by parts in the first integration with $u = \sinh^{-1}(x)$ and $v' = x$, we get

$$\begin{aligned}\int x\sinh^{-1}(x)dx &= \frac{x^2}{2}\sinh^{-1}(x) - \frac{1}{2}\int \frac{x^2+1-1}{\sqrt{1+x^2}}dx \\ &= \frac{1+x^2}{2}\sinh^{-1}(x) - \frac{1}{2}\int \sqrt{x^2+1}dx\end{aligned}$$

$$
\begin{aligned}
\int \sqrt{x^2+1}dx &\overset{x=\tan(t)}{=} \int \sec^3(t)dt \\
&= \frac{1}{2}\sec(t)\tan(t) - \frac{1}{2}\ln|\sec(t)+\tan(t)| + c \\
&= \frac{x}{2}\sqrt{1+x^2} - \frac{1}{2}\sin^{-1}(x) + c,
\end{aligned}
$$

In second integration, we use integration by parts with $u = \sec(t)$, $v' = \sec^2(t)$.

Exercises

3-2-1 Evaluate the following integrals:

1) $\displaystyle\int \sinh(ax)\cosh(bx)dx$, for $|a| \neq |b|$

2) $\displaystyle\int \cosh^3(x)dx$,

3) $\displaystyle\int \sinh^3(x)dx$,

4) $\displaystyle\int \sinh^7(x)\ \cosh^3(x)dx$,

5) $\displaystyle\int \sinh^5(x)\ \cosh^4(x)dx$,

6) $\displaystyle\int \sinh^3(x)\ \cosh^2(x)dx$,

7) $\displaystyle\int \sinh^2(x)dx$,

8) $\displaystyle\int \sinh^4(x)dx$,

9) $\displaystyle\int \operatorname{sech}^5(x)\tanh^3(x)dx$,

10) $\displaystyle\int \tanh^3(x)\ \operatorname{sech}^3(x)dx$,

Solutions of Exercises

3-2-1 1)

$$\begin{aligned}\int \sinh(ax)\cosh(bx)dx &= \frac{1}{b^2-a^2}\left[b\cosh(bx)\sinh(ax)\right.\\ &\quad \left.-a\cosh(ax)\sinh(bx)\right]+c\\ &= \frac{1}{2(a+b)}\cosh((a+b)x)\\ &\quad +\frac{1}{2(a-b)}\cosh((a-b)x)+c,\end{aligned}$$

2) $\displaystyle\int \cosh^3(x)dx \overset{u=\sinh(x)}{=} \int (u^2-1)du = \frac{1}{3}\sinh^3(x)-\sinh(x)+c,$

3) $\displaystyle\int \sinh^3(x)dx \overset{u=\cosh(x)}{=} \int (u^2-1)du = \frac{1}{3}\cosh^3(x)-\cosh(x)+c,$

4)

$$\begin{aligned}\int \sinh^7(x)\ \cosh^3(x)dx &\overset{u=\sinh(x)}{=} \int u^7\ (1+u^2)\ du\\ &= \frac{\sinh^8(x)}{8}+\frac{\sinh^{10}(x)}{10}+c,\end{aligned}$$

5) Using the change of variable $u=\cosh(x)$, we get

$$\begin{aligned}\int \sinh^5(x)\ \cosh^4(x)dx &= \int (u^2-1)^2u^4\ du\\ &= \frac{\cosh^5(x)}{5}-\frac{2\cosh^7(x)}{7}+\frac{\cosh^9(x)}{9}+c,\end{aligned}$$

6)

$$\begin{aligned}\int \sinh^3(x)\ \cosh^2(x)dx &\overset{u=\cosh(x)}{=} \int (u^2-1)\ u^2\ du\\ &= \frac{\cosh^5(x)}{5}-\frac{\cosh^3(x)}{3}+c,\end{aligned}$$

7)

$$\begin{aligned}\int \sinh^2(x)dx &= \int (\cosh^2(x)-1)dx\\ &= \frac{\sinh(2x)}{4}-\frac{3}{2}x+c,\end{aligned}$$

8)

$$\begin{aligned}\int \sinh^4(x)dx &= \int (\cosh^2(x)-1)^2 dx \\ &= -\frac{11\sinh(2x)}{16}+\frac{\sinh(x)\cosh^3(x)}{4}+\frac{23}{8}x+c,\end{aligned}$$

9)

$$\begin{aligned}\int \operatorname{sech}^5(x)\tanh^3(x)dx &\overset{u=\operatorname{sech}(x)}{=} -\int (u^4-u^6)du \\ &= \frac{1}{7}\operatorname{sech}^7(x)-\frac{1}{5}\operatorname{sech}^5(x)+c\end{aligned}$$

10)

$$\begin{aligned}\int \tanh^3(x)\ \operatorname{sech}^3(x)dx &\overset{u=\operatorname{sech}(x)}{=} -\int (1-u^2)u^2\ du \\ &= \frac{\operatorname{sech}^5(x)}{5}-\frac{\operatorname{sech}^3(x)}{3}+c,\end{aligned}$$

Exercises

3-3-1 Compute the following integrals:

1) $\displaystyle\int \frac{x-3}{x+5}dx,$

2) $\displaystyle\int \frac{x^2+x-5}{x^2+2x-35}dx,$

3) $\displaystyle\int \frac{dx}{x^2(x-1)^2},$

4) $\displaystyle\int \frac{3x^2+x+4}{x^4+3x^2+2}dx.$

5) $\displaystyle\int \frac{dx}{1+x+x^2},$

6) $\displaystyle\int \frac{dx}{(1+x+x^2)^2},$

7) $\displaystyle\int \frac{dx}{(x+1)(x^2+x+1)},$

8) $\displaystyle\int \frac{dx}{(x-1)^2(1+x+x^2)^2},$

Solutions of Exercises

3-3-1 1)

$$\int \frac{2x-3}{x+5}dx \overset{t=x+5}{=} \int \frac{2(t-5)-3}{t}dt$$
$$= 2t - 13\ln|t| + c = 2x - 13\ln|x+5| + c.$$

Also we have $\frac{2x-3}{x+5} = 2 - \frac{13}{x+5}$, then
$\int \frac{2x-3}{x+5}dx = 2x - 13\ln|x+5| + c,$

2)

$$\int \frac{x^2+x-5}{x^2+2x-35}dx \overset{t=x+1}{=} \int (1 - \frac{t}{t^2-36} + \frac{30}{t^2-36})dt$$
$$= t + \frac{1}{2}\ln|t^2-36| + \frac{5}{4}\ln\left|\frac{t-6}{t+6}\right| + c$$
$$= x + \frac{1}{2}\ln|x^2+2x-35| + \frac{5}{4}\ln\left|\frac{x-6}{x+7}\right| + c,$$

3) $\frac{1}{x^2(x-1)^2} = \frac{2}{x} + \frac{1}{x^2} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$. Then

$$\int \frac{dx}{x^2(x-1)^2} = 2\ln\left|\frac{x}{x-1}\right| - \frac{1}{x} - \frac{1}{x-1} + c,$$

4) $\frac{3x^2+x+4}{x^4+3x^2+2} = \frac{3x^2+x+4}{(x^2+1)(x^2+2)} = \frac{x+1}{x^2+1} + \frac{-x+2}{x^2+2}$.
Then $\int \frac{3x^2+x+4}{x^4+3x^2+2}dx = \frac{1}{2}\ln(\frac{x^2+1}{x^2+2}) + \tan^{-1}(x) + \sqrt{2}\tan^{-1}(\frac{x}{\sqrt{2}}) + c,$

5) $\int \frac{dx}{1+x+x^2} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c.$

6) $I = \int \frac{dx}{(1+x+x^2)^2}$, we set $t = \frac{2x+1}{\sqrt{3}}$. We have:
$I = \frac{8}{3\sqrt{3}}\int \frac{dt}{(1+t^2)^2} = \frac{4}{\sqrt{3}}\tan^{-1}(t) + \frac{4t}{3\sqrt{3}(1+t^2)}$.
Then $I = \frac{4}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + \frac{2x+1}{3(1+x+x^2)} + c,$

7) $\dfrac{1}{(x+1)(x^2+x+1)} = \dfrac{1}{x+1} - \dfrac{x}{x^2+x+1}$, then

$$\begin{aligned}
\int \frac{dx}{(x+1)(x^2+x+1)} &= \int \frac{dx}{x+1} - \frac{1}{2}\int \frac{(2x+1)-1}{x^2+x+1}dx \\
&= \ln|x+1| - \frac{1}{2}\ln(x^2+x+1) \\
&\quad + \frac{1}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c,
\end{aligned}$$

8) $J = \displaystyle\int \frac{dx}{(x-1)^2(1+x+x^2)^2}$

$$\begin{aligned}
J &= -\frac{2}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} + \frac{1}{9}\int \frac{(2x+1)dx}{x^2+x+1} \\
&\quad + \frac{1}{6}\int \frac{2x+1+1}{(x^2+x+1)^2}dx \\
&= -\frac{2}{9}\ln|x-1| - \frac{1}{3(x-1)} + \frac{1}{9}\ln(1+x+x^2) - \frac{1}{6(1+x+x^2)} \\
&\quad + \int \frac{dx}{(1+x+x^2)^2} + c.
\end{aligned}$$

Exercises

3-4-1 Simplify each of the following expressions by eliminating the radical by using an appropriate trigonometric substitution.

1) $\dfrac{x}{\sqrt{9-x^2}}$, 3) $\dfrac{x-2}{x\sqrt{x^2-25}}$, 5) $\dfrac{2-2x}{\sqrt{x^2-2x-3}}$.

2) $\dfrac{3+x}{\sqrt{16+x^2}}$, 4) $\dfrac{1+x}{\sqrt{x^2+2x+2}}$,

3-4-2 Evaluate the following integrals:

1) $\displaystyle\int \frac{3+x}{\sqrt{16+x^2}}dx$,

2) $\displaystyle\int \frac{x-2}{x\sqrt{x^2-25}}dx$,

3) $\displaystyle\int \sqrt{x^2+a^2}dx$, for $a>0$,

4) $\displaystyle\int \sqrt{x^2-a^2}dx$, for $a>0$,

5) $\displaystyle\int \sqrt{x^2+3x+1}dx$,

6) $\displaystyle\int \sqrt{x^2+2x+5}dx$,

7) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+3}}$,

8) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x-3}}$,

9) $\displaystyle\int \frac{dx}{(4+x^2)^{\frac{3}{2}}}$,

Solutions of Exercises

3-4-1 1) $\dfrac{x}{\sqrt{9-x^2}} \overset{x=3\sin(\theta)}{=} \dfrac{\sin(\theta)}{|\cos(\theta)|}$,

2) $\dfrac{3+x}{\sqrt{16+x^2}} \overset{x=4\tan(\theta)}{=} \dfrac{3+4\tan(\theta)}{4|\sec(\theta)|}$,

3) $\dfrac{x-2}{x\sqrt{x^2-25}} \overset{x=5\sec(\theta)}{=} \dfrac{5\sec(\theta)-2}{5|\tan(\theta)|}$,

4) $\dfrac{1+x}{\sqrt{x^2+2x+2}} = \dfrac{1+x}{\sqrt{(x+1)^2+1}} \overset{x=\tan(\theta)-1}{=} \dfrac{\tan(\theta)}{|\sec(\theta)|}$,

5) $\dfrac{2-2x}{\sqrt{x^2-2x-3}} = \dfrac{2-2x}{\sqrt{(x-1)^2-4}} \overset{x=2\sec(\theta)+1}{=} \dfrac{-2\sec(\theta)}{|\tan(\theta)|}$.

3-4-2 1)

$$\begin{aligned}
\int \frac{3+x}{\sqrt{16+x^2}}dx &\overset{x=4\tan(\theta)}{=} \int (3\sec(\theta)+4\sec(\theta)\tan(\theta))d\theta \\
&= 3\ln|\sec(\theta)+\tan(\theta)|+4\sec(\theta)+c \\
&= 3\ln(x+\sqrt{16+x^2})+\sqrt{16+x^2}+c,
\end{aligned}$$

2)

$$\begin{aligned}
\int \frac{x-2}{x\sqrt{x^2-25}}dx &\overset{x=5\sec(\theta)}{=} \frac{1}{5}\int (5\sec(\theta)-2)d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)|-\frac{2}{5}\theta+c \\
&= \ln(x+\sqrt{x^2-25})-\frac{2}{5}\sec^{-1}(\frac{x}{5})+c,
\end{aligned}$$

3) Using the change of variable $x = a\tan(\theta)$, we get

$$\begin{aligned}
\int \sqrt{x^2+a^2}dx &= a^2\int \sec^3(\theta)d\theta \\
&= a^2\left(\frac{1}{2}\sec(\theta)\tan(\theta)+\frac{1}{2}\ln|\sec(\theta)+\tan(\theta)|\right)+c \\
&= \frac{1}{2}x\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c
\end{aligned}$$

4)

$$\begin{aligned}\int \sqrt{x^2-a^2}dx &\overset{x=a\sec(\theta)}{=} a^2\int \sec(\theta)\tan^2(\theta)d\theta \\ &= a^2\int (\sec^3(\theta)-\sec(\theta))d\theta \\ &= \frac{1}{2}x\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+c.\end{aligned}$$

Then

$$\int \sqrt{x^2\pm a^2}dx = \frac{1}{2}x\sqrt{x^2\pm a^2}\pm\frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|+c$$

5)

$$\begin{aligned}\int \sqrt{x^2+3x+1}dx &\overset{t=x+\frac{3}{2}}{=} \int \sqrt{t^2-\frac{5}{4}}dt \\ &= \frac{1}{2}t\sqrt{t^2-\frac{5}{4}}-\frac{5}{8}\ln\left|t+\sqrt{t^2-\frac{5}{4}}\right|+c \\ &= \frac{1}{2}(x+\frac{3}{2})\sqrt{x^2+3x+1} \\ &\quad -\frac{5}{8}\ln\left|(x+\frac{3}{2})+\sqrt{x^2+3x+1}\right|+c\end{aligned}$$

6)

$$\begin{aligned}\int \sqrt{x^2+2x+5}dx &\overset{t=x+1}{=} \int \sqrt{t^2+4}dt \\ &= \frac{1}{2}t\sqrt{t^2+4}+2\ln\left|t+\sqrt{t^2+4}\right|+c \\ &= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} \\ &\quad +2\ln\left|(x+1)+\sqrt{x^2+2x+5}\right|+c\end{aligned}$$

7)

$$\int \frac{dx}{\sqrt{x^2+2x+3}} \overset{t=x+1}{=} \int \frac{dt}{\sqrt{t^2+2}}$$
$$= \sinh^{-1}\left(\frac{t}{\sqrt{2}}\right)$$
$$= \ln\left|x+1+\sqrt{x^2+2x+3}\right| + c$$

8)

$$\int \frac{dx}{\sqrt{x^2+2x-3}} \overset{t=x+1}{=} \int \frac{dt}{\sqrt{t^2-4}}$$
$$= \cosh^{-1}\left(\frac{t}{2}\right) = \ln\left|x+1+\sqrt{x^2+2x-3}\right| + c$$

9)

$$\int \frac{dx}{(4+x^2)^{\frac{3}{2}}} \overset{x=2\tan(\theta)}{=} \frac{1}{4}\int \cos(\theta)d\theta$$
$$= \frac{1}{4}\sin(\theta) + c = \frac{x}{4\sqrt{4+x^2}} + c$$

Exercises

$\boxed{\text{3-5-1}}$ Compute the following integrals:

$$1)\ \int \frac{dx}{\sin^2(x)\cos(x)}, \quad 2)\ \int \frac{\sin(x)dx}{\sin(x)-\cos(x)}.$$

Solutions of Exercises

3-5-1 1) $\displaystyle\int \frac{dx}{\sin^2(x)\cos(x)} = \int \frac{dx}{\cos(x)} + \int \frac{\cos(x)dx}{\sin^2(x)} = \ln|\tan(\frac{x}{2} + \frac{\pi}{4})| - \frac{1}{\sin(x)} + c,$

2)

$$\begin{aligned}\int \frac{\sin(x)dx}{\sin(x)-\cos(x)} &\overset{t=\tan(x)}{=} \int \frac{tdt}{(t-1)(1+t^2)} \\ &= \frac{1}{2}\ln|\tan(x)-1| + \frac{1}{2}\ln|\cos(x)| + \frac{x}{2} + c.\end{aligned}$$

Exercises

3-6-1 Evaluate the following integrals:

1) $\displaystyle\int \frac{\sqrt{2x-1}}{2x+3}dx,$

2) $\displaystyle\int_0^1 x\sqrt{2-\sqrt{1-x^2}}dx,$

3) $\displaystyle\int \frac{dx}{\sqrt{x}+x\sqrt{x}},$

4) $\displaystyle\int_{1/3}^3 \frac{\sqrt{x}}{x^2+x}dx,$

5) $\displaystyle\int \frac{dx}{x^2\sqrt{4x^2-1}}.$

Solutions of Exercises

3-6-1 1)

$$\int \frac{\sqrt{2x-1}}{2x+3}dx \overset{t=\sqrt{2x-1}}{=} \int \frac{t^2}{t^2+4}dt = t - 2\tan^{-1}(\frac{t}{2}) + c$$
$$= \sqrt{2x-1} - 2\tan^{-1}(\frac{\sqrt{2x-1}}{2}) + c,$$

2) $\displaystyle\int_0^1 x\sqrt{2-\sqrt{1-x^2}}dx \overset{t=\sqrt{1-x^2}}{=} \int_0^1 \sqrt{2-t}dt = \frac{2^{\frac{5}{2}}}{3} - \frac{2}{3},$

3) $\displaystyle\int \frac{dx}{\sqrt{x}+x\sqrt{x}} \overset{t=\sqrt{x}}{=} \int \frac{2dt}{1+t^2} = \tan^{-1}(\sqrt{x}) + c,$

4) $\displaystyle\int_{\frac{1}{3}}^{3} \frac{\sqrt{x}}{x^2+x}dx \overset{t=\sqrt{x}}{=} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dt}{1+t^2} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.,$

5) using the substitution $2x = \sec(\theta)$, we get
$$\int \frac{dx}{x^2\sqrt{4x^2-1}} = 2\int \cos(\theta)d\theta = 2\sin(\theta) + c = \frac{\sqrt{1-4x^2}}{x} + c.$$

Exercises

3-7-1 Prove that the following improper integrals are convergent and compute the value of these integrals.

1) $\displaystyle\int_0^{+\infty} xe^{-2x}dx,$

2) $\displaystyle\int_0^{+\infty} e^{-x}\sin(x)dx,$

3) $\displaystyle\int_0^{1} \frac{x+1}{\sqrt{x}}dx,$

4) $\displaystyle\int_1^{+\infty} \frac{dx}{x^2\sqrt{x-1}},$

5) $\displaystyle\int_1^{2} \frac{2x^3}{\sqrt{x^4-1}}dx$

6) $\displaystyle\int_0^{1} \frac{\ln(1-x^2)}{x^2}dx,$

7) $\displaystyle\int_0^{1} \frac{x\ln x}{(1-x^2)^{3/2}}dx,$

8) $\displaystyle\int_0^{1} \frac{\ln x}{(1-x)^{\frac{3}{2}}}dx,$

9) $\displaystyle\int_1^{+\infty} \frac{x^4+1}{x^3(x+1)(1+x^2)}dx,$

10) $\displaystyle\int_1^{+\infty} \frac{dx}{x\ \sqrt[4]{1+x^2}},$

11) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\ln(\tan(x))dx.$

3-7-2 Determine whether the following integrals are convergent or divergent:

1) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt[4]{1+x}}dx,$

2) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt[4]{(1+x)^5}}dx,$

3) $\displaystyle\int_{-\infty}^{0} 2^x dx,$

4) $\displaystyle\int_{-\infty}^{+\infty} \cos(\pi x)dx,$

5) $\displaystyle\int_6^{8} \frac{4}{\sqrt[3]{x-6}}dx,$

6) $\displaystyle\int_1^{3} \frac{1}{\sqrt{3-x}}dx,$

7) $\displaystyle\int_1^{+\infty} \frac{\ln(x)}{x^4}dx,$

8) $\displaystyle\int_0^{1} \frac{1}{2-3x}dx,$

9) $\displaystyle\int_1^{+\infty} \frac{\tan^{-1}(x)}{x^2}dx,$

Solutions of Exercises

3-7-1 1) Using integration by parts,
$$\int_0^c xe^{-2x}dx = -\frac{1}{2}ce^{-2c}+\frac{1}{2}\int_0^c e^{-2x}dx = -\frac{1}{2}ce^{-2c}-\frac{1}{4}e^{-2c}+\frac{1}{4}.$$
Since $\lim\limits_{c\to+\infty}\frac{1}{2}ce^{-2c} = 0$ and $\lim\limits_{c\to+\infty}\frac{1}{4}e^{-2c} = 0$, the integral
$$\int_0^{+\infty} xe^{-2x}dx$$
is convergent and $\int_0^{+\infty} xe^{-2x}dx = \frac{1}{4}$,

2) Using integration by parts,
$\int_0^c e^{-x}\sin(x)dx = e^{-c}\sin(c)+\int_0^c e^{-x}\cos(x)dx$ and by another integration by parts,
$$\int_0^c e^{-x}\sin(x)dx = e^{-c}\sin(c)+1-e^{-c}\cos(c)-\int_0^c e^{-x}\sin(x)dx.$$
Then $\int_0^c e^{-x}\sin(x)dx = \frac{1}{2}e^{-c}\sin(c)+\frac{1}{2}-\frac{1}{2}e^{-c}\cos(c)$.

Since $\lim\limits_{c\to+\infty} e^{-c}\frac{\sin(c)}{2} = \lim\limits_{c\to+\infty} e^{-c}\frac{\cos(c)}{2} = 0$, the integral
$\int_0^{+\infty} e^{-2x}\sin(x)dx$ converges to $\frac{1}{2}$.

3) For $0 < a < 1$,
$\int_a^1 \frac{x+1}{\sqrt{x}}dx \overset{t=\sqrt{x}}{=} 2\int_{\sqrt{a}}^1 (t^2+1)dt = 2(\frac{4}{3}-\frac{a^{\frac{3}{2}}}{3}-\sqrt{a})$, then
$\int_0^1 \frac{x+1}{\sqrt{x}}dx$ is converges to $\frac{8}{3}$.

4) $$\int_1^c \frac{dx}{x^2\sqrt{x-1}} \overset{x=1+t^2}{=} \int_0^{\sqrt{c-1}} \frac{2dt}{(1+t^2)^2}.$$
Using integration by parts:
$$\int \frac{dt}{1+t^2} = \frac{t}{1+t^2}+2\int \frac{t^2}{(1+t^2)^2}dt = \frac{t}{1+t^2}+2\int \frac{t^2+1-1}{(1+t^2)^2}dt.$$
Then $2\int \frac{dt}{(1+t^2)^2} = \tan^{-1}(t)+\frac{t}{1+t^2}$. Then the integral
$\int_1^{+\infty} \frac{dx}{x^2\sqrt{x-1}}$ converges to $\frac{\pi}{2}$.

5) For $1 < a < 2$, $\displaystyle\int_a^2 \frac{2x^3}{\sqrt{x^4-1}}dx = \sqrt{15} - \sqrt{a^4-1}$ and $\displaystyle\int_1^2 \frac{2x^3}{\sqrt{x^4-1}}dx = \sqrt{15}$,

6) Using integration by parts, for $0 < x < 1$,

$$\begin{aligned}\int \frac{\ln(1-x^2)}{x^2}dx &= -\frac{\ln(1-x^2)}{x} - 2\int \frac{dx}{1-x^2} \\ &= -\frac{\ln(1-x^2)}{x} - \ln(1+x) + \ln(1-x) \\ &= -\frac{(1+x)\ln(1+x)}{x} - \frac{(1-x)\ln(1-x)}{x}.\end{aligned}$$

Since $\displaystyle\lim_{x\to 1} -\frac{(1+x)\ln(1+x)}{x} - \frac{(1-x)\ln(1-x)}{x} = -2\ln 2$ and $\displaystyle\lim_{x\to 0} -\frac{(1+x)\ln(1+x)}{x} - \frac{(1-x)\ln(1-x)}{x} = 0$, then the integral $\displaystyle\int_0^1 \frac{\ln(1-x^2)}{x^2}dx$ converges.

7) Using integration by parts, for $0 < x < 1$, $u = \ln x$ and $v' = \frac{x}{(1-x^2)^{3/2}}$, we get

$$\begin{aligned}\int \frac{x\ln x}{(1-x^2)^{3/2}}dx &= \frac{\ln(x)}{\sqrt{1-x^2}} - \int \frac{dx}{x\sqrt{1-x^2}} \\ &= \frac{(1-\sqrt{1-x^2})\ln(x)}{\sqrt{1-x^2}} + \ln(1+\sqrt{1-x^2}) \\ &= \frac{x^2\ln(x)}{\sqrt{1-x^2}(1+\sqrt{1-x^2})} + \ln(1+\sqrt{1-x^2})\end{aligned}$$

Using l'Hôpital rule,
$\displaystyle\lim_{x\to 0^+} \frac{x^2\ln(x)}{\sqrt{1-x^2}(1+\sqrt{1-x^2})} + \ln(1+\sqrt{1-x^2}) = \ln 2$ and
$\displaystyle\lim_{x\to 1^-} \frac{x^2\ln(x)}{\sqrt{1-x^2}(1+\sqrt{1-x^2})} + \ln(1+\sqrt{1-x^2}) = 0$, then the integral $\displaystyle\int_0^1 \frac{x\ln(x)}{(1-x^2)^{3/2}}dx$ converges to $-\ln 2$.

8)

$$\begin{aligned}\int_0^1 \frac{\ln(x)}{(1-x)^{\frac{3}{2}}}dx &= 2\int_0^1 \frac{\ln(1-t^2)}{t^2}dt \\ &= \left[\frac{(t-1)\ln(1-t)}{t} - \frac{(t+1)\ln(1+t)}{t}\right]_0^1 \\ &= -2\ln 2.\end{aligned}$$

In the first step, we take the change of variable, $x = 1-t^2$ and in the second step, we use an integration by parts with $u = \ln(1-t^2)$ and $v' = \frac{1}{t^2}$.

9)

$$\begin{aligned}\int_1^{+\infty} \frac{x^4+1}{x^3(x+1)(1+x^2)}dx &= \int_1^{+\infty}\left(-\frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x+1} + \frac{x+1}{1+x^2}\right)dx \\ &= \frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\ln 2.\end{aligned}$$

10)

$$\begin{aligned}\int_1^{+\infty} \frac{dx}{x\sqrt[4]{1+x^2}} &\overset{t^4=1+x^2}{=} \int_{2^{\frac{1}{4}}}^{+\infty} \frac{2t^2dt}{(t^4-1)} \\ &= \int_{2^{\frac{1}{4}}}^{+\infty}\left(\frac{1}{2(t-1)} - \frac{1}{2(t+1)} + \frac{1}{1+t^2}\right)dt \\ &= \frac{1}{2}\ln\left(\frac{2^{\frac{1}{4}}+1}{2^{\frac{1}{4}}-1}\right) + \frac{\pi}{2} - \tan^{-1}(2^{\frac{1}{4}}).\end{aligned}$$

11)

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(x)\ln(\tan(x))dx &= \int_0^{\frac{\pi}{4}} \cos(x)\ln(\tan(x))dx \\ &\quad - \int_0^{\frac{\pi}{4}} \sin(x)\ln(\tan(x))dx.\end{aligned}$$

An integration by parts yields

$$\int_0^{\frac{\pi}{4}} \cos(x)\ln(\tan(x))dx = -\int_0^{\frac{\pi}{4}} \frac{dx}{\cos(x)} = -\ln(1+\sqrt{2}).$$

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \sin(x)\ln(\tan(x))dx &= [(1-\cos(x))\ln(\sin(x)) \\
&\quad -\cos(x)\ln(\cos(x)) - \ln(1+\cos(x))]_0^{\frac{\pi}{4}} \\
&= \frac{1}{2}(1+\sqrt{2})\ln 2 - \ln(1+\sqrt{2}).
\end{aligned}
$$

Then $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\ln(\tan(x))dx = -\frac{1}{2}(1+\sqrt{2})\ln 2.$

3-7-2 1)

$$
\begin{aligned}
\int_0^{+\infty} \frac{1}{\sqrt[4]{1+x}}dx &= \int_0^{+\infty} (1+x)^{-\frac{1}{4}}dx = \frac{4}{3}\left.(1+x)^{\frac{3}{4}}\right]_0^{+\infty} \\
&= +\infty,
\end{aligned}
$$

the integral diverges.

2)

$$
\begin{aligned}
\int_0^{+\infty} \frac{1}{\sqrt[4]{(1+x)^5}}dx &= \int_0^{+\infty} (1+x)^{-\frac{5}{4}}dx \\
&= -4\left.(1+x)^{-\frac{1}{4}}\right]_0^{+\infty} = 4
\end{aligned}
$$

the integral converges.

3) $\displaystyle\int_{-\infty}^0 2^x dx = \left.\frac{2^r}{\ln 2}\right]_{-\infty}^0 = \frac{1}{\ln 2}$, the integral converges.

4) $\displaystyle\int_a^b \cos(\pi t)dt = \left(\frac{1}{\pi}\sin(\pi b) - \frac{1}{\pi}\sin(\pi a)\right)$. Since $\displaystyle\lim_{a\to-\infty}\lim_{b\to+\infty}\left(\frac{1}{\pi}\sin(\pi b) - \frac{1}{\pi}\sin(\pi a)\right)$ does not exists, the integral diverges.

5) $\displaystyle\int_6^8 \frac{4}{\sqrt[3]{x-6}}dx = 6(x-6)^{\frac{2}{3}}\Big]_6^8 = 6.2^{\frac{2}{3}}$, the integral converges.

6) $\displaystyle\int_1^3 \frac{1}{\sqrt{3-x}}dx = -2(3-x)^{\frac{1}{2}}\Big]_1^3 = 2\sqrt{2}$, the integral converges.

7) Using integration by parts

$$\begin{aligned}\int_1^{+\infty} \frac{\ln(x)}{x^4} dx &= \left[-\frac{\ln x}{3x^3}\right]_1^{+\infty} + \frac{1}{3}\int_1^{+\infty} \frac{dx}{x^4} dx \\ &= \left[-\frac{1}{9x^3}\right]_1^{+\infty} = \frac{1}{9}\end{aligned}$$

8) $\displaystyle\int_0^1 \frac{1}{2-3x} dx = \int_0^{\frac{2}{3}} \frac{1}{2-3x} dx + \int_{\frac{2}{3}}^1 \frac{1}{2-3x} dx.$

Since $\displaystyle\int_0^{\frac{2}{3}} \frac{1}{2-3x} dx = +\infty$, the integral $\displaystyle\int_0^1 \frac{1}{2-3x} dx$ diverges.

9) Using integration by parts

$$\begin{aligned}\int_1^{+\infty} \frac{\tan^{-1}(x)}{x^2} dx &= \left[-\frac{\tan^{-1}(x)}{x}\right]_1^{+\infty} + \int_1^{+\infty} \frac{dx}{x(1+x^2)} dx \\ &= \frac{\pi}{4} + \int_1^{+\infty} \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx \\ &= \frac{\pi}{4} + \left[\ln(\frac{x}{\sqrt{1+x^2}})\right]_1^{+\infty} = \frac{\pi}{4} + \frac{1}{2}\ln 2.\end{aligned}$$

The integral $\displaystyle\int_1^{+\infty} \frac{\tan^{-1}(x)}{x^2} dx$ converges.

4 Exercises on Chapter IV

Exercises

4-1-1 Set up integrals to evaluate the areas bounded by the graphs of the following curves

1) $y = \ln x$, $y = 0$ and $x = 2$,
2) $y = e^x$, $x = \ln 4$, $x = 0$ and $y = 0$,
3) $y = x^2$ and $y = -x^2 + 2$,
4) $y = \dfrac{4}{x}$, $x = 0$, $y = 1$ and $y = 2$.

4-1-2 Find the area of the region between the graphs of the functions $y = e^x, y = 4e^{-x}$ and $y = 1$.

4-1-3 Find the area of the region bounded by the curves $x = y^2$, $x + y = 6$, $y = -4$, $y = 2$.

4-1-4 Find the area between the curves: $y = \cos(x), y = \sin(x)$, $0 \leq x \leq \frac{\pi}{2}$,

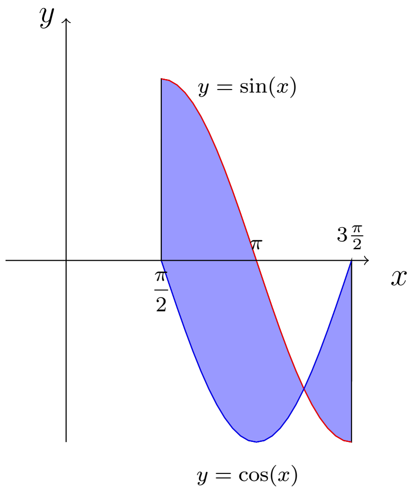
4-1-5 Sketch the region bounded by the curves and find its area

1) $4x = 4y - y^2$, $4x - y = 0$,
2) $y = e^x$, $y = e$, $y = x$, $x = 0$,
3) $4y = x^2$ and $y = \frac{8}{x^2+4}$,
4) The region in the first quadrant bounded by the x-axis, the parabola $y = \frac{x^2}{3}$, and the circle $x^2 + y^2 = 4$,
5) $y = \sin^2(x)$, $y = \tan^2 x$, $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

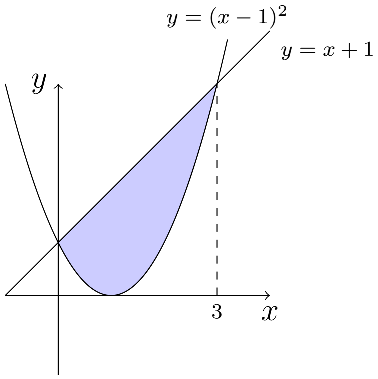
4-1-6 Sketch the region bounded by the curves and find its area in the following :

1) $y = \frac{1}{3}x^2$ and $y = 2x - \frac{1}{3}x^2$.
2) $y = -x$ and $x = y^2 + 2y$.

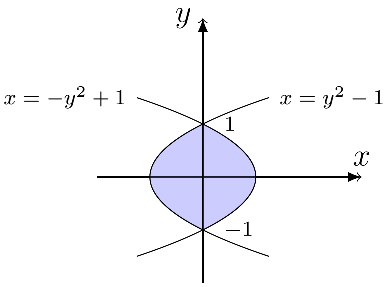
4-1-7 Find the area of the shaded regions:



1)



2)



3)

4-1-8 Find the area of the region bounded by the graphs of the curves of $y = x^2 - 4x$ and $y = 0$

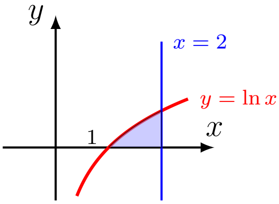
4-1-9 Find the area of the region bounded by the graphs of the curves $y = x^2 + 2x + 1$, $y = 1 - x$ and $y = 0$

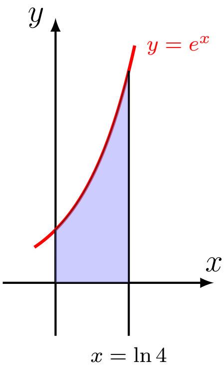
4-1-10 Find the area of the region bounded by the graphs of the curves $y = x^2$, $y = x^2 + 1$, $x = 0$ and $x = 1$.

Solutions of Exercises

4-1-1

1) Note that $y = \ln x$ intersects the $x-$axis at $x = 1$. The desired area is $\displaystyle\int_1^2 \ln x \, dx = [x \ln x - x]_1^2 = \ln(\frac{4}{e})$.

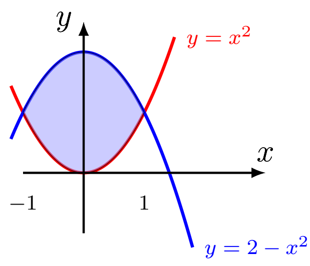




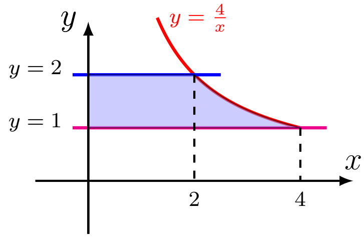
2) The desired area is $\displaystyle\int_0^{\ln 4} e^x \, dx = 3$.

3) Note that $y = x^2$ is a parabola opens upward with vertex $(0, 0)$ and $y = -x^2 + 2$ is another parabola opens downward with vertex $(0, 2)$.

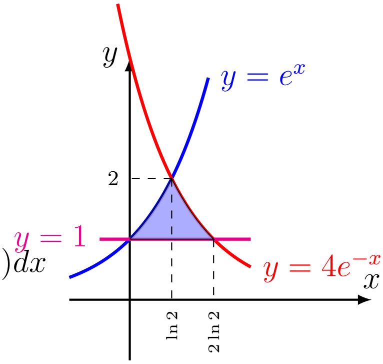
$x^2 = -x^2 + 2 \iff 2x^2 = 2$, then $x = \pm 1$. The intersection points of $y = x^2$ and $y = -x^2 + 2$ are: $(1, 1)$ and $(-1, 1)$. The desired area is $\displaystyle\int_{-1}^{1} [(-x^2 + 2) - x^2] \, dx = \frac{8}{3}$.



4) The desired area is $\displaystyle\int_1^2 \frac{4}{y} \, dy = 4 \ln 2$.

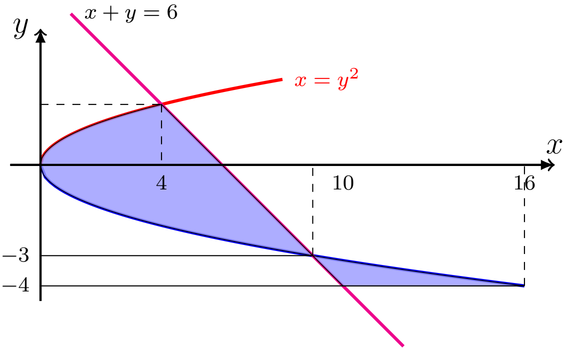


4-1-2 $e^x = 4e^{-x} \iff e^{2x} = 4$, then $x = \ln 2$. The area of the region between the graphs of the functions $y = e^x, y = 4e^{-x}$ and $y = 1$ is:

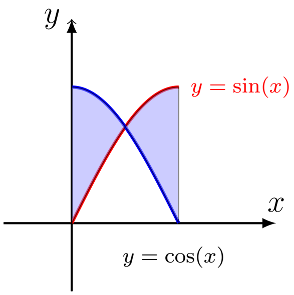


$$
\begin{aligned}
A &= \int_0^{\ln 2} (e^x - 1)dx + \int_{\ln 2}^{2\ln 2} (4e^{-x} - 1)dx \\
&= 2 - 2\ln 2.
\end{aligned}
$$

4-1-3 The area of the region bounded by the curves $x = y^2$, $x + y = 6$, $y = -4$ is $A = \displaystyle\int_{-3}^{2} \left(6 - y - y^2\right) dy + \int_{-4}^{-3} (y^2 - 6 + y)dy = \frac{71}{3}$.



4-1-4 The area between the curves: $y = \cos(x), y = \sin(x)$, for $0 \le x \le \frac{\pi}{2}$ is:

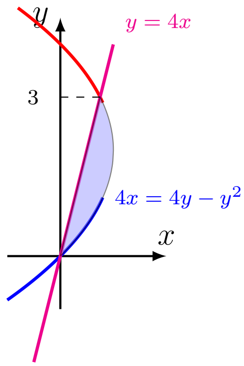


$$
\begin{aligned}
A &= \int_0^{\frac{\pi}{4}} (\cos(x) - \sin(x))dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin(x) - \cos(x))dx \\
&= 2(\sqrt{2} - 1).
\end{aligned}
$$

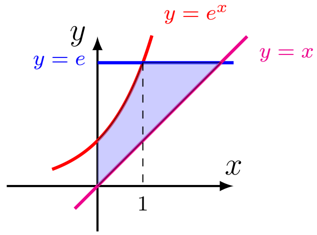
4-1-5

1) The area between the curves: $4x = 4y - y^2$, $4x - y = 0$ is:

$$A = \int_0^3 (y - \frac{1}{4}y^2 - \frac{1}{4}y)dy = \frac{9}{8}.$$

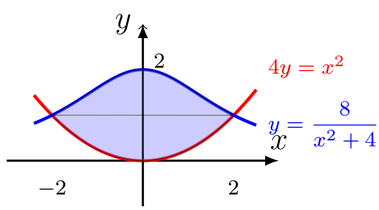


2) The area of the region bounded by the curves $y = e^x$, $y = e$, $y = x$ and $x = 0$, is



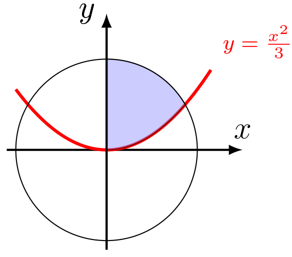
$$A = \int_0^1 (e^x - x)dx + \int_1^e (e - x)dx = \frac{e^2}{2} - 1.$$

3) The area of the region bounded by the curves $4y = x^2$ and $y = \dfrac{8}{x^2+4}$ is



$$\begin{aligned} A &= \int_{-2}^{2} \left(\frac{8}{x^2+4} - \frac{x^2}{4} \right) dx \\ &= \left[4\tan^{-1}(\frac{x}{2}) - \frac{x^3}{12} \right]_{-2}^{2} = 2\pi - \frac{4}{3}. \end{aligned}$$

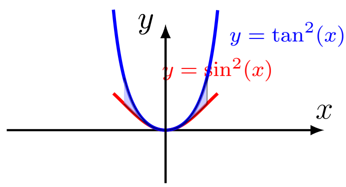
4) The region in the first quadrant bounded by the x-axis, the parabola $y = \frac{x^2}{3}$, and the circle $x^2 + y^2 = 4$. The area of the given region is:



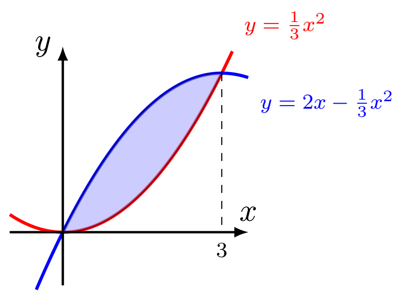
$$
\begin{aligned}
A &= \int_0^{\sqrt{3}} \left(\sqrt{4-x^2} - \frac{x^2}{3}\right) dx \\
&\overset{x=2\sin(\theta)}{=} 2\int_0^{\frac{\pi}{3}} \left(2\cos(\theta) - \frac{4\sin^2(\theta)}{3}\right)\cos(\theta)d\theta \\
&= 2\left[\theta + \frac{1}{2}\sin(2\theta) - \frac{4}{9}\sin^3(\theta)\right]_0^{\frac{\pi}{3}} = \frac{1}{6}(4\pi + \sqrt{3}).
\end{aligned}
$$

5) $y = \sin^2(x)$, $y = \tan^2 x$, $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
The area of the given region is:

$$
\begin{aligned}
A &= 2\int_0^{\frac{\pi}{4}} (\tan^2(x) - \sin^2 x)dx \\
&= 2\left[\tan(x) - x - \frac{x}{2} + \frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{4}} \\
&= \frac{5}{2} - \frac{3\pi}{4}.
\end{aligned}
$$

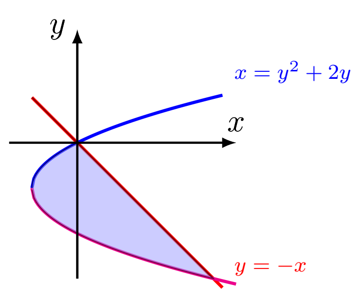


$x^2 = 6x - x^2 \iff x = 0$ or $x = 3$.
The area of the given region is:



4-1-6 1)

$$
\begin{aligned}
A &= \frac{1}{3}\int_0^3 (6x - 2x^2)dx \\
&= \frac{1}{3}\left[3x^2 - \frac{2}{3}x^3\right]_0^3 = 3.
\end{aligned}
$$

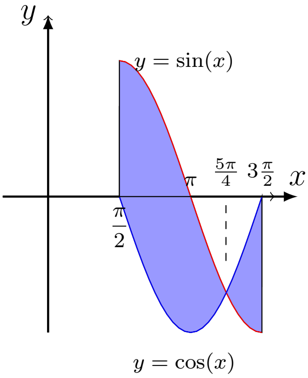


The area of the region is:

2) $A = \int_{-3}^{0} (-y - y^2 - 2y)dy = \frac{9}{2}$.

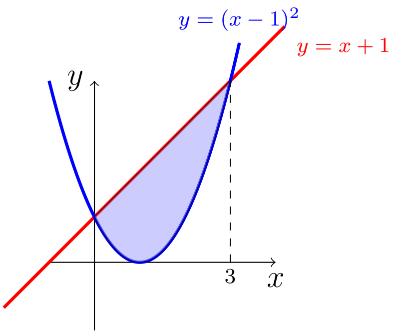
4-1-7 1) The area of the region is

$$\begin{aligned} A &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\sin(x) - \cos(x)| dx \\ &= \int_{\frac{\pi}{2}}^{\frac{5\pi}{4}} \sin(x) - \cos(x) dx \\ &\quad + \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} \cos(x) - \sin(x) dx \\ &= 2\sqrt{2}. \end{aligned}$$

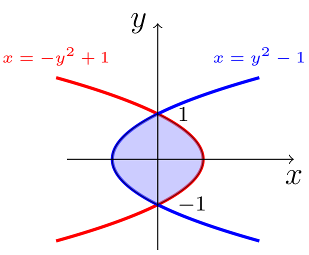


2) The area of the shaded region is equal to

$$\begin{aligned} A &= \int_0^3 (x+1) - (x-1)^2 dx \\ &= \frac{9}{2}. \end{aligned}$$



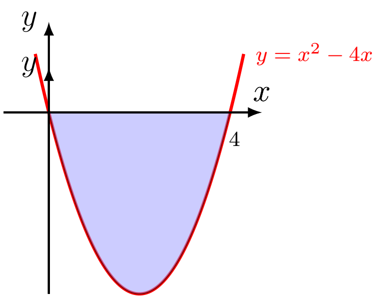
3) $1 - y^2 = y^2 - 1 \iff y = \pm 1.$
The area of the shaded region is equal to
$A = 2\int_{-1}^{1} (1 - y^2) dy = \dfrac{8}{3}.$



4-1-8 Note that $x^2 - 4x = (x^2 - 4x + 4) - 4 = (x-2)^2 + 4$ is a parabola opens upward with vertex $(2, -4)$ and $y = 0$ is the $x-$axis.

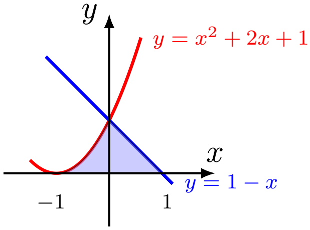
$x^2 - 4x = 0 \iff x = 0$ or $x = 4$.
The desired area is

$$\begin{aligned} A &= \int_0^4 \left[0 - (x^2 - 4x)\right] dx \\ &= \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \frac{32}{3}. \end{aligned}$$



4-1-9 Note that $y = x^2 + 2x + 1 = (x+1)^2$ is a parabola opens upward with vertex (-1,0), $y = 1 - x$ is a straight line and $y = 0$ is the $x-$axis.

$x^2 + 2x + 1 = 1 - x \iff x = 0$ or $x = -3$. The intersection point of $y = 1 - x$ and $y = 0$ is $x = 1$. The area is

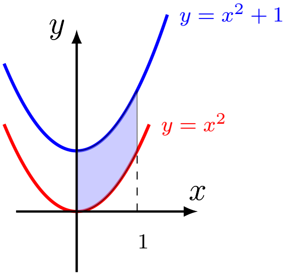


$$\begin{aligned} A &= \int_{-1}^{0} (x+1)^2 \, dx + \int_0^1 (1-x) \, dx \\ &= \frac{1}{3}\left[(x+1)^3\right]_{-1}^{0} - \frac{1}{2}\left[(x-1)^2\right]_0^1 = \frac{5}{6}. \end{aligned}$$

4-1-10 Note that $x^2 + 1$ is a parabola opens upward with vertex $(0,1)$, $y = x^2$ is another parabola opens upward with vertex $(0,0)$, $x = 0$ is the $y-$axis and $x = 1$ is a straight line parallel to the $y-$axis and passing through the point $(1,0)$.

Note also that $y = x^2 + 1$ and $y = x^2$ do not intersect.

The desired area is
$A = \int_0^1 \left[(x^2+1) - x^2\right] \, dx = 1.$



Exercises

4-2-1 Find the volume of a ball of radius R.

4-2-2 Find the volume between the sphere of center $(0,0,0)$ and radius R and the sphere of center $(0,0,0)$ and radius $R+r$.

4-2-3 Find the volume of the solid obtained by rotating the region bounded by $y=\sqrt{36-x^2}$, $y=0, x=2, x=4$, about the x-axis.

4-2-4 Sketch the region bounded by the curves and find the volume of the solid generated by revolving the region about the x- or y- axis, as specified below.

1) $y=1-|x|$, $y=0$, revolved around the x-axis,
2) $y=x^2$, $y=2-x$, revolved around the x-axis,
3) $y=|x|$, $y=2-x^2$, revolved around the x-axis,
4) $f(x)=\cos(\frac{\pi}{2}x)$, $y=0$, $x\in[0,1]$, revolved around the x-axis,
5) $x=\sqrt{9-y^2}$, $x=0$, revolved around the y-axis.

4-2-5 Find the volume of the solid obtained by rotating the region bounded by $y=1+\sec(x)$, $y=3$, about the line $y=1$.

4-2-6 Set up an integration to find the volume and draw an illustration each of the solid obtained by rotating the region bounded by
$y=0, y=\cos^2 x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$

1) About the x-axis,
2) About the line $y=1$.

4-2-7 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specific axis

1) $y=\cos(x^2)$, $y=0$, $x=0$, $x=\sqrt{\frac{\pi}{2}}$ about the y-axis.
2) $y=x^2$, $y=4-x^2$ about the y-axis.
3) $x=y^2+1$, $x=2$, about the line $y=-2$.
4) $x^2-y^2=1$, $x=2$, about the line $y=3$.

5) $y = x^2 + 1$ and $y = 4 - x^2$ about the line $y = -2$

6) $x^2 - y^2 = 5$, $x = 4$ about the y-axis,

7) $y = \sqrt{x}$, $y = x^2$ about the line $y = 2$,

8) $y = \sqrt{x}$, the x-axis for $0 \leq x \leq 4$ about the line $y = 2$.

4-2-8 Let $R = \{(x, y) \in \mathbb{R}^2;\ \frac{(x-3)^2}{4} + \frac{y^2}{9} \leq 1;\ y > 0\}$. This region is bounded by $x = 1$, $x = 5$, $y = 0$ and the graph of the function $f(x) = 3\sqrt{1 - \dfrac{(x-3)^2}{4}}$. ($R$ is also the region included in the upper half of the ellipse with center $(3, 0)$ and its left vertex is $(1, 0)$, right vertex is $(5, 0)$ and upper vertex is $(3, 3)$).

1) Find the volume of the solid of revolution of R around the x-axis.

2) Find the volume of the solid of revolution of R around the y-axis.

3) Find the volume of the solid of revolution of R around the $x = 1$.

4) Find the volume of the solid of revolution of R around the $x = 6$.

5) Find the volume of the solid of revolution of R around the $y = 4$.

6) Find the volume of the solid of revolution of R around the $y = -2$.

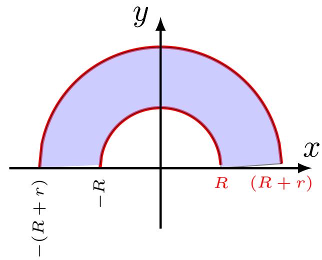
Solutions of Exercises

$\boxed{4\text{-}2\text{-}1}$ Let $f(x) = \sqrt{R^2 - x^2}$, $x \in [-R, R]$.
The volume of a ball of radius R is the volume of the solid of revolution about the $x-$ axis of the surface under the graph of f. Then

$$V = \pi \int_{-R}^{R} f^2(x)dx = 2\pi \int_0^R (R^2 - x^2)dx = 2\pi(R^3 - \frac{R^3}{3}) = \frac{4}{3}\pi R^3.$$

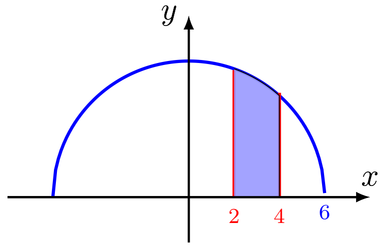
$\boxed{4\text{-}2\text{-}2}$

$$\begin{aligned} V &= \pi \int_{-(R+r)}^{R+r} ((R+r)^2 - x^2)dx \\ &\quad -\pi \int_{-R}^{R} ((R+r)^2 - x^2)dx \\ &= \frac{4}{3}\pi((R+r)^3 - R^3). \end{aligned}$$



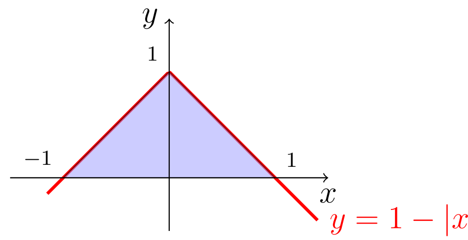
$\boxed{4\text{-}2\text{-}3}$

$$\begin{aligned} V &= \pi \int_2^4 (36 - x^2)dx \\ &= \pi(72 - \frac{56}{3}) = \frac{160}{3}\pi. \end{aligned}$$



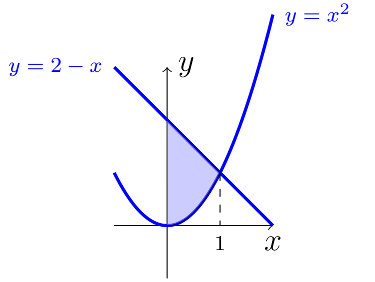
$\boxed{4\text{-}2\text{-}4}$ 1)

$$\begin{aligned} V &= \pi \int_{-1}^{1} (1 - |x|)^2 dx \\ &= 2\pi \int_0^1 (1 - x)^2 dx = \frac{2\pi}{3}. \end{aligned}$$



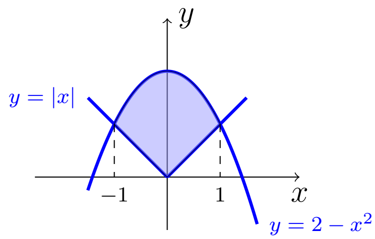
2) $y = x^2$, $y = 2 - x$, revolved around the x-axis,

$$
\begin{aligned}
V &= \pi \int_0^1 (2-x)^2 - x^4 dx \\
&= \frac{32\pi}{15}.
\end{aligned}
$$



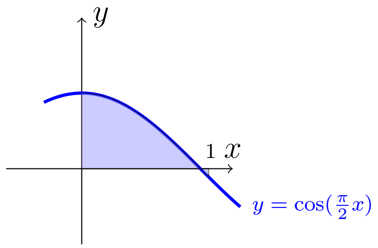
3) $y = |x|$, $y = 2 - x^2$, revolved around the x-axis,

$$
\begin{aligned}
V &= \pi \int_{-1}^1 (2-x^2)^2 - x^2 dx \\
&= \frac{76\pi}{15}.
\end{aligned}
$$



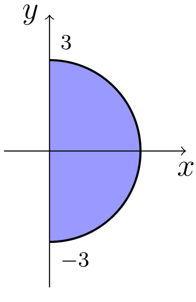
4) $f(x) = \cos(\frac{\pi}{2}x)$, $y = 0$, $x \in [0, 1]$, revolved around the x-axis,

$$
\begin{aligned}
V &= \pi \int_0^1 \cos^2(\frac{\pi}{2}x) dx \\
&= \frac{\pi}{2}.
\end{aligned}
$$

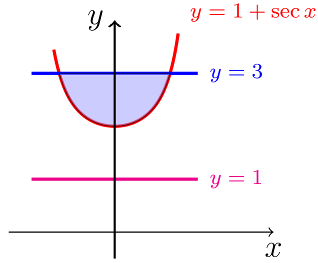


5)

$$
\begin{aligned}
V &= \pi \int_{-3}^3 (9 - y^2) dy \\
&= 2\pi \int_0^3 (9 - y^2) dy = 36\pi.
\end{aligned}
$$

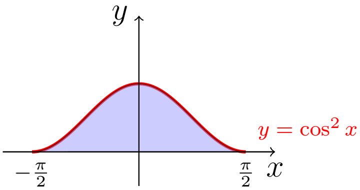


$1+\sec x = 3 \iff \cos x = \frac{1}{2} \iff x = \pm\frac{\pi}{3}.$



4-2-5

$$\begin{aligned} V &= \pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4 - \sec^2 x)dx \\ &= \frac{8\pi^2}{3} - 2\pi\sqrt{3}. \end{aligned}$$



4-2-6

1) About the x-axis:

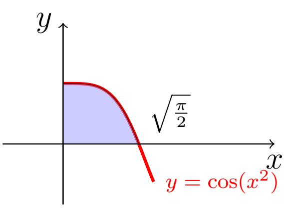
$$\begin{aligned} V &= 2\pi \int_0^{\frac{\pi}{2}} \cos^4 x dx = 2\pi \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos(2x)}{2}\right)^2 dx \\ &= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos(2x) + \frac{\cos(4x)}{2}\right) dx = \frac{3\pi^2}{8}. \end{aligned}$$

2) About the line $y = 1$.

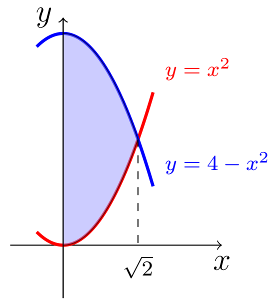
$$\begin{aligned} V &= 2\pi \int_0^{\frac{\pi}{2}} (1 - (1-\cos^2(x))^2)dx = 2\pi \int_0^{\frac{\pi}{2}} (1 - \sin^4(x))dx \\ &= 2\pi \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{4}(1 - 2\cos(2x) + \frac{1+\cos(4x)}{2})\right) dx = \frac{5\pi^2}{8}. \end{aligned}$$

4-2-7 1)

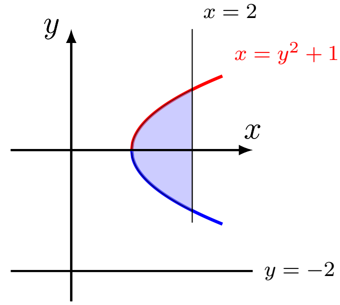
$$\begin{aligned} V &= 2\pi \int_0^{\sqrt{\frac{\pi}{2}}} x\cos(x^2)dx \\ &\overset{t=x^2}{=} \pi \int_0^{\frac{\pi}{2}} \cos(t)dt = \pi. \end{aligned}$$

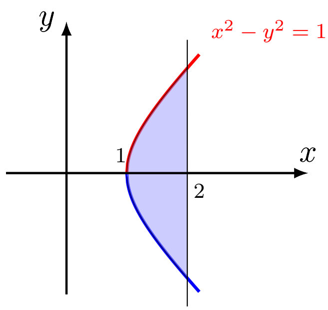


2)
$$4-x^2=x^2 \iff x=\pm\sqrt{2}.$$
$$V=2\pi\int_0^{\sqrt{2}} x(4-2x^2)dx=4\pi.$$



3)
$$\begin{aligned} V &= \pi\int_0^1 ((2+\sqrt{x-1})^2dx \\ &\quad -\pi\int_0^1 (2-\sqrt{x-1})^2)dx \\ &\overset{t^2=x-1}{=} \pi\int_0^1 8t^2dt=\frac{8\pi}{3}. \end{aligned}$$

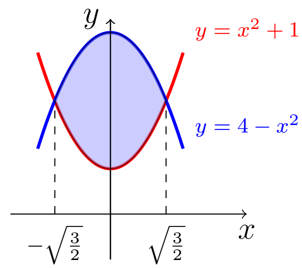




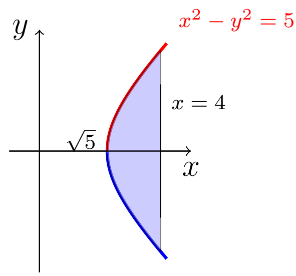
4)

$$\begin{aligned} V &= \pi\int_1^2 \left((3+\sqrt{x^2-1})^2-(3-\sqrt{x^2-1})^2\right)dx \\ &= 12\pi\int_1^2 \sqrt{x^2-1}dx \\ &\overset{x=\sec(\theta)}{=} 12\pi\int_0^{\frac{\pi}{3}} \tan^2(\theta)\sec(\theta)d\theta \\ &= 12\pi\int_0^{\frac{\pi}{3}} \left(\sec^3(\theta)-\sec(\theta)\right)d\theta \\ &\overset{u=\sec(\theta),v'=\sec^2(\theta)}{=} 6\pi\left[\sec(\theta)\tan(\theta)-\ln|\sec(\theta)+\tan(\theta)|\right]_0^{\frac{\pi}{3}} \\ &= 6\pi(2\sqrt{3}-\ln(2+\sqrt{3})). \end{aligned}$$

$$x^2+1=4-x^2 \iff x=\pm\sqrt{\tfrac{3}{2}}.$$

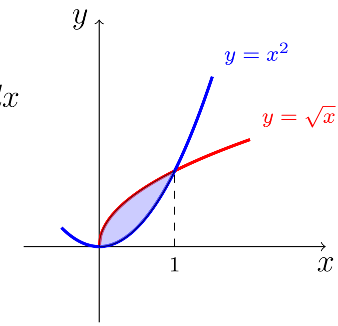


5) $$\begin{aligned} V &= \pi\int_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}}((6-x^2)^2-(x^2+3)^2)dx \\ &= 18\pi\int_0^{\sqrt{\frac{3}{2}}}(3-2x^2)dx = 36\sqrt{\frac{3}{2}}\pi. \end{aligned}$$



6) $$\begin{aligned} V &= 4\pi\int_{\sqrt{5}}^{4} x\sqrt{x^2-5}dx \\ &= \frac{44\sqrt{11}}{3}\pi. \end{aligned}$$

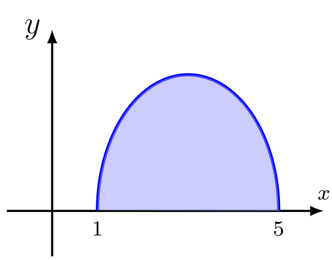
7) $y=\sqrt{x}$, $y=x^2$ about the line $y=2$.



$$\begin{aligned} V &= \pi\int_0^1((2-x^2)^2-(2-\sqrt{x})^2)dx \\ &= \pi\int_0^1(x^4-4x^2-x+4\sqrt{x})dx \\ &= \frac{31\pi}{30}. \end{aligned}$$

8) $V=\pi\displaystyle\int_0^4\left(4-(2-\sqrt{x})^2\right)dx=\pi\int_0^4\left(4\sqrt{x}-x\right)dx=\frac{40\pi}{3}.$

4-2-8



1) $V = 9\pi \int_1^5 \left(1 - \frac{(x-3)^2}{4}\right) dx \overset{t=\frac{x-3}{2}}{=} 18\pi \int_{-1}^1 (1-t^2)dt = 24\pi.$

2)

$$\begin{aligned} V &= 2\pi \int_1^5 3x\sqrt{1 - \frac{(x-3)^2}{4}}dx \\ &\overset{t=\frac{x-3}{2}}{=} 12\pi \int_{-1}^1 (3+2t)\sqrt{1-t^2}dt \\ &= 12\pi \int_{-1}^1 3\sqrt{1-t^2}dt \\ &\overset{t=\sin(\theta)}{=} 18\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos(2\theta))d\theta = 18\pi^2 \end{aligned}$$

3)

$$\begin{aligned} V &= 2\pi \int_1^5 3(x-1)\sqrt{1 - \frac{(x-3)^2}{4}}dx \\ &\overset{t=\frac{x-3}{2}}{=} 24\pi \int_{-1}^1 (1+t)\sqrt{1-t^2}dt \\ &= 24\pi \int_{-1}^1 \sqrt{1-t^2}dt = 12\pi^2. \end{aligned}$$

4)

$$\begin{aligned} V &= 2\pi \int_1^5 3(6-x)\sqrt{1 - \frac{(x-3)^2}{4}}dx \\ &\overset{t=\frac{x-3}{2}}{=} 12\pi \int_{-1}^1 (3-2t)\sqrt{1-t^2}dt \\ &= 36\pi \int_{-1}^1 \sqrt{1-t^2}dt = 18\pi^2. \end{aligned}$$

5)

$$
\begin{aligned}
V &= \pi \int_1^5 \left(16 - \left(4 - 3\sqrt{1 - \frac{(x-3)^2}{4}}\right)^2\right) dx \\
&= \pi \int_1^5 \left(24\sqrt{1 - \frac{(x-3)^2}{4}} - 9\left(1 - \frac{(x-3)^2}{4}\right)\right) dx \\
&\overset{t=\frac{x-3}{2}}{=} 2\pi \int_{-1}^1 \left(24\sqrt{1-t^2} - 9(1-t^2)\right) dt \\
&= 24\pi^2 - 24\pi = 24\pi(\pi - 1).
\end{aligned}
$$

6)

$$
\begin{aligned}
V &= \pi \int_1^5 \left(\left(2 + 3\sqrt{1 - \frac{(x-3)^2}{4}}\right)^2 - 4\right) dx \\
&= \pi \int_1^5 \left(9\left(1 - \frac{(x-3)^2}{4}\right) + 12\sqrt{1 - \frac{(x-3)^2}{4}}\right) dx \\
&\overset{t=\frac{x-3}{2}}{=} 2\pi \int_{-1}^1 \left(9(1-t^2) + 12\sqrt{1-t^2}\right) dt \\
&= 24\pi + 12\pi^2 = 12\pi(2\pi + 1).
\end{aligned}
$$

Exercises

4-3-1 Find the arc length of the following graphs.

1) $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $x \in [1,5]$
2) $f(x) = \ln(\sin x)$, $x \in [\frac{\pi}{6}, \frac{\pi}{2}]$
3) $f(x) = \cosh x$ on the interval $[0, \ln 2]$
4) $f(x) = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x$ on the interval $[1,2]$
5) $f(x) = \pi + \dfrac{2}{3}x\sqrt{x}$ on the interval $[0,8]$
6) $f(x) = \ln|\sec x|$ on the interval $\left[0, \dfrac{\pi}{4}\right]$
7) $f(x) = \dfrac{1}{3}\left(x^2+2\right)^{\frac{3}{2}}$ on the interval $[0,1]$
8) $f(x) = \dfrac{e^{2x}+e^{-2x}}{4}$ on the interval $[0,1]$
9) $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $[1,3]$
10) $f(x) = \dfrac{1}{3}x^{\frac{3}{2}} - \sqrt{x}$ on the interval $[1,4]$

4-3-2 Find the length of the following curves

1) $f(x) = e^x$ from $x = 0$ to $x = \dfrac{\ln 3}{2}$.
2) $x = \ln(\cos(y))$, $0 \le y \le \frac{\pi}{3}$,
3) $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$,
4) $x = 1 - e^{-y}$, $0 \le y \le 2$.

4-3-3 Find the area of the surface obtained by revolving the following curves about the x-axis:

1) $y = \sqrt{1+e^x}$, $0 \le x \le 1$,
2) $y = \dfrac{1}{x}$, $1 \le x \le 2$,

3) $y = \dfrac{x^3}{3}$, $0 \le x \le 1$

4) $y = \sqrt{x}$, $0 \le x \le 4$

5) $y = \sqrt{9 - x^2}$, $0 \le x \le 4$

6) $y = \dfrac{1}{3}\left(3\sqrt{x} - x^{\frac{3}{2}}\right)$, $1 \le x \le 3$

7) $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$, $1 \le x \le 2$

8) $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$, $1 \le x \le 3$

4-3-4 Find the area of the surface obtained by revolving the following curves about the y-axis:

1) $y = 1 - x^2$, $0 \le x \le 1$,

2) $y = \dfrac{x^2}{4} - \dfrac{\ln(x)}{2}$, $1 \le x \le 2$.

Solutions of Exercises

4-3-1 1)

$$\begin{aligned} L &= \int_1^5 \sqrt{1+(\frac{x}{2}-\frac{1}{2x})^2}dx \\ &= \int_1^5 \frac{x^2+1}{2x}dx = \left[\frac{x^2}{4}+\frac{1}{2}\ln x\right]_1^5 \\ &= 6+\frac{1}{2}\ln 5. \end{aligned}$$

2)

$$\begin{aligned} L &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{1+\cot^2(x)}dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \csc(x)dx \\ &= [\ln|\csc(x)-\cot(x)|]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \ln(2-\sqrt{3}). \end{aligned}$$

3) $f(x)=\cosh x \implies f'(x)=\sinh x$

$$\begin{aligned} L = \int_0^{\ln 2} \sqrt{1+(\sinh x)^2}\ dx &= \int_0^{\ln 2} \sqrt{1+\sinh^2 x}\ dx \\ &= \int_0^{\ln 2} \sqrt{\cosh^2 x}\ dx \\ &= \int_0^{\ln 2} \cosh x\ dx \\ &= \sinh(\ln 2) = \sinh(\ln 2) \\ &= \frac{e^{\ln 2}-e^{-\ln 2}}{2} \\ &= \frac{2-\frac{1}{2}}{2} = \frac{3}{4} \end{aligned}$$

4) $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x \implies f'(x) = \frac{1}{2}x - \frac{1}{2}\frac{1}{x} = \frac{x}{2} - \frac{1}{2x}$

$$
\begin{aligned}
\mathrm{L} = \int_1^2 \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\, dx \quad &= \quad \int_1^2 \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}\right)}\, dx \\
&= \quad \int_1^2 \sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}}\, dx \\
&= \quad \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx \\
&= \quad \int_1^2 \left|\frac{x}{2} + \frac{1}{2x}\right|\, dx \\
&= \quad \int_1^2 \left(\frac{x}{2} + \frac{1}{2x}\right)\, dx \\
&= \quad \left[\frac{x^2}{4} + \frac{1}{2}\ln x\right]_1^2 \\
&= \quad \left[\left(1 + \frac{1}{2}\ln 2\right) - \left(\frac{1}{4} + \frac{1}{2}\ln 1\right)\right] \\
&= \quad 1 + \frac{1}{2}\ln 2 - \frac{1}{4} = \frac{3}{4} + \frac{\ln 2}{2}
\end{aligned}
$$

5) $L = \int_0^8 \sqrt{x+1}dx = \frac{52}{3}$

6) $L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}dx = \ln(1 + \sqrt{2})$

7) $L = \int_0^1 \sqrt{1 + x^2(x^2+2)}dx = \frac{4}{3}$

8) $L = \int_0^1 \sqrt{1 + \sinh^2(2x)}dx = \frac{\sinh^2 2}{3}$

9) $L = \int_1^3 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}dx = \int_1^3 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)dx = \frac{14}{3}$

10)

$$\begin{aligned} L &= \int_1^4 \sqrt{1+\left(\frac{\sqrt{x}}{2}-\frac{1}{2\sqrt{x}}\right)^2}\,dx \\ &= \int_1^4 \left(\frac{\sqrt{x}}{2}+\frac{1}{2\sqrt{x}}\right) dx = \frac{10}{3} \end{aligned}$$

4-3-2 1)

$$\begin{aligned} L &= \int_0^{\frac{\ln 3}{2}} \sqrt{1+e^{2x}}dx \overset{1+e^{2x}=t^2}{=} \int_{\sqrt{2}}^2 \frac{t^2}{t^2-1}dt \\ &= \int_{\sqrt{2}}^2 (1-\frac{1}{1-t^2})dt = \left[t-\frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|\right]_{\sqrt{2}}^2 \\ &= (2-\sqrt{2})-\frac{\ln 3}{2}+\ln(1+\sqrt{2}) \\ &= 2-\sqrt{2}+\frac{1}{2}\ln\left(\frac{3+\sqrt{2}}{3}\right). \end{aligned}$$

2)

$$\begin{aligned} L &= \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 y}dy \\ &= \int_0^{\frac{\pi}{3}} \sec y dy = [\ln|\sec y+\tan y|]_0^{\frac{\pi}{3}} = \ln(2+\sqrt{3}) \end{aligned}$$

3) $L = \int_0^1 \sqrt{1+\frac{1-x}{x}}dx = \int_0^1 \frac{dx}{\sqrt{x}} = 2,$

4) $L = \int_0^2 \sqrt{1+e^{-2y}}dy.$

$$\begin{aligned} \int \sqrt{1+e^{-2y}}dy &\overset{e^{-y}=\tan(\theta)}{=} -\int \frac{\sec(\theta)}{\tan(\theta)}\sec^2(\theta)d\theta \\ &= -\int \csc(\theta)d\theta - \int \sec(\theta)\tan(\theta)d\theta \\ &= \ln(\csc(\theta)+\cot(\theta))-\sec(\theta)+c \\ &= y+\ln(\sqrt{1+e^{-2y}}+1)-\sqrt{1+e^{-2y}}+c \end{aligned}$$

Then

$$\begin{aligned} L &= \int_0^2 \sqrt{1+e^{-2y}}dy \\ &= 2-\sqrt{2}+\ln(1+\sqrt{1+e^{-4}})-\ln(1+\sqrt{2})-\sqrt{1+e^{-4}}. \end{aligned}$$

4-3-3 1)

$$\begin{aligned} A &= 2\pi\int_0^1 \sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}dx \\ &= \pi\int_0^1 (e^x+2)dx = \pi(1+e) \end{aligned}$$

2) $A = 2\pi\displaystyle\int_1^2 x^{-2}\sqrt{x^2+1}dx.$

$$\begin{aligned} \int x^{-2}\sqrt{x^2+1}dx &\overset{x=\tan(\theta)}{=} \int \frac{1}{\cos(\theta)\sin^2(\theta)}d\theta \\ &= \int \frac{\cos(\theta)}{\sin^2(\theta)(1-\sin^2(\theta))}d\theta \\ &= -\frac{1}{\sin(\theta)}+\frac{1}{2}\ln(\frac{1+\sin(\theta)}{1-\sin(\theta)})+c \\ &= -\frac{\sqrt{1+x^2}}{x}+\ln(x+\sqrt{1+x^2})+c \end{aligned}$$

Then

$$A = 2\pi\int_1^2 x^{-2}\sqrt{x^2+1}dx = 2\pi\left(\sqrt{2}-\frac{\sqrt{5}}{2}+\ln(2+\sqrt{5})-\ln(\sqrt{2}+1)\right).$$

3)

$$\begin{aligned} A &= 2\pi\int_0^1 \frac{x^3}{3}\sqrt{1+x^4}dx \\ &\overset{t^2=1+x^4}{=} \frac{2\pi}{3}\int_1^{\sqrt{2}} t^2dt = \frac{\pi}{9}(2\sqrt{2}-1). \end{aligned}$$

4)

$$
\begin{aligned}
A = 2\pi \int_1^4 \sqrt{x}\, \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\, dx &= 2\pi \int_1^4 \sqrt{x}\, \sqrt{1+\frac{1}{4x}}\, dx \\
&= 2\pi \int_1^4 \sqrt{x}\, \sqrt{\frac{4x+1}{4x}}\, dx \\
&= 2\pi \int_1^4 \sqrt{x}\, \frac{\sqrt{4x+1}}{2\sqrt{x}}\, dx \\
&= 2\pi\, \frac{1}{2} \int_1^4 \sqrt{4x+1}\, dx \\
&= \pi \int_1^4 (4x+1)^{\frac{1}{2}}\, dx \\
&= \pi\, \frac{1}{4} \int_1^4 (4x+1)^{\frac{1}{2}}(4)\, dx \\
&= \frac{\pi}{4} \left[\frac{2}{3}(4x+1)^{\frac{3}{2}}\right]_1^4 \\
&= \frac{\pi}{4} \left[\frac{2}{3}(17)^{\frac{3}{2}} - \frac{2}{3}(5)^{\frac{3}{2}}\right] \\
&= \frac{\pi}{6} \left[(17)^{\frac{3}{2}} - (5)^{\frac{3}{2}}\right]
\end{aligned}
$$

5)

$$
\begin{aligned}
A &= 2\pi \int_{-3}^3 \sqrt{9-x^2}\, \sqrt{1+\left(\frac{-x}{\sqrt{9-x^2}}\right)^2}\, dx \\
&= 2\pi \int_{-3}^3 \sqrt{9-x^2}\, \sqrt{1+\frac{x^2}{9-x^2}}\, dx \\
&= 2\pi \int_{-3}^3 \sqrt{9-x^2}\, \sqrt{\frac{(9-x^2)+x^2}{9-x^2}}\, dx \\
&= 2\pi \int_{-3}^3 \sqrt{9-x^2}\, \sqrt{\frac{9}{9-x^2}}\, dx \\
&= 2\pi \int_{-3}^3 \sqrt{9-x^2}\, \frac{3}{\sqrt{9-x^2}}\, dx = 6\pi \int_{-3}^3 1\, dx \\
&= 6\pi\, [x]_{-3}^3 = 6\pi[3-(-3)] = 6\pi\, (6) = 36\pi
\end{aligned}
$$

6)

$$\begin{aligned}A &= 2\pi\int_1^3 \frac{1}{3}\left(3\sqrt{x}-x^{\frac{3}{2}}\right)\sqrt{1+(\frac{1}{2\sqrt{x}}-\frac{\sqrt{x}}{2})^2}dx\\ &= \frac{\pi}{3}\int_1^3(3-x)(1+x)dx=\frac{16\pi}{9}\end{aligned}$$

7)

$$\begin{aligned}A &= 2\pi\int_1^2(\frac{x^3}{6}+\frac{1}{2x})\sqrt{1+(\frac{x^2}{2}-\frac{1}{2x^2})^2}dx\\ &= \frac{47\pi}{16}\end{aligned}$$

8)

$$\begin{aligned}A &= 2\pi\int_1^3(\frac{x^4}{4}+\frac{1}{8x^2})\sqrt{1+(x^3-\frac{1}{4x^3})^2}dx\\ &= 2\pi\int_1^3(\frac{x^4}{4}+\frac{1}{8x^2})(x^3+\frac{1}{4x^3})dx\\ &= \frac{\pi}{4}\int_1^3(2x^7+\frac{3}{2}x+\frac{1}{4x^5})dx\end{aligned}$$

4-3-4 1) $A=2\pi\int_0^1 x\sqrt{1+4x^2}dx=\frac{\pi}{6}(5^{\frac{3}{2}}-1)$,

2) $A=2\pi\int_0^1 x\sqrt{1+(\frac{x}{2}-\frac{1}{2x})^2}dx=\frac{\pi}{2}\int_0^1(x^2+1)dx=\frac{2\pi}{3}$.

5 Exercises on Chapter V

Exercises

5-1-1 Explain when a differentiable parametric curve $\gamma(t) = (x(t), y(t))$ has a

1) horizontal tangent at $\gamma(a)$,
2) vertical tangent at $\gamma(a)$.

5-1-2 Find the length of the following curves:

1) $x = 2 + 3t, y = \cosh(3t)$, $0 \leq t \leq 1$,
2) $x = e^t + e^{-t}$, $y = 5 - 2t$ for $0 \leq t \leq 3$,

5-1-3 Find the area of the surface obtained by rotating the curve $x = t^3, y = t^2$ for $0 \leq t \leq 1$ around the $x-$axis.

5-1-4 Find the area of the surface obtained by rotating the curve about the x-axis:

1) $E = \{(x, y) \in \mathbb{R}^2;\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ y \geq 0\}$,
2) $\mathcal{C} = \{(x, y) \in \mathbb{R}^2;\ x^2 + (y - b)^2 = a^2\}$, $0 < a < b$.

Solutions of Exercises

$\boxed{5\text{-}1\text{-}1}$

1) The parametric curve $\gamma(t) = (x(t), y(t))$ has an horizontal tangent at $\gamma(a)$ if $\lim\limits_{t\to a} \dfrac{y'(t)}{x'(t)} = 0$.
2) The parametric curve $\gamma(t) = (x(t), y(t))$ has a vertical tangent at $\gamma(a)$ if $\lim\limits_{t\to a} \dfrac{x'(t)}{y'(t)} = 0$.

$\boxed{5\text{-}1\text{-}2}$

1) $L = \displaystyle\int_0^1 \sqrt{9 + 9\sinh^2(3t)}dt = 3\int_0^1 \cosh(3t)dt = \sinh(3)$,
2) $L = \displaystyle\int_0^3 \sqrt{4\sinh^2(t) + 4}dt = 2\int_0^3 \cosh(t)dt = 2\sinh(3)$,

$\boxed{5\text{-}1\text{-}3}$

$$\begin{aligned} A &= 2\pi\int_0^1 t^2\sqrt{9t^4 + 4t^2}dt = 2\pi\int_0^1 t^3\sqrt{9t^2 + 4}dt \\ &\overset{x^2=9t^2+4}{=} \frac{2}{9}\pi\int_2^{\sqrt{13}} x^2\Big(\frac{x^2-4}{9}\Big)dx = \frac{2\pi}{81}\left(\frac{(13)^{\frac{5}{2}}}{5} - \frac{4(13)^{\frac{3}{2}}}{3} + \frac{2^6}{15}\right). \end{aligned}$$

$\boxed{5\text{-}1\text{-}4}$

1) $x = a\cos t$, $y = b\sin(t)$, $t \in [0, \pi]$ is a parametrization of the curve.

$$\begin{aligned} A &= 2\pi\int_0^{\pi} b\sin(t)\sqrt{a^2\sin^2(t) + b^2\cos^2(t)}dt \\ &= 2ab\pi\int_0^{\pi} \sin(t)\sqrt{1 + \Big(\frac{b^2-a^2}{a^2}\Big)\cos^2(t)}dt \\ &\overset{x=\cos(t)}{=} 2\pi ab\int_{-1}^1 \sqrt{1 + \Big(\frac{b^2-a^2}{a^2}\Big)x^2}dx \\ &= 4\pi ab\int_0^1 \sqrt{1 + \Big(\frac{b^2-a^2}{a^2}\Big)x^2}dx. \end{aligned}$$

If $b > a > 0$,

$$\begin{aligned} A &= 4\pi ab \int_0^1 \sqrt{1 + (\frac{b^2 - a^2}{a^2})x^2} dx \\ &\overset{x = \frac{a\tan(\theta)}{\sqrt{b^2 - a^2}}}{=} \frac{4\pi a^2 b}{\sqrt{b^2 - a^2}} \int_0^{\tan^{-1}(\frac{\sqrt{b^2 - a^2}}{a})} \sec^3(\theta) d\theta \\ &= \frac{2\pi a^2 b}{\sqrt{b^2 - a^2}} \left(\frac{\sqrt{b^4 - a^4}}{a^2} + \ln(\frac{\sqrt{b^2 - a^2}}{a} + \frac{\sqrt{b^2 + a^2}}{a}) \right). \end{aligned}$$

If $a > b > 0$,

$$\begin{aligned} A &= 4\pi ab \int_0^1 \sqrt{1 - (\frac{a^2 - b^2}{a^2})x^2} dx \\ &\overset{x = \frac{a\sin(\theta)}{\sqrt{a^2 - b^2}}}{=} \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \int_0^{\sin^{-1}(\frac{\sqrt{a^2 - b^2}}{a})} (1 + \cos(2\theta)) d\theta \\ &= \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}} \left(\sin^{-1}(\frac{\sqrt{a^2 - b^2}}{a}) + \frac{b\sqrt{a^2 - b^2}}{a^2} \right). \end{aligned}$$

2) $x = a\cos(t)$, $y = b + a\sin(t)$, $t \in [0, 2\pi]$ is a parametrization of the curve.
$A = 2a\pi \int_0^{2\pi} (b + a\sin(t)) dt = 4\pi^2 ab.$

Exercises

$\boxed{5\text{-}2\text{-}1}$ Find the rectangular coordinates of the following points

1) $(3, \frac{3\pi}{4})$,
2) $(-3, \frac{3\pi}{4})$,
3) $(2, \frac{7\pi}{6})$,
4) $(-2, \frac{7\pi}{6})$,
5) $(-2, \frac{8\pi}{3})$.

$\boxed{5\text{-}2\text{-}2}$ Find the polar coordinates of the following points with $0 \leq \theta < 2\pi$ and $r > 0$

1) $(-3, -3)$, 2) $(1, -\sqrt{3})$, 3) $(3, 3)$, 4) $(-\sqrt{3}, 1)$.

$\boxed{5\text{-}2\text{-}3}$ Find the polar coordinates of the following points with $0 \leq \theta < 2\pi$ and $r < 0$

1) $(-3, -3)$, 2) $(1, -\sqrt{3})$, 3) $(3, 3)$, 4) $(-\sqrt{3}, 1)$.

Solutions of Exercises

5-2-1 1) $x = 3\cos(\frac{3\pi}{4}) = -\frac{3\sqrt{2}}{2}$, $y = 3\sin(\frac{3\pi}{4}) = \frac{3\sqrt{2}}{2}$,

2) $x = -3\cos(\frac{3\pi}{4}) = \frac{3\sqrt{2}}{2}$, $y = -3\sin(\frac{3\pi}{4}) = -\frac{3\sqrt{2}}{2}$,

3) $x = 2\cos(\frac{7\pi}{6}) = -\sqrt{3}$, $y = 2\sin(\frac{7\pi}{6}) = -1$,

4) $x = -2\cos(\frac{7\pi}{6}) = \sqrt{3}$, $y = -2\sin(\frac{7\pi}{6}) = 1$,

5) $x = -2\cos(\frac{8\pi}{3}) = -2\cos(\frac{\pi}{3}) = -1$,
$y = -2\sin(\frac{8\pi}{3}) = 2\sin(\frac{\pi}{3}) = \sqrt{3}$.

5-2-2 1) $r = 3\sqrt{2}$, $\theta = \frac{5\pi}{4}$,

2) $r = 2$, $\theta = \frac{5\pi}{6}$,

3) $r = 3\sqrt{2}$, $\theta = \frac{\pi}{4}$,

4) $r = 2$, $\theta = \frac{2\pi}{3}$.

5-2-3 1) $r = -3\sqrt{2}$, $\theta = \frac{\pi}{4}$,

2) $r = -2$, $\theta = \frac{11\pi}{6}$,

3) $r = -3\sqrt{2}$, $\theta = \frac{5\pi}{4}$,

4) $r = -2$, $\theta = \frac{5\pi}{3}$.

Exercises

5-3-1 Sketch the curve with the given polar equation:

1) $r = 1 + 2\cos(\theta)$ 2) $r = 3 + \sin(\theta)$ 3) $r = 2\cos(4\theta)$

5-3-2 Find the polar equations of the following Cartesian equations:

1) $y = x$,
2) $4y^2 = x$,
3) $xy = 4$.
4) $(x^2+y^2)^2 = 2xy$

5-3-3 Find the Cartesian equations of the following polar equations:

1) $r = 2$;
2) $r = -3$;
3) $r = 2\cos(2\theta)$;
4) $r = 2\sin(2\theta)$;
5) $r = 2\sec(\theta)$;
6) $r = 2\csc(2\theta)$;
7) $r = 4\sec(\theta)$;
8) $\theta = \dfrac{\pi}{3}$;
9) $r = \tan(\theta)\sec(\theta)$.

5-3-4 Find the area of the region bounded by the curve

1) $r = \tan(\theta)$, $\theta \in [\frac{\pi}{6}, \frac{\pi}{3}]$;
2) $r = 1 - \sin(\theta)$, $\theta \in [0, \pi]$;
3) $r = 1 - \sin(\theta)$, $\theta \in [0, 2\pi]$;
4) $r^2 = 9\sin(2\theta)$, $\theta \in [0, \frac{\pi}{2}]$;
5) $r = 2\tan\theta$, $\theta \in [0, \frac{\pi}{8}]$;
6) $r = 2(4\cos\theta - \sec\theta)$, $\theta \in [0, \frac{\pi}{4}]$.

5-3-5 Find the area of the region that lies inside both curves: $r = 3 - 2\cos(\theta)$, $r = 3 - 2\sin(\theta)$.

5-3-6 Find the area of the region that lies inside the curve $r = 2 + \sin\theta$ and outside the curve $r = 3\sin\theta$.

5-3-7
1) Sketch the region inside the curve $r = 3$ and outside the curve $r = 2$ and find its area;
2) Sketch the region inside the curve $r = 2$ and over the straight line $r = -\csc\theta$ and find its area;

3) Sketch the region inside the curve $r = 4$ and outside the curve $r = 4\sin\theta$ and find its area;

4) Sketch the region inside the curve $r = 4\cos\theta$ and outside the curve $r = 2\cos\theta$ and find its area;

5) Sketch the region inside the curve $r = 1$ and outside the curve $r = 1 - \cos\theta$ and find its area;

6) Sketch the region inside the curve $r = 2 + 2\cos\theta$ and outside the curve $r = 3$ and find its area;

7) Sketch the region inside the curve $r = 3\sin\theta$ and outside the curve $r = 1 + \sin\theta$ and find its area:

8) Sketch the region inside the curve $r = 1 + \cos\theta$ and outside the curve $r = 1 - \cos\theta$ and find its area;

9) Sketch the region inside the curve $r = 1 + \cos\theta$ and outside the curve $r = 3\cos\theta$ and find its area;

10) Sketch the region inside the curve $r = \cos(\theta)$ and outside the curve $r = 1 - \cos(\theta)$ and find its area;

11) Sketch the region inside the curve $r = 1$ and outside the curve $r = 1 - \cos(\theta)$ and find its area;

12) Sketch the common region between the curves $r = 2\sin(\theta)$ and $r = 2\cos(\theta)$ and find its area;

5-3-8 Find the area enclosed by the curve $r = 1 - 2\sin(\theta)$.

5-3-9 Set up the integral that gives the area of the region that lies inside the curve $r = 2 + \cos(2\theta)$ and outside the curve $r = 2 + \sin(\theta)$.

5-3-10 Find the length of the polar curves

1) $r = 5^\theta$, $\theta \in [0, 2\pi]$;

2) $r = \sin^3(\dfrac{\theta}{3})$, $\theta \in [0, \pi]$;

3) $r = 2(1 + \cos(\theta))$, $\theta \in [0, 2\pi]$.

5-3-11 Find the surface area of the curve $r = e^{5\theta}$ rotated around the line $\theta = \frac{\pi}{2}$ from 0 to $\frac{\pi}{4}$.

5-3-12 Find the surface area of the revolution of the curve $r = 5\cos\theta$ revolved around the polar axis (the $x-$axis) from 0 to $\frac{\pi}{3}$.

5-3-13 Find the surface area of the curve $r = -3 - 3\sin\theta$ revolved around the line $\theta = \frac{\pi}{2}$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

5-3-14 Find the surface area of the curve $r = \sin\theta$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

5-3-15 Find the surface area of the curve $r = \cos\theta$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \pi$.

5-3-16 Find the surface area of the curve $r = e^{\theta}$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

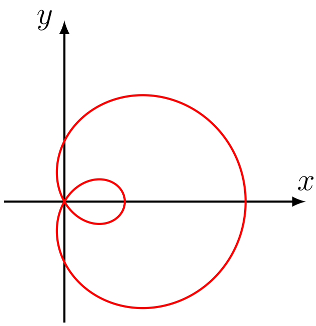
5-3-17 Find the surface area of the curve $r = 1 + \cos\theta$ rotated about the $x-$axis from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

5-3-18 Set up, but do not evaluate, an integral that gives the surface area of the curve rotated about the given axis.

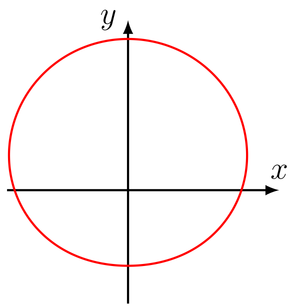
1) $r = 2 + 2\sin\theta$ rotated about the $x-$axis from $\theta = 0$ to $\theta = \pi$;
2) $r = \cos\theta\sin\theta$ rotated about the $x-$axis from $\theta = 0$ to $\theta = \frac{\pi}{2}$;
3) $r = \sin(2\theta)$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{3}$;
4) $r = \cos\theta\sin(2\theta)$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{3}$;
5) $r = 5 - 4\cos\theta$ rotated about the line $\theta = \frac{\pi}{2}$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$;
6) $r = \theta\cos\theta$ rotated about the $x-$axis from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$.

Solutions of Exercises

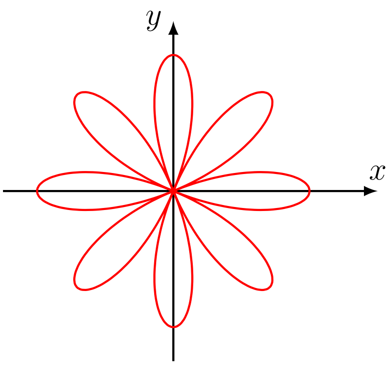
5-3-1



(a) $r = 1 + 2\cos(\theta)$



(b) $r = 3 + \sin(\theta)$



(c) $r = 2\cos(4\theta)$

5-3-2 1) $y = x \iff \tan(\theta) = 1$, then $\theta = \dfrac{\pi}{4}$ is a polar equation,

2) $4y^2 = x \iff 4r\sin^2(\theta) = \cos(\theta) \iff r = \dfrac{1}{4}\csc(\theta)\cot(\theta)$,

3) $xy = 4 \iff r^2 = \dfrac{4}{\sin(\theta)\cos(\theta)} = \dfrac{8}{\sin(2\theta)} = 8\csc(2\theta)$.

4) $(x^2 + y^2)^2 = 2xy \iff r^2 = \sin(2\theta)$.

5-3-3 1) $x^2 + y^2 = 4$

2) $x^2 + y^2 = 9$

3)

$$\begin{aligned} r = 2\cos(2\theta) \quad &\Longleftrightarrow \quad r = 2(\cos^2\theta - \sin^2\theta) \\ &\Longleftrightarrow \quad r^3 = 2(x^2 - y^2) \iff (x^2+y^2)^{\frac{3}{2}} = 2(x^2-y^2) \end{aligned}$$

4) $r = 2\sin(2\theta) \iff r = 4\cos\theta\sin\theta) \iff (x^2+y^2)^{\frac{3}{2}} = 4xy$

5) $x = 2$

6) $r = 2\csc(2\theta) \iff \sqrt{x^2+y^2} = xy$

7) $x = 4$, $y = 4\tan(\theta)$. Then $x = 4$ is the Cartesian equation,

8) $x = \frac{r}{2}$, $y = \frac{r\sqrt{3}}{2}$. Then $y = \sqrt{3}x$ is the Cartesian equation,

9) $x = \tan(\theta)$, $y = \tan^2(\theta)$. Then $y = x^2$ is the Cartesian equation.

5-3-4 1) $A = \dfrac{1}{2}\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2(\theta)d\theta = \dfrac{1}{2}\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2(\theta){-}1)d\theta = \dfrac{1}{2}\left(\sqrt{3} - \dfrac{1}{\sqrt{3}} - \dfrac{\pi}{6}\right)$,

2) $A = \dfrac{1}{2}\displaystyle\int_0^{\pi} (1-\sin(\theta))^2 d\theta = \dfrac{3\pi}{2} - 4$,

3) $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (1-\sin(\theta))^2 d\theta = \dfrac{3\pi}{2}$,

4) $A = \dfrac{9}{2}\displaystyle\int_0^{\frac{\pi}{2}} \sin(2\theta)d\theta = \dfrac{9}{2}$,

5) $A = \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{8}} 4\tan^2(\theta)d\theta = 2\displaystyle\int_0^{\frac{\pi}{8}} (\sec^2(\theta){-}1)d\theta = \left[2\tan(\theta) - 2\theta\right]_0^{\frac{\pi}{8}}$.

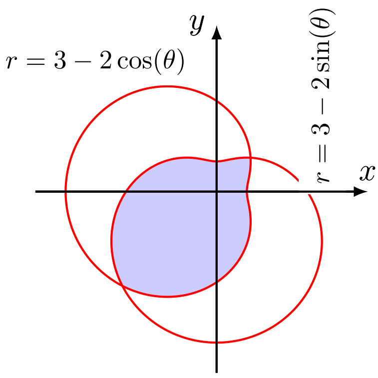
Since $\sin^2(\theta) = \dfrac{1}{2}(1-\cos(2\theta))$ and $\cos^2(\theta) = \dfrac{1}{2}(1+\cos(2\theta))$, then $\sin(\dfrac{\pi}{8}) = \dfrac{1}{2}\sqrt{2-\sqrt{2}}$ and $\cos(\frac{\pi}{8}) = \frac{1}{2}\sqrt{2+\sqrt{2}}$.

$A = \dfrac{2-\sqrt{2}}{2+\sqrt{2}} - \dfrac{\pi}{4} = 3 - 2\sqrt{2} - \dfrac{\pi}{4}$,

6)

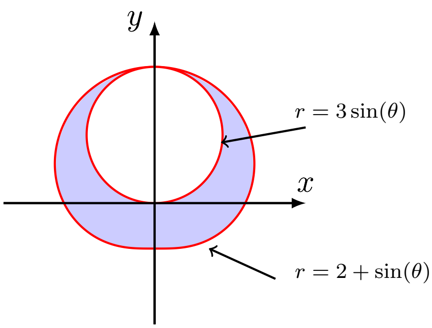
$$\begin{aligned} A &= \frac{1}{2}\int_0^{\frac{\pi}{4}} 4(4\cos(\theta) - \sec(\theta))^2 d\theta \\ &= 2\left[4\sin(2\theta) + \theta + \tan(\theta)\right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{2} + 8 + \sqrt{2}. \end{aligned}$$

$3+2\cos(\theta) = 3+2\sin(\theta)$, then $\theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{5\pi}{4}$.



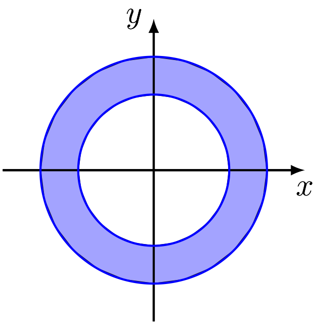
5-3-5

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (3-2\sin(\theta))^2 \\ &\quad + \frac{1}{2}\int_{\frac{5\pi}{4}}^{2\pi+\frac{\pi}{4}} (3-2\cos(\theta))^2 d\theta \\ &= 11\pi - 12\sqrt{2} \end{aligned}$$



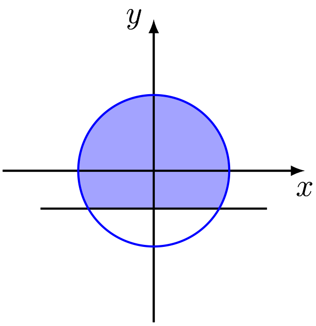
5-3-6

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} (2+\sin(\theta))^2 d\theta \\ &\quad - \frac{9}{2}\int_0^{\pi} \sin^2(\theta) d\theta \\ &= \frac{9\pi}{4}. \end{aligned}$$



5-3-7

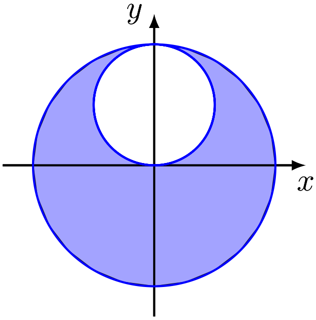
1) $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} (9-4) d\theta = 5\pi$



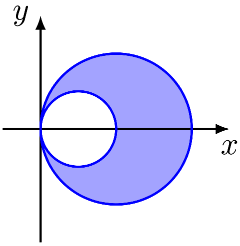
2)

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/6}^{7\pi/6} 4 d\theta \\ &\quad + \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \csc^2\theta d\theta \\ &= \frac{8}{3}\pi + \sqrt{3} \end{aligned}$$

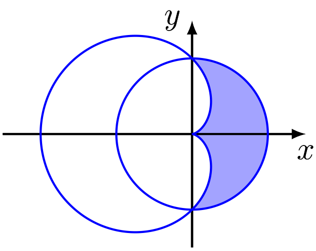
3)
$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} 16 d\theta \\ &\quad -\frac{1}{2}\int_0^{\pi} 16\sin^2\theta d\theta \\ &= 12\pi \end{aligned}$$



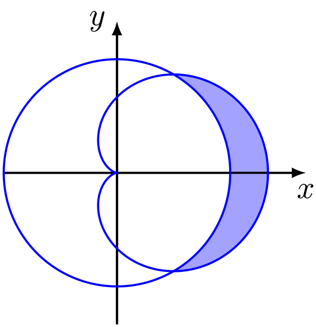
4)
$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi} 16\cos^2\theta d\theta \\ &\quad -\frac{1}{2}\int_0^{\pi} 4\cos^2\theta d\theta \\ &= 3\pi \end{aligned}$$



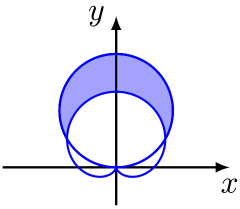
5)
$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \\ &\quad -\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos\theta)^2 d\theta \\ &= 2-\frac{\pi}{4} \end{aligned}$$



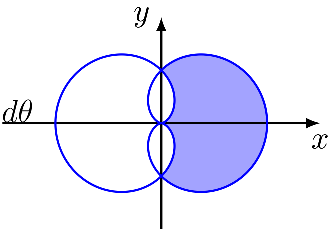
6)
$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(4(1+\cos\theta)^2-9\right) d\theta \\ &= \frac{9}{2}\sqrt{3}-\pi \end{aligned}$$



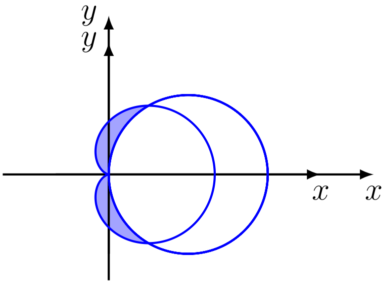
7)
$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (9\sin^2\theta-(1+\sin\theta)^2) d\theta \\ &= \pi \end{aligned}$$



8) $$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos\theta)^2 - (1-\cos\theta)^2 d\theta \\ &= 4 \end{aligned}$$



9) $$\begin{aligned} A &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos\theta)^2 - 9\cos^2\theta d\theta \\ &= 2 - \frac{\pi}{2} \end{aligned}$$



10) The area inside $r = \cos(\theta)$ and outside $r = 1-\cos(\theta)$ is
$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} \cos^2(\theta) - (1-\cos(\theta))^2 d\theta\right) = \sqrt{3} - \frac{\pi}{3}.$$

11) The area inside $r = 1$ and outside $r = 1-\cos(\theta)$ is
$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1 - (1-\cos(\theta))^2\right) d\theta\right) = 2 - \frac{\pi}{4}.$$

12) The area of the common region between the curves $r = 2\sin(\theta)$ and $r = 2\cos(\theta)$ is

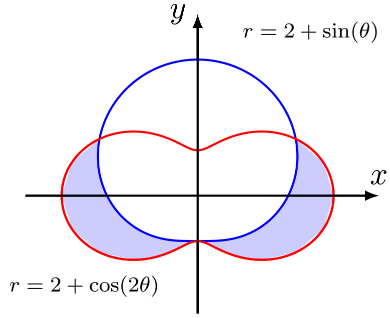
$$\begin{aligned} A &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin^2(\theta)d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2(\theta)d\theta \\ &= \int_0^{\frac{\pi}{4}} \sin^2(\theta)d\theta = \frac{\pi}{8} - \frac{1}{4}. \end{aligned}$$

5-3-8

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} (1-2\sin(\theta))^2 d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (3 - 2\cos(2\theta) - 4\sin(\theta))d\theta = \frac{3\pi}{2}. \end{aligned}$$

5-3-9 $2+\cos(2\theta) = 2+\sin(\theta) \iff \theta = -\frac{\pi}{2}$ or $\theta = \frac{\pi}{6}$.

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (2+\cos(2\theta))^2 - (2+\sin(\theta))^2 d\theta.$$



5-3-10 1)

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r^2(\theta)+r'^2(\theta)}d\theta \\ &= \int_0^{2\pi} \sqrt{5^{2\theta}+(\ln^2 5)5^{2\theta}}d\theta \\ &= \int_0^{2\pi} 5^\theta\sqrt{1+\ln^2 5}d\theta = \frac{5^{2\pi}-1}{\ln 5}\sqrt{1+\ln^2 5}. \end{aligned}$$

2)

$$\begin{aligned} L &= \int_0^{\pi} \sqrt{\sin^6(\frac{\theta}{3})+\sin^4(\frac{\theta}{3})\cos^2(\frac{\theta}{3})}d\theta \\ &= \int_0^{\pi} \sin^2(\frac{\theta}{3})d\theta \\ &= \frac{1}{2}\int_0^{\pi} 1-\cos(\frac{2\theta}{3})d\theta = \frac{\pi}{2}-\frac{3\sqrt{3}}{8}. \end{aligned}$$

3) $L = \int_0^{2\pi} 2\sqrt{2(1+\cos(\theta))}d\theta = 4\int_0^{2\pi} \left|\cos(\frac{\theta}{2})\right| d\theta = 16.$

$\boxed{\text{5-3-11}}$

$$\begin{aligned} A &= 2\pi \int_0^{\frac{\pi}{4}} (e^{5\theta}) \cos\theta \sqrt{(e^{5\theta})^2 + (5e^{5\theta})^2} d\theta \\ &= 2\sqrt{26}\pi \int_0^{\frac{\pi}{4}} e^{10\theta} \cos\theta d\theta \\ &= \frac{\sqrt{26}}{101}\pi \left(11e^{5\frac{\pi}{2}}\sqrt{2} - 20\right). \end{aligned}$$

$\boxed{\text{5-3-12}}$

$$\begin{aligned} A &= 2\pi \int_0^{\frac{\pi}{3}} 5\cos\theta \sin\theta \sqrt{(5\cos\theta)^2 + (-5\sin\theta)^2} d\theta \\ &= 25\pi \int_0^{\frac{\pi}{3}} \sin(2\theta) d\theta \\ &= \frac{25}{2}\pi[-\cos(2\theta)]_0^{\frac{\pi}{3}} = \frac{75}{4}\pi. \end{aligned}$$

$\boxed{\text{5-3-13}}$

$$\begin{aligned} A &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3 + 3\sin\theta) \cos\theta \sqrt{(-3 - 3\sin\theta)^2 + (-3\cos\theta)^2} d\theta \\ &= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3(1 + \sin\theta) \cos\theta \sqrt{9(1 + \sin\theta)^2 + 9\cos^2\theta} d\theta \\ &= 18\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin\theta)^{\frac{3}{2}} \cos\theta d\theta \\ &= \frac{576\pi}{5}. \end{aligned}$$

$\boxed{\text{5-3-14}}$

$$\begin{aligned} A &= 2\pi \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \sqrt{\sin^2\theta + \cos^2\theta} d\theta \\ &= \pi \int_0^{\frac{\pi}{2}} \sin(2\theta) d\theta = \pi. \end{aligned}$$

5-3-15

$$A = 2\pi \int_0^{\pi} \cos^2\theta d\theta = \pi^2.$$

5-3-16

$$\begin{aligned} A &= 2\pi \int_0^{\frac{\pi}{2}} e^{\theta} \cos\theta \sqrt{2e^{2\theta}} d\theta = 2\sqrt{2}\pi \int_0^{\frac{\pi}{2}} e^{2\theta} \cos\theta d\theta \\ &= 2\pi \left[e^{2\theta} (\frac{3}{8} \cos\theta + \frac{1}{4} \sin\theta) \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}(2e^{\pi} - 3). \end{aligned}$$

5-3-17

$$\begin{aligned} A &= 2\pi \int_0^{\frac{\pi}{2}} (1+\cos\theta)\sin\theta \sqrt{(1+\cos\theta)^2 + \sin^2\theta} d\theta \\ &= 2\sqrt{2}\pi \int_0^{\frac{\pi}{2}} (1+\cos\theta)^{\frac{3}{2}} \sin\theta d\theta \\ &= \frac{4\sqrt{2}\pi}{5}(2^{\frac{5}{2}} - 1). \end{aligned}$$

5-3-18 1)

$$\begin{aligned} A &= 2\pi \int_0^{\pi} (2 + 2\sin\theta)\sin\theta \sqrt{4(1+\sin\theta)^2 + 4\cos^2\theta} d\theta \\ &= 8\sqrt{2}\pi \int_0^{\pi} (1+\sin\theta)^{\frac{3}{2}} \sin\theta d\theta. \end{aligned}$$

2)

$$\begin{aligned} A &= 2\pi \int_0^{\frac{\pi}{2}} \cos\theta \sin^2\theta \sqrt{\frac{1}{4}\sin^2(2\theta) + \cos^2(2\theta)} d\theta \\ &= \pi \int_0^{\frac{\pi}{2}} \cos\theta \sin^2\theta \sqrt{1 + 3\cos^2(2\theta)} d\theta. \end{aligned}$$

3)

$$\begin{aligned} A &= 2\pi \int_0^{\frac{\pi}{3}} \sin(2\theta)\cos\theta \sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)} d\theta \\ &= 2\pi \int_0^{\frac{\pi}{3}} \sin(2\theta)\cos\theta \sqrt{1 + 3\cos^2(2\theta)} d\theta. \end{aligned}$$

4)

$$\begin{aligned} A &= 2\pi \int_0^{\frac{\pi}{3}} \cos^2\theta \sin(2\theta) \sqrt{(4\sin\theta\cos^2\theta - 2\sin^3\theta)^2 + 4\sin^4\theta\cos^2\theta} d\theta \\ &= 4\pi \int_0^{\frac{\pi}{3}} \cos^2\theta \sin\theta \sin(2\theta) \sqrt{1 + 3\cos^4\theta - 3\cos^2\theta\sin^2\theta} d\theta. \end{aligned}$$

5)

$$\begin{aligned} A &= 2\pi \int_0^{\frac{\pi}{2}} (5 - 4\cos\theta)\cos\theta \sqrt{(5 - 4\cos\theta)^2 + 16\sin^2\theta} d\theta \\ &= 2\pi \int_0^{\frac{\pi}{2}} (5 - 4\cos\theta)\cos\theta \sqrt{41 - 40\cos\theta} d\theta. \end{aligned}$$

6)

$$\begin{aligned} A &= 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \theta\cos\theta\sin\theta \sqrt{\theta^2\cos^2\theta + (\cos\theta - \theta\sin\theta)^2} d\theta \\ &= 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \theta\cos\theta\sin\theta \sqrt{\theta^2 + \cos^2\theta - 2\theta\cos\theta\sin\theta} d\theta. \end{aligned}$$

6 Exercises on Chapter VI

Exercises

6-1-1 Study the following parametric curves:

1) $x(t) = \cos(3t), y(t) = \sin(2t),$
2) $x(t) = 3\cos(t) - \cos(3t), y(t) = 3\sin(t) - \sin(3t),$
3) $x(t) = \cos(2t), y(t) = \sin(3t),$
4) $x(t) = \dfrac{t}{1+t^3}, y(t) = \dfrac{t^2}{1+t^3},$
5) $x(t) = t - \sin(t), y(t) = 1 - \cos(t),$
6) $x(t) = \sin(\dfrac{t}{2}), y(t) = \tan(t),$
7) $x(t) = \dfrac{t^2}{t-1}, y(t) = \dfrac{t}{t^2-1},$
8) $x(t) = t^2 + \dfrac{1}{t}, y(t) = e^t + 2t^2,$

6-1-2 Study the following curves in polar coordinates:

1) $r(\theta) = 1 + \cos(\theta),$
2) $r(\theta) = \cos(2\theta),$
3) $r(\theta) = \sin(\theta)\cos(2\theta),$
4) $r(\theta) = \cos\left(\dfrac{\theta}{3}\right) + \dfrac{\sqrt{2}}{2},$
5) $r(\theta) = \sin^3(\dfrac{\theta}{3}),$
6) $r(\theta) = \sec(\theta) + \csc(\theta),$
7) $r(\theta) = 2 + \sec(\theta),$
8) $r(\theta) = \sqrt{\cos(2\theta)},$
9) $r(\theta) = \dfrac{\sin^2(\theta)}{\cos(\theta)},$
10) $r(\theta) = \dfrac{1}{\sin\frac{\theta}{2}} = \csc(\dfrac{\theta}{2}).$

Solutions of Exercises

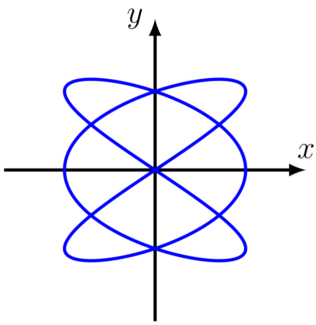
6-1-1

1) $x(t) = \cos(3t), y(t) = \sin(2t), x'(t) = -2\sin(3t), y'(t) = 2\cos(2t)$. The curve $f(t) = (x(t), y(t))$ is $2\pi-$periodic.

$x(-t) = x(t)$, $y(-t) = -y(t)$, thus we study the curve on $[0, \pi]$ and we take a symmetry with respect to the $x-$axis.

$x(\pi - t) = -x(t)$, $y(\pi - t) = -y(t)$, thus we study the curve on $[0, \frac{\pi}{2}]$ and we take a symmetry with respect to the origin and a symmetry with respect to the $x-$axis.

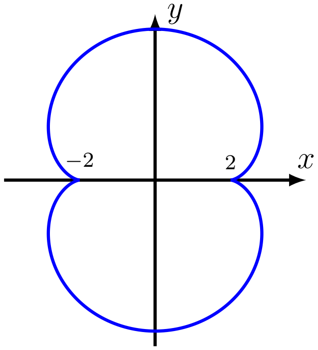
t	0		$\frac{\pi}{4}$		$\frac{\pi}{3}$		$\frac{\pi}{2}$
$x'(t)$	0		$-$		0	$+$	2
x	1	$\searrow$			-1	$\nearrow$	0
$y'(t)$		$+$	0		$-$		-2
y	0	$\nearrow$	1	$\searrow$			0



2) $x(t) = 3\cos(t) - \cos(3t)$, $y(t) = 3\sin(t) - \sin(3t)$. The curve is $2\pi-$periodic and $x(-t) = x(t)$, $y(-t) = -y(t)$. Then we study the curve on the interval $[0, \pi]$ and we do a symmetry with respect to $x-$axis. Also $x(\pi - t) = -x(t)$, $y(\pi - t) = y(t)$. Then we study the curve on the interval $[0, \frac{\pi}{2}]$ and we do a symmetry with respect to $y-$axis and a symmetry with respect to $x-$axis.

$x'(t) = -3\sin(t) + 3\sin(3t) = 3(\sin(3t) - \sin(t)) = 6\cos(2t)\sin(t)$,
$y'(t) = 3(\cos(t) - \cos(3t)) = 6\sin(2t)\sin(t)$.

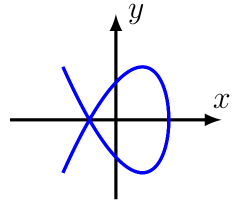
t	0		$\frac{\pi}{4}$		$\frac{\pi}{2}$
$x'(t)$	0	$+$	0	$-$	
x	2		$\nearrow$		$3\sqrt{2}$
$y'(t)$	0		$+$		0
y	0		$\nearrow$		4



3) $x(t) = \cos(2t), y(t) = \sin(3t)$.

The curve is $2\pi-$periodic and $x(-t) = x(t)$, $y(-t) = -y(t)$. Then we study the curve on the interval $[0, \pi]$ and we do a symmetry with respect to $x-$axis. Also $x(\pi - t) = x(t)$, $y(\pi - t) = -y(t)$. Then we study the curve on the interval $[0, \frac{\pi}{2}]$ and we do a symmetry with respect to $y-$axis.
$x'(t) = -2\sin(2t)$, $y'(t) = 3\cos(3t)$.

t	0		$\frac{\pi}{6}$		$\frac{\pi}{2}$
$x'(t)$	0		$-$		0
x	1		$\searrow$		-1
$y'(t)$		$+$	0	$-$	0
y	0	$\nearrow$	1	$\searrow$	-1



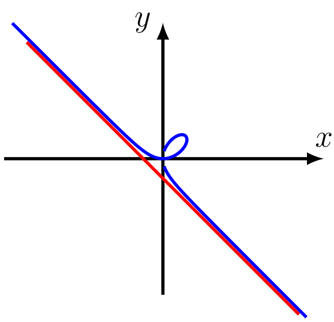
4) $x(t) = \dfrac{t}{1+t^3}, y(t) = \dfrac{t^2}{1+t^3}$. The curve is defined on $\mathbb{R}\backslash\{-1\}$.

$x'(t) = \dfrac{1-2t^3}{(1+t^3)^2}, y'(t) = \dfrac{t(2-t^3)}{(1+t^3)^2}$.

$\lim\limits_{t\to-\infty} \dfrac{y(t)}{x(t)} = -\infty$ and $\lim\limits_{t\to+\infty} \dfrac{y(t)}{x(t)} = +\infty$. Then the $y-$axis is an asymptotic direction of the curve for $t \to \pm\infty$.

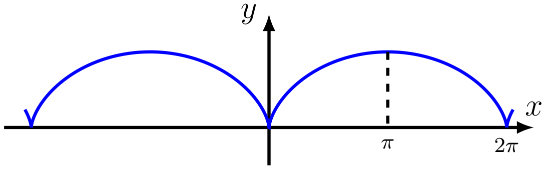
$\lim\limits_{t\to-1} \dfrac{y(t)}{x(t)} = -1$ and $\lim\limits_{t\to-1} y(t)+x(t) = -\dfrac{1}{3}$ and $y(t)+x(t)+\frac{1}{3} \geq 0$ for t in a neighborhood of -1. Then the equation of the asymptote is $y = -x-\dfrac{1}{3}$ and the curve is above the asymptote.

t	$-\infty$		-1		0		$2^{-\frac{1}{3}}$		$2^{\frac{1}{3}}$		$+\infty$
$x'(t)$		$+$	‖	$+$			0	$-$			
x	0	$\nearrow$	$+\infty$ ‖ $-\infty$	$\nearrow$		$\nearrow$	$\frac{2^{\frac{2}{3}}}{3}$	$\searrow$		$\searrow$	0
$y'(t)$		$-$	‖	$-$	0	$+$		$+$	0	$-$	
y	0	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	0	$\nearrow$		$\nearrow$	$\frac{2^{\frac{2}{3}}}{3}$	$\searrow$	0



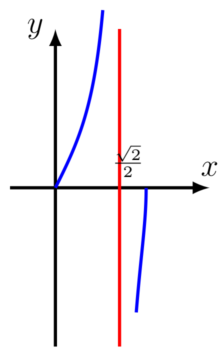
5) $x(t) = t - \sin(t), y(t) = 1 - \cos(t)$, $x(t+2\pi) = x(t) + 2\pi$, $y(t+2\pi) = y(t)$, $x(-t) = -x(t)$, $y(-t) = y(t)$, $x'(t) = 1 - \cos(t)$, $y'(t) = \sin(t)$. $\dfrac{y'(t)}{x'(t)} = \dfrac{\sin(t)}{1-\cos(t)} = \cot(\dfrac{t}{2})$. Then the tangent to the curve at the point $(0,0)$ is parallel to the $y-$axis.

t	0		π		2π
$x'(t)$	0		$+$		0
x	0	$\nearrow$			2π
$y'(t)$	0	$+$	0	$-$	0
y	0	$\nearrow$	2	$\searrow$	0



6) $x(t) = \sin(\dfrac{t}{2}), y(t) = \tan(t)$.
The curve is periodic of period 2π. $x(-t) = -x(t)$, $y(-t) = -y(t)$. We can study the curve on the interval $[0, \pi]$ and do do a symmetry with respect to the origin. $x'(t) = \frac{1}{2}\cos(\dfrac{t}{2})$, $y'(t) = \sec^2 t$.

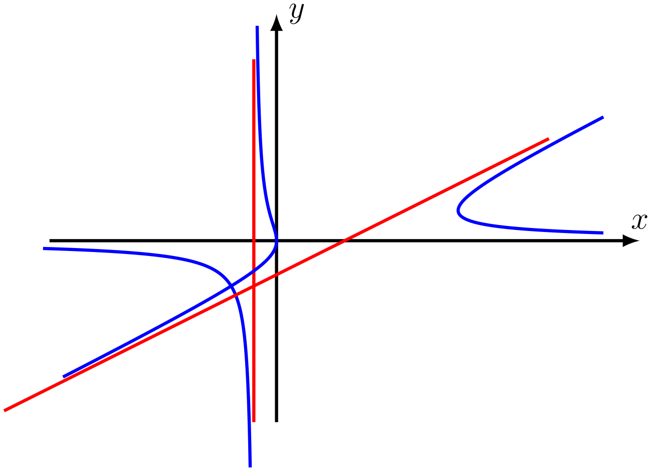
t	0		$\dfrac{\pi}{2}$		π
$x'(t)$			$+$		
x	0	$\nearrow$			1
$y'(t)$		$+$	‖	$+$	
y	0	$\nearrow$ $+\infty$	‖	-∞ $\nearrow$	0



7) $x(t) = \dfrac{t^2}{t-1}, y(t) = \dfrac{t}{t^2-1}, x'(t) = \dfrac{t(t-2)}{(t-1)^2}, y'(t) = \frac{-}{1+t^2}(t^2-1)^2$.
$\lim\limits_{t\to 1} \dfrac{y(t)}{x(t)} = \dfrac{1}{2}, \lim\limits_{t\to 1} y(t) - \dfrac{1}{2}x(t) = -\dfrac{3}{4}$ and
$y(t) - \dfrac{1}{2}x(t) + \dfrac{3}{4} = \dfrac{(1-t)(2t+3)}{4(1+t)}$.
The line of equation $y = \frac{1}{2}x - \frac{3}{4}$ is an asymptote to the curve. The curve is over the asymptote for $t < 1$ and below the

asymptote for $t > 1$ (In the neighborhood of 1.

t	$-\infty$		-1		0		1		2		$+\infty$
$x'(t)$			$+$		0	$-$	$\Vert$	$-$	0	$+$	
x	$-\infty$	$\nearrow$			0	$\searrow$	$-\infty \Vert +\infty$	$\searrow$	$\frac{2}{3}$	$\nearrow$	$+\infty$
$y'(t)$		$-$	$\Vert$		$-$		$\Vert$		$-$		
y	0	$\searrow$	$-\infty \Vert +\infty$		$\searrow$		$-\infty \Vert +\infty$		$\searrow$		0



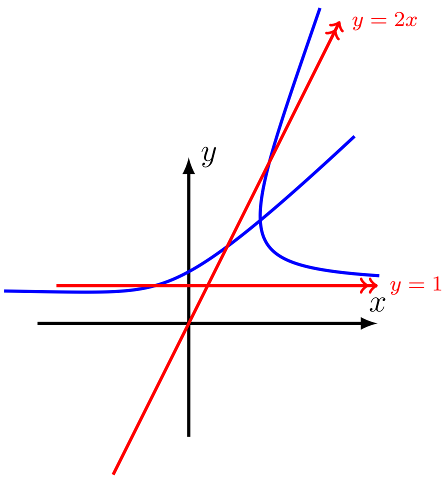
8) $x(t) = t^2 + \dfrac{1}{t}, y(t) = e^t + 2t^2$. $x'(t) = 2t - \dfrac{1}{t^2} = \dfrac{2t^3 - 1}{t^2}$, $y'(t) = e^t + 4t$

$y = 1$ is an asymptote of the curve.

$\lim\limits_{t \to -\infty} \dfrac{y(t)}{x(t)} = 2$ and $\lim\limits_{t \to -\infty} y(t) - 2x(t) = 0$. Then the line of equation $y = 2x$ is an asymptote to the curve.

For t is a neighborhood of $+\infty$, we have a parabolic branch parallel to the $y-$axis.

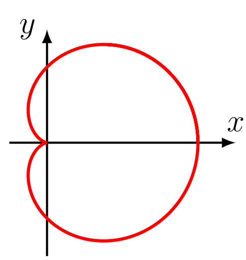
t	$-\infty$		a		0		$2^{-\frac{1}{3}}$		$+\infty$
$x'(t)$			$-$		$\|$	$-$	0	$+$	
x	$+\infty$	$\searrow$		$-\infty$	$\|$ $+\infty$	$\searrow$	$x(2^{-\frac{1}{3}})$	$\nearrow$	$+\infty$
$y'(t)$		$-$	0				$+$		
y	$+\infty$	$\searrow$	$y(a)$			$\nearrow$			$+\infty$



$x(2^{-\frac{1}{3}}) = 2^{\frac{1}{3}} + 2^{-\frac{2}{3}}$, a is the zeros of the function $e^t + 4t$.

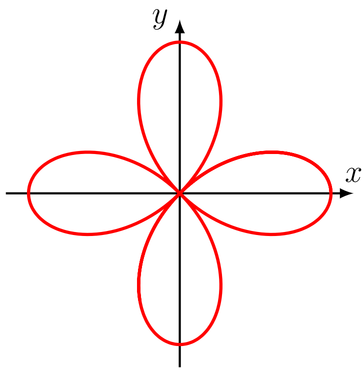
6-1-2 1) $r(\theta) = 1 + \cos(\theta)$.
r is $2\pi-$periodic, $r(-\theta) = r(\theta)$ thus it suffices to study the curve on the interval $[0, \pi]$ and make a symmetry with respect to the $x-$axis.
$r'(\theta) = -\sin(\theta)$.

θ	0		π
$r'(\theta)$	0	$-$	0
$r(\theta)$	2	$\searrow$	0



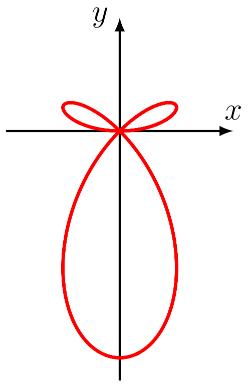
2) $r(\theta) = \cos(2\theta)$. r is $\pi-$periodic, $r(-\theta) = r(\theta)$ and $r(\frac{\pi}{2} - \theta) = -r(\theta)$ thus it suffices to study the curve on the interval $[0, \frac{\pi}{4}]$ and make a symmetry with respect to the $x-$axis and the exis of equation $y = x$ and symmetry with respect to O.
$r'(\theta) = -2\sin(2\theta)$.

θ	0		$\frac{\pi}{4}$
$r'(\theta)$	0	$-$	
$r(\theta)$	1	$\searrow$	0



3) $r(\theta) = \sin(\theta)\cos(2\theta)$.
r is $2\pi-$periodic and $r(-\theta) = -r(\theta)$, thus it suffices to study the curve on the interval $[0, \pi]$ and make a symmetry with respect to the $y-$axis. Also $r(\pi - \theta) = r(\theta)$, thus it suffices to study the curve on the interval $[0, \pi]$ and make a symmetry with respect to the $y-$axis.
$r'(\theta) = \cos(\theta)\cos(2\theta) - 2\sin(\theta)\sin(2\theta) = \cos(\theta)(1 - 6\sin^2(\theta))$.
Let $\alpha \in [0, \frac{\pi}{2}]$ such that $\sin^2(\alpha) = \frac{1}{6}$, then

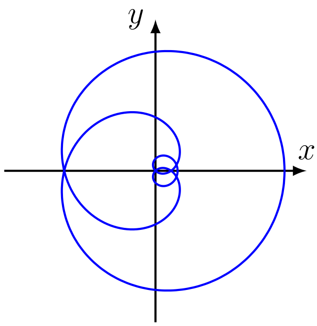
θ	0		α		$\frac{\pi}{2}$		$\pi - \alpha$		π
$r'(\theta)$		$+$	0	$-$	0	$+$	0	$-$	
$r(\theta)$	0	$\nearrow$	$\frac{2}{3\sqrt{6}}$	$\searrow$	-1	$\nearrow$	$\frac{2}{3\sqrt{6}}$	$\searrow$	0



4) $r(\theta) = \cos\left(\dfrac{\theta}{3}\right) + \dfrac{\sqrt{2}}{2}$.

r is $6\pi-$periodic and $r(-\theta) = r(\theta)$, thus it suffices to study the curve on the interval $[0, 3\pi]$ and make a symmetry with respect to the $x-$axis.

θ	0		3π
$r'(\theta)$	0	$-$	0
$r(\theta)$	$1+\frac{\sqrt{2}}{2}$	$\searrow$	$\frac{\sqrt{2}}{2}-1$

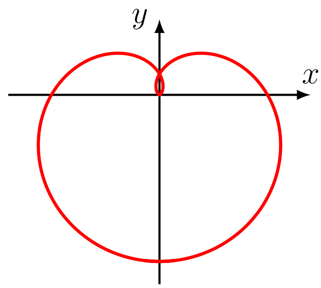


5) $r(\theta) = \sin^3(\dfrac{\theta}{3})$.

r is $6\pi-$periodic, $r(-\theta) = -r(\theta)$ and $r(3\pi-\theta) = r(\theta)$ thus it suffices to study the curve on the interval $[0, \frac{3\pi}{2}]$ and make a symmetry with respect to the $y-$axis.

$r'(\theta) = \sin^2(\dfrac{\theta}{3})\cos(\dfrac{\theta}{3})$.

θ	0		$3\pi/2$
$r'(\theta)$	0	$+$	0
$r(\theta)$	0	$\nearrow$	1

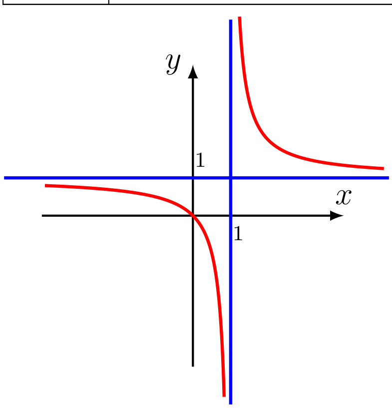


6) $r(\theta) = \sec(\theta) + \csc(\theta)$.
r is $2\pi-$periodic, $r(\pi + \theta) = -r(\theta)$, then we study the curve on the interval $[0, \pi]$ and we make a symmetry with respect to O.

$$\begin{aligned} r'(\theta) &= \sec(\theta)\tan(\theta) - \csc(\theta)\cot(\theta) \\ &= \frac{(\sin(\theta) - \cos(\theta))(1 + \sin(\theta)\cos(\theta)}{\sin^2(\theta)\cos^2(\theta)}. \end{aligned}$$

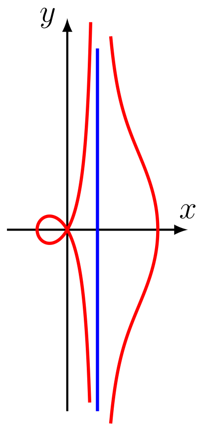
$\lim\limits_{\theta \to 0} r(\theta)\sin(\theta) = 1$, then $y = 1$ is an asymptote.
$\lim\limits_{\theta \to 0} r(\theta)\sin(\theta - \dfrac{\pi}{2}) = -1$, then $x = 1$ is an asymptote.

θ	0		$\frac{\pi}{4}$		$\frac{\pi}{2}$		π
$r'(\theta)$	$\Vert$	$-$	0	$+$	$\Vert$	$+$	$\Vert$
$r(\theta)$	$+\infty$	$\searrow$	$2\sqrt{2}$	$\nearrow$	$+\infty \Vert -\infty$	$\nearrow$	$+\infty$



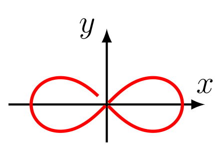
7) $r(\theta) = 2 + \sec(\theta)$.
r is $2\pi-$periodic, $r(-\theta) = r(\theta)$, then we study the curve on the interval $[0, \pi]$ and we make a symmetry with respect to $x-$axis.
$r'(\theta) = \sec(\theta)\tan(\theta)$.
$\lim\limits_{\theta \to 0} r(\theta)\sin(\theta - \dfrac{\pi}{2}) = -1$, then $x = 1$ is an asymptote.

θ	0		$\frac{\pi}{2}$		π
$r'(\theta)$	0	$+$	$\|$	$+$	0
$r(\theta)$	3	$\nearrow$	$+\infty \;\|\; -\infty$	$\nearrow$	1



8) $r(\theta) = \sqrt{\cos(2\theta)}$.
r is $2\pi-$periodic, $r(-\theta) = r(\theta)$ and $r(\pi - \theta) = r(\theta)$, thus it suffices to study the curve on the interval $[0, \pi]$ and make a symmetry with respect to the $x-$axis and a symmetry with respect to the $y-$axis.
$r'(\theta) = -\dfrac{\sin(2\theta)}{\sqrt{\cos(2\theta)}}$.

θ	0		$\frac{\pi}{4}$	$\frac{\pi}{2}$
$r'(\theta)$	0	$-$		
$r(\theta)$	1	$\searrow$	0	

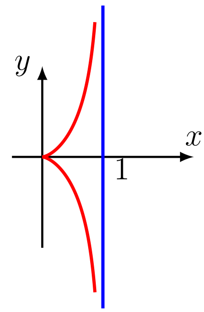


9) $r(\theta) = \dfrac{\sin^2(\theta)}{\cos(\theta)}$.
r is $2\pi-$periodic, $r(-\theta) = r(\theta)$ and $r(\pi - \theta) = -r(\theta)$ thus it suffices to study the curve on the interval $[0, \pi]$ and make a symmetry with respect to the $x-$axis and a symmetry with respect to the $y-$axis.

$r'(\theta) = \dfrac{\sin(\theta)(1+\cos^2(\theta))}{\cos^2(\theta)}$.

$\lim\limits_{\theta\to\frac{\pi}{2}} r(\theta)\sin(\theta - \dfrac{\pi}{2}) = 1$. Then $x = 1$ is an asymptote to the curve.

θ	0		$\frac{\pi}{2}$		π
$r'(\theta)$	0	$+$	$\Vert$	$+$	0
$r(\theta)$	0	$\nearrow$	$+\infty \ \Vert$ -∞	$\nearrow$	0



10) $r(\theta) = \frac{1}{\sin(\frac{\theta}{2})} = \csc(\frac{\theta}{2})$.

r is $4\pi-$periodic, $r(-\theta) = -r(\theta)$ and $r(2\pi - \theta) = r(\theta)$ thus it suffices to study the curve on the interval $[0, \frac{3\pi}{2}]$ and make a symmetry with respect to the $y-$axis.

$r'(\theta) = -\dfrac{1}{2}\csc(\dfrac{\theta}{2})\cot(\dfrac{\theta}{2})$.

$\lim\limits_{\theta\to 0} r(\theta)\sin(\theta) = 2$. Then $y = 2$ and $y = -2$ are asymptote to the curve.

θ	0		π
$r'(\theta)$	$\Vert$	$-$	0
$r(\theta)$	$+\infty$	$\searrow$	1