SEMESTER I FINAL EXAMINATION, 1443
Dept. Math., College of Science, KSU
Math: 107 Full Mark: 40 Time: 3 Hours

Q1. [3+2] (a) Find all values of x, y, and z for which the matrix A is *symmetric*

$$A = \begin{bmatrix} 2 & x-2y+2z & 2x+y+z \\ 3 & 5 & x+z \\ 0 & -2 & 7 \end{bmatrix}$$

(b) Let

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

and $f(x) = x^2 + 3x + 2$. Find $f(B)$.

Q2. [2+2+4] (a) Show that $\det(A) = (\alpha - 1)(\beta - 1)(\gamma - 1)$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & \beta & 1 \\ 1 & 1 & 1 & \gamma \end{bmatrix}$$

(b) Suppose that D and E are 3×3 matrices with $\det(D) = 4$ such that $D(E - 2D) = 0$, find $\det(E)$.

(c) Let

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Find $adj(C)$, and hence check that $C adj(C) = -6I_3$.

Q3. [2+3+2] (a) Find the volume of the parallelepiped (box) having $\mathbf{a} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{c} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ as adjacent edges.

(b) Find the distance between the point $A(3, -1, 4)$ and the line given by $x = -2 + 3t$ $y = -2t$, and $z = 1 + 4t$.

(c) Find a set of parametric equations for the line of intersection of the planes: $x - 2y + z = 0$, $2x + 3y - 2z = 0$.

Q4. [2+2+5] (a) Find the velocity, speed and acceleration of a particle that moves along the plane curve described by $\mathbf{r}(t) = 2\sin\frac{t}{2}\mathbf{i} + 2\cos\frac{t}{2}\mathbf{j}$.

(b) Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = \cos 2t\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$ that satisfies the initial condition $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(c) Find the tangential and normal components of acceleration, and curvature for the position vector given by $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}$.

Q5. [2+3+3+3]

(a) Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + (y - 3x)^2}$ does not exist.

(b) Find an equation of the tangent plane to the surface of equation $x \ln y + y \ln z + xz = 1$ at the point $P_0(1, 1, 1)$.

(c) Find the local extrema and saddle points, if any, of the function $f(x, y) = 6xy + 3x^2 - y^3 + 1$

(d) Use Lagrange multipliers, find extrema of $f(x, y) = x - 4y$ subject to the constraint $x^2 + 2y^2 = 9$.

Solution M 107 Final Exam SI/43

Q1 (a)

$$A = A^T \Rightarrow \begin{pmatrix} 2 & x-2y+2z & 2x+y+z \\ 3 & 5 & x+z \\ 0 & -2 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ x-2y+2z & 5 & -2 \\ 2x+y+z & x+z & 7 \end{pmatrix}$$

gives

$$\begin{cases} x - 2y + 2z = 3 \\ 2x + y + z = 0 \\ x + z = -2 \end{cases}$$

Marks 3

AM :

$$\begin{pmatrix} 1 & -2 & 2 & 3 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & -2 \end{pmatrix} \xrightarrow[-r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & -2 & 2 & 3 \\ 0 & 5 & -3 & -6 \\ 0 & 2 & -1 & -5 \end{pmatrix} \xrightarrow{\frac{1}{5}r_2}$$

$$\begin{pmatrix} 1 & -2 & 2 & 3 \\ 0 & 1 & \frac{-3}{5} & \frac{-6}{5} \\ 0 & 2 & -1 & -5 \end{pmatrix} \xrightarrow{-2r_2+r_3} \begin{pmatrix} 1 & -2 & 2 & 3 \\ 0 & 1 & -\frac{3}{5} & \frac{-6}{5} \\ 0 & 0 & \frac{1}{5} & \frac{-13}{5} \end{pmatrix} \xrightarrow{5r_3}$$

$$\begin{pmatrix} 1 & -2 & 2 & 3 \\ 0 & 1 & \frac{-3}{5} & \frac{-6}{5} \\ 0 & 0 & 1 & -13 \end{pmatrix}$$

so, $z = -13$

$$y = \frac{-6}{5} + \frac{3}{5} z = \frac{-6}{5} - \frac{39}{5} = \frac{-6-39}{5} = \frac{-45}{5} = -9$$

done

$x = 11$
$y = -9$
$z = -13$

Q1 (b) $f(x) = x^2 + 3x + 2$ Marks 2

$$f(B) = B^2 + 3B + 2I_2$$
$$= \begin{pmatrix}1&2\\0&3\end{pmatrix}\begin{pmatrix}1&2\\0&3\end{pmatrix} + 3\begin{pmatrix}1&2\\0&3\end{pmatrix} + 2\begin{pmatrix}1&0\\0&1\end{pmatrix}$$
$$= \begin{pmatrix}1&8\\0&9\end{pmatrix} + \begin{pmatrix}3&6\\0&9\end{pmatrix} + \begin{pmatrix}2&0\\0&2\end{pmatrix}$$
$$= \begin{pmatrix}6&14\\0&20\end{pmatrix}$$

done ✓

Q2 (a)

$$\begin{vmatrix}1&1&1&1\\1&\alpha&1&1\\1&1&\beta&1\\1&1&1&\gamma\end{vmatrix} \xrightarrow[-r_1+r_3,\ -r_1+r_4]{-r_1+r_2} \begin{vmatrix}1&1&1&1\\0&\alpha-1&0&0\\0&0&\beta-1&0\\0&0&0&\gamma-1\end{vmatrix} = (\alpha-1)(\beta-1)(\gamma-1)$$

done ✓ Marks 2

(b) $\det(D) = 4 \Rightarrow D$ is invertible so

$$D(E-2D) = 0 \Rightarrow E = 2D$$
$$\Rightarrow \det(E) = \det(2D) = 2^3 \det(D) = 32$$

done ✓ Marks 2

(c) $\operatorname{adj}(C) = \begin{pmatrix}1&-3&1\\-5&-3&7\\1&3&-5\end{pmatrix}$ 2 where $C = \begin{pmatrix}1&2&3\\3&1&2\\2&1&3\end{pmatrix}$

Thus, $C \operatorname{adj}(C) = \begin{pmatrix}1&2&3\\3&1&2\\2&1&3\end{pmatrix}\begin{pmatrix}1&-3&1\\-5&-3&7\\1&3&-5\end{pmatrix} = \begin{pmatrix}-6&0&0\\0&-6&0\\0&0&-6\end{pmatrix}$ 2

done ✓

$$C \operatorname{adj}(C)) = \begin{pmatrix}-6&0&0\\0&-6&0\\0&0&-6\end{pmatrix} = -6I_3$$

1

Marks $1+2+1 = 4$

Q_3 (a) $\underline{a} = \langle 3, -5, 1 \rangle$, $\underline{b} = \langle 0, 2, -2 \rangle$

$\underline{c} = \langle 3, 1, 1 \rangle$

$$V = \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} = 3(2+2) + 5(0+6) + 1(0-6)$$

$$= 12 + 30 - 6 = 42 - 6 = 36$$

done ✓

$\Rightarrow V = 36$. (Marks 2)

(b) $A(3, -1, 4)$ $\quad x = -2 + 3t,\ y = -2t,\ z = 1 + 4t$

The direction vector for the line is

$$\underline{a} = \langle 3, -2, 4 \rangle$$

Put $t = 0$ to get a point on the line,

$$P(-2, 0, 1).$$

Thus

$$\vec{PA} = \langle 3 - (-2), -1 - 0, 4 - 1 \rangle$$

✓ $$= \langle 5, -1, 3 \rangle$$

and

✓ $$\vec{PA} \times \underline{a} = \begin{vmatrix} i & j & k \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix} = \langle 2, -11, -7 \rangle$$

Hence the distance is

done ✓ $$d = \frac{\| \vec{PA} \times \underline{a} \|}{\| \underline{a} \|} = \frac{\sqrt{174}}{\sqrt{29}} = \sqrt{6}.$$

(Marks 3)

(c) $x - 2y + z = 0$

$2x + 3y - 2z = 0$

AM: $$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 3 & -2 & 0 \end{array}\right) \xrightarrow{-2r_1 + r_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 7 & -4 & 0 \end{array}\right) \xrightarrow{\frac{1}{7} r_2}$$

(P.T.O)

Q_3 (c) $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & 0 \end{array}\right)$ gives $x = 2y - z$, $y = \frac{4}{7} z$

Take $z = 7t$ then $y = \frac{4}{7}(7t) = 4t$

$x = 2(4t) - 7t = 8t - 7t = t$

Thus,

done ✓ $\left.\begin{array}{l} x = t \\ y = 4t \\ z = 7t \end{array}\right\}$ OR $\left\{\begin{array}{l} z = t \\ y = \frac{4}{7} t \\ x = 2\left(\frac{4}{7} t\right) - t = \frac{8}{7} t - t = \frac{1}{7} t \end{array}\right.$

(Marks 2)

Q_4 (a) $\underline{r}(t) = 2 \sin \frac{t}{2} \underline{i} + 2 \cos \frac{t}{2} \underline{j}$ (Marks 2)

- $\underline{v}(t) = \underline{r}'(t) = \cos \frac{t}{2} \underline{i} - \sin \frac{t}{2} \underline{j}$ (velocity)

done ✓ • $\|\underline{r}'(t)\| = \sqrt{\cos^2 \frac{t}{2} + \sin^2 \frac{t}{2}} = 1$ (Speed)

- $\underline{a}(t) = -\frac{1}{2} \sin \frac{t}{2} \underline{i} - \frac{1}{2} \cos \frac{t}{2} \underline{j}$ (Acc.)

(b) (Marks 2)

$$\underline{r}(t) = \int \underline{r}'(t)\, dt$$

$$= \left(\int \cos 2t \, dt\right) \underline{i} + \left(\int -2 \sin t \, dt\right) \underline{j} + \left(\int \frac{1}{1+t^2} dt\right) \underline{k}$$

$$= \left(\frac{1}{2} \sin 2t + c_1\right) \underline{i} + \left(2 \cos t + c_2\right) \underline{j} + \left(\tan^{-1} t + c_3\right) \underline{k}$$

$t = 0 \Rightarrow \underline{r}(0) = 3\underline{i} - 2\underline{j} + \underline{k} \Rightarrow$

(P.T.O)

$\underline{r}(0) = 3\underline{i} + (-2)\underline{j} + \underline{k}$

$\Rightarrow c_1 = 3, \; 2 + c_2 = -2, \text{ and } c_3 = 1$

done Thus

$$\underline{r}(t) = \left(\tfrac{1}{2}\sin 2t + 3\right)\underline{i} + (2\cos t - 4)\underline{j} + (\tan^{-1} t + 1)\underline{k}$$

Q4 (c) $\underline{r}(t) = 3t\underline{i} - t\underline{j} + t^2\underline{k}$

$\Rightarrow \underline{r}'(t) = 3\underline{i} - \underline{j} + 2t\underline{k}$

$\Rightarrow \|\underline{r}'(t)\| = \sqrt{9+1+4t^2} = \sqrt{10+4t^2}$

$\Rightarrow \underline{r}''(t) = 2\underline{k}$

Marks 1+ 2+2=5

done $a_{\underline{T}} = \dfrac{\underline{r}'(t)\cdot\underline{r}''(t)}{\|\underline{r}'(t)\|} = \dfrac{4t}{\sqrt{10+4t^2}}$ (tangential component of acc.)

$$\underline{r}'(t) \times \underline{r}''(t) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2(-\underline{i} - 3\underline{j}) = -2\underline{i} - 6\underline{j}$$

Normal component of acceleration

$$a_{\underline{N}} = \frac{\|\underline{r}'(t) \times \underline{r}''(t)\|}{\|\underline{r}'(t)\|} = \frac{\sqrt{4+36}}{\sqrt{10+4t^2}} = \frac{2\sqrt{10}}{\sqrt{10+4t^2}}$$

curvature k

$$k = \frac{\|\underline{r}'(t) \times \underline{r}''(t)\|}{\|\underline{r}'(t)\|^3} = \frac{2\sqrt{10}}{(10+4t^2)^{3/2}}$$

Q_5 (a) Along x-axis   Marks 2

$$\lim_{x \to 0} \frac{0}{x^4 + 9x^2} = 0$$

Along $y = 3x$, $\lim_{x \to 0} \frac{3x^4}{x^4} = 3$

$\Rightarrow$ Limit DNE!

(b) Let $F(x, y, z) = x \ln y + y \ln z + xz - 1 = 0$

$F_x(x, y, z) = \ln y + z$ and $F_x(1,1,1) = 1$

$F_y(x, y, z) = \frac{x}{y} + \ln z$ and $F_y(1,1,1) = 1$

$F_z(x, y, z) = \frac{y}{z} + x$ and $F_z(1,1,1) = 2$

So the reqd. eq. of the tangent plane is

$$F_x(1,1,1)(x-1) + F_y(1,1,1)(y-1) + F_z(1,1,1)(z-1) = 0$$

$\Rightarrow (x-1) + (y-1) + 2(z-1) = 0$

$\Rightarrow x + y + 2z = 4.$ Marks 3

(c) $f(x, y) = 6xy + 3x^2 - y^3 + 1$

Critical points: $\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 6y + 6x = 0 \\ 6x - 3y^2 = 0 \end{cases}$

$\Rightarrow \begin{cases} y = -x \\ x(2-x) = 0 \end{cases} \Rightarrow (0,0), (2,-2)$ are critical points

Q5 (c) ~~At~~ $f_{xx}(x,y) = 6$, $f_{xy}(x,y) = 6$, $f_{yy}(x,y) = -6y$

At $(0,0)$ $\quad \begin{vmatrix} 6 & 6 \\ 6 & 0 \end{vmatrix} = -36 < 0 \Rightarrow$ saddle point

At $(2,-2)$

$$\begin{vmatrix} 6 & 6 \\ 6 & 12 \end{vmatrix} = 36 > 0 \text{ and } f_{xx}(2,-2) = 6 > 0$$

$\Rightarrow$ at $(2,-2)$, local minimum.

Marks 3

(d) Let $g(x,y) = x^2 + 2y^2 - 9$

$\nabla f(x,y) = \langle 1, -4 \rangle$

$\nabla g(x,y) = \langle 2x, 4y \rangle$

Marks 3

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 1 = 2\lambda x \\ -4 = 4\lambda y \\ x^2 + y^2 = 9 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \\ \frac{1}{4\lambda^2} + \frac{2}{\lambda^2} = 0 \end{cases}$$

$$\Rightarrow \left. \begin{array}{l} x = \frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \\ \lambda = \pm \frac{1}{2} \end{array} \right\}$$

with $\lambda = \frac{1}{2}$, $(x,y) = (1,-2)$

$f(1,-2) = 9$ Max. value,

with $\lambda = -\frac{1}{2}$, $(x,y) = (-1,2)$

$f(-1,2) = -9$ Min. value.