Second Semester 1442	**First Exam**	**King Saud University**
(without calculators)	**Time: 8 - 9:30 am**	**College of Science**
Wednesday 5-7-1442	**240 Math**	**Math. Department**

Q1: If $A=\begin{bmatrix}2&4\\1&2\end{bmatrix}$, $B^T=\begin{bmatrix}1&2\\0&2\\-1&0\end{bmatrix}$, $C=\begin{bmatrix}1&1&1\\2&3&2\\1&2&2\end{bmatrix}$ and $P(x)=\frac{1}{4}x^2-x+2$, then find the following:

(a) $P(A)=\frac{1}{4}A^2-A+2I$

$$\frac{1}{4}\begin{bmatrix}2&4\\1&2\end{bmatrix}\begin{bmatrix}2&4\\1&2\end{bmatrix}-\begin{bmatrix}2&4\\1&2\end{bmatrix}+2\begin{bmatrix}1&0\\0&1\end{bmatrix}$$
$$=\frac{1}{4}\begin{bmatrix}8&16\\4&8\end{bmatrix}-\begin{bmatrix}2&4\\1&2\end{bmatrix}+\begin{bmatrix}2&0\\0&2\end{bmatrix}$$
$$=\begin{bmatrix}2&4\\1&2\end{bmatrix}-\begin{bmatrix}2&4\\1&2\end{bmatrix}+\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}2&0\\0&2\end{bmatrix}$$

(b) adj(A)=

$$adj\left(\begin{bmatrix}2&4\\1&2\end{bmatrix}\right)=\begin{bmatrix}2&-1\\-4&2\end{bmatrix}^T=\begin{bmatrix}2&-4\\-1&2\end{bmatrix}$$

$$adj(A)=C^T=\begin{bmatrix}C_{11}&C_{12}\\C_{21}&C_{22}\end{bmatrix}^T$$

(c) the inverse of C

$$[C\,|\,I]=\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\2&3&2&0&1&0\\1&2&2&0&0&1\end{array}\right]\xrightarrow[(-1)R_{13}]{(-2)R_{12}}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&0&-2&1&0\\0&1&1&-1&0&1\end{array}\right]$$

$$\xrightarrow[(-1)R_{23}]{(-1)R_{21}}\left[\begin{array}{ccc|ccc}1&0&1&3&-1&0\\0&1&0&-2&1&0\\0&0&1&1&-1&1\end{array}\right]\xrightarrow{(-1)R_{31}}\left[\begin{array}{ccc|ccc}1&0&0&2&0&-1\\0&1&0&-2&1&0\\0&0&1&1&-1&1\end{array}\right]=[I\,|\,C^{-1}]$$

(d) Solution of Bx=0 by Gauss-Jordan Elimination

$$\begin{bmatrix}1 & 0 & -1\\ 2 & 2 & 0\end{bmatrix}\xrightarrow{(-2)R_{12}}\begin{bmatrix}1 & 0 & -1\\ 0 & 2 & 2\end{bmatrix}\xrightarrow{(\frac{1}{2})R_2}\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 1\end{bmatrix}$$

$$x - z = 0 \,\&\, y + z = 0$$

$$z = t, x = t, y = -t, t \in \mathbb{R}$$

(e) $T_B(1,2,3)$

$$T_B(1,2,3) = B\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1 & 0 & -1\\ 2 & 2 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-2\\6\end{bmatrix}$$

Q2: Find the determinant of the following matrix, then find the cofactor C_{12}:

(4 marks)

$$\begin{bmatrix}1 & 2 & 2 & 2\\ 2 & 5 & 4 & 4\\ 3 & 6 & 6 & 7\\ 4 & 8 & 10 & 8\end{bmatrix}$$

$$\begin{vmatrix}1 & 2 & 2 & 2\\ 2 & 5 & 4 & 4\\ 3 & 6 & 6 & 7\\ 4 & 8 & 10 & 8\end{vmatrix} \overset{\substack{(-2)R_{12}\\(-3)R_{13}}}{\underset{(-4)R_{14}}{=}} \begin{vmatrix}1 & 2 & 2 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 2 & 0\end{vmatrix} \overset{R_{34}}{=} - \begin{vmatrix}1 & 2 & 2 & 2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 1\end{vmatrix} = -1(1)(2)(1) = -2$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix}2 & 4 & 4\\ 3 & 6 & 7\\ 4 & 10 & 8\end{vmatrix} = -4\begin{vmatrix}1 & 2 & 2\\ 3 & 6 & 7\\ 2 & 5 & 4\end{vmatrix}$$

$$\overset{(-3)R_{12}}{\underset{(-2)R_{13}}{=}} -4\begin{vmatrix}1 & 2 & 2\\ 0 & 0 & 1\\ 0 & 1 & 0\end{vmatrix} \overset{R_{23}}{=} 4\begin{vmatrix}1 & 2 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = 4(1)(1)(1) = 4$$

Q3: (a) Prove that if A is an invertible matrix, then $\det(A^{-1})=(\det(A))^{-1}$. (2 marks)

$$AA^{-1} = I \Rightarrow \left|AA^{-1}\right| = |I| = 1$$

$$\Rightarrow |A|\left|A^{-1}\right| = 1 \Rightarrow \left|A^{-1}\right| = \frac{1}{|A|}$$

$$as\ |A| \neq 0$$

(b) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

(2 marks)

$$(A^{-1})^T=(A^T)^{-1}=A^{-1}.$$

(c) If $B = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$, then find tr(B). (1 mark)

tr(B)=1+2=3.

(d) If A is a square matrix of order 2 such that det(A)= 3, then find $det(2(A^T)^{-1})$. (2 marks)

$$det\left(2\left(A^T\right)^{-1}\right) = 2^2 det\left(\left(A^T\right)^{-1}\right) = 4det\left(\left(A^{-1}\right)^T\right)$$

$$= 4det\left(A^{-1}\right) = \frac{4}{3}$$

(e) If the solution set of the system Ax=b is {(2r+1,s-1):r,s∈ℝ}, then find the solution set of the system Ax=0. (2 marks)

The system Ax=b is consistent. So the number of free variables is 2. But the system has already 2 variables since the solution set is a subset of $\mathbb{R}^2$. So the system does not have any leading variable. So the (REF) of A is 0 and hence A is 0 also. Thus, b=0 as b=Ax=0x=0. So Ax=b is already a homogeneous linear system and the has the same solution set of A.

(We can also write the solution set by {(r,s):r,s∈ℝ}).