Second Semester	**Final Exam**	**King Saud University**
(two pages)	**Time allowed: 3 hours**	**College of Science**
Sunday 13-9-1442	**240 Math**	**Math. Department**

Q1: Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find:

(i) A^2. (2 marks)

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 5 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

(ii) 2A+I. (2 marks)

$$2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

(iii) A^T. (1 mark)

$$A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(iv) tr(A). (1 mark)

= 1+2+1=4

(v) the inverse of A. (3 marks)

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_{12}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-1)R_{23}]{(-1)R_{21}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] = [I \mid A^{-1}]$$

(vi) $T_A(1,1,1)$. (1 mark)

$$= A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\1 & 2 & 0\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\3\\2\end{bmatrix}$$

(vii) the solution set of Ax=0. (2 marks)

Since A is invertible, So x=0.

Q2: Let V be the subspace of $\mathbb{R}^3$ **spanned** by the set S={v_1=(1, 2,2), v_2=(2, 4,4), v_3=(4, 9, 8)}. Find a **subset** of S that forms a basis for V. (4 marks)

$$\begin{bmatrix}1 & 2 & 4\\2 & 4 & 9\\2 & 4 & 8\end{bmatrix} \xrightarrow[(-2)R_{13}]{(-2)R_{12}} \begin{bmatrix}1 & 2 & 4\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

So {v_1=(1, 2,2), v_3=(4, 9, 8)} is a basis for V.

Q3: Show that $A = \begin{bmatrix}-1 & 1 & -1\\0 & 1 & 0\\0 & 1 & 0\end{bmatrix}$ is diagonalizable and find a matrix P that diagonalizes A. (8 marks)

$$0 = \left|\lambda I - A\right| = \begin{vmatrix}\lambda+1 & -1 & +1\\0 & \lambda-1 & 0\\0 & -1 & \lambda\end{vmatrix} = (\lambda+1)(\lambda-1)\lambda$$

So the eigenvalues are 0, 1, −1. Since they are different, A is diagonalizable.
Now, for the eigenvalue λ=0

$$\begin{bmatrix}1 & -1 & 1\\0 & -1 & 0\\0 & -1 & 0\end{bmatrix} \xrightarrow[(-1)R_{23}]{(-1)R_{21}} \begin{bmatrix}1 & 0 & 1\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix} \xrightarrow{(-1)R_2} \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

So x=−z=−t and y=0. Hence, {(−1,0,1)} is a basis of E_0. Now, for the eigenvalue λ=1

$$\begin{bmatrix}2 & -1 & 1\\0 & 0 & 0\\0 & -1 & 1\end{bmatrix} \xrightarrow{(-1)R_{31}} \begin{bmatrix}2 & 0 & 0\\0 & 0 & 0\\0 & -1 & 1\end{bmatrix} \xrightarrow{(-1)R_3} \begin{bmatrix}2 & 0 & 0\\0 & 0 & 0\\0 & 1 & -1\end{bmatrix}$$

So x=0 and y=z=t. Hence, {(0,1,1)} is a basis of E_1. Now, for the eigenvalue λ=−1

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow[(-1)R_{13}]{(-2)R_{12}} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[(-1)R_{23}]{(-1)R_1} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{2}R_2)} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(1)R_{21}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So x=t and y=z=0. Hence, {(1,0,0)} is a basis of E_{-1}. Therefore,

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Q4: Assume that the vector space $\mathbb{R}^3$ has the Euclidean inner product. Apply the Gram-Schmidt process to transform the following basis vectors (1,0,0), (1,1,0), (1,1,1) into an **orthonormal basis**. (8 marks)

Let $v_1=(1,0,0)$, $v_2=(1,1,0)$, $v_3=(1,1,1)$.
Now define u_1, u_2 and u_3 as follows:

$$u_1 = v_1 = (1,0,0)$$

$$u_2 = v_2 - \frac{<v_2,u_1>}{\|u_1\|^2}u_1 = (1,1,0) - (1,0,0) = (0,1,0)$$

$$u_3 = v_3 - \frac{<v_3,u_2>}{\|u_2\|^2}u_2 - \frac{<v_3,u_1>}{\|u_1\|^2}u_1$$

$$= (1,1,1) - (0,1,0) - (1,0,0) = (0,0,1)$$

$$w_1 = \frac{u_1}{\|u_1\|} = (1,0,0)$$

$$w_2 = \frac{u_2}{\|u_2\|} = (0,1,0)$$

$$w_3 = \frac{u_3}{\|u_3\|} = (0,0,1)$$

So $\{w_1, w_2, w_3\}$ is the wanted orthonormal basis.

Q5: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by:
$T\,(x_1,x_2) = (x_1,x_2,x_1+x_2)$.

(a) Show that T is a linear transformation. (3 marks)

For all real numbers x_1, x_2, y_1, y_2 and k, we have

$$T\left((x_1,x_2)+(y_1,y_2)\right)=T(x_1+y_1,x_2+y_2)$$
$$=\left(x_1+y_1,x_2+y_2,(x_1+y_1)+(x_2+y_2)\right)$$
$$=\left(x_1+y_1,x_2+y_2,(x_1+x_2)+(y_1+y_2)\right)$$
$$=\left(x_1,x_2,x_1+x_2\right)+\left(y_1,y_2,y_1+y_2\right)$$
$$=T\left(x_1,x_2\right)+T\left(y_1,y_2\right)$$
$$Also,$$
$$T\left(k(x_1,x_2)\right)=T\left((kx_1,kx_2)\right)=\left(kx_1,kx_2,kx_1+kx_2\right)$$
$$=\left(kx_1,kx_2,k(x_1+x_2)\right)=k\left(x_1,x_2,x_1+x_2\right)$$
$$=kT\left(x_1,x_2\right)$$

(b) Show that T is one-to-one. (2 marks)

Suppose $T\left(x_1,x_2\right)=0$. So

$$0=T\left(x_1,x_2\right)=(x_1,x_2,x_1+x_2)$$
$$\Rightarrow x_1=x_2=0$$
$$\Rightarrow \left(x_1,x_2\right)=\left(0,0\right)$$
$$\Rightarrow \ker(T)=\{0\}$$

So T is 1-1.

(c) Find $[T]_{S,B}$ where S is the standard basis for $\mathbb{R}^3$ and B={v_1=(1,1),v_2=(1,0)}. (3 marks)

T(1,1)=(1,1,2) and T(1,0)=(1,0,1). Hence

$$[T]_{S,B}=\left[[T(1,1)]_S \mid [T(1,0)]_S\right]=\begin{bmatrix}1 & 1\\ 1 & 0\\ 2 & 1\end{bmatrix}$$