[Solution Key] **KING SAUD UNIVERSITY**

COLLEGE OF SCIENCES

DEPARTMENT OF MATHEMATICS

Mid-term Exam / MATH-244 (Linear Algebra) / Semester 443

Max. Marks: 30 **Max.Time: 2 hrs**

Question 1: [Marks: 3+3+3]**:**

a) Find λ that satisfies the matrix equation $\mathbf{X^8} - 8\lambda\mathbf{I} = \mathbf{O}$ where $\mathbf{X} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution: $\mathbf{X^2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow 2^4 - 8\lambda = 0 \Rightarrow \lambda=2$.

b) Let **A**, **B** and **C** be $\mathbf{3} \times \mathbf{3}$ matrices such that $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $|\mathbf{B}| = 1$ and $|\mathbf{C}| = 2$.

Then evaluate the determinant $|(\mathbf{A}^{-2}\mathbf{C})^{-1}\mathbf{A}^{-1}\mathbf{B} - 2\mathbf{C}^{-1}\mathbf{B}|$.

Solution: $|(\mathbf{A}^{-2}\mathbf{C})^{-1}\mathbf{A}^{-1}\mathbf{B} - 2\mathbf{C}^{-1}\mathbf{B}| = |\mathbf{C}^{-1}\mathbf{AB} - 2\mathbf{C}^{-1}\mathbf{B}| = |\mathbf{C}|^{-1}(|\mathbf{A} - 2\mathbf{I}||\mathbf{B}| = \frac{|\mathbf{B}|}{|\mathbf{C}|}\begin{vmatrix} -1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \frac{1}{2}(5) = \frac{5}{2}$.

c) Find the matrix **B** such that $(\mathbf{2A} - \mathbf{B})^{-1} = adj(\mathbf{A})$ where $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution: $(2\mathbf{A} - \mathbf{B})^{-1} = adj(\mathbf{A}) = |\mathbf{A}|\mathbf{A}^{-1} = \mathbf{A}^{-1} \Rightarrow 2\mathbf{A} - \mathbf{B} = -\mathbf{A} \Rightarrow \mathbf{B} = 3\mathbf{A} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 0 & 3 \\ 0 & 3 & 3 \end{bmatrix}$.

Question 2: [Marks: 3+3+3]**:**

a) Solve the following system of linear equations:

$$\begin{aligned} 2x + y + z &= 1 \\ x + 2y + z &= -1 \\ x + y + 2z &= 0. \end{aligned}$$

Solution: $|\mathbf{A}| = 4$, $|\boldsymbol{A_x}| = 4$, $|\boldsymbol{A_y}| = -4$ and $|\boldsymbol{A_z}| = 0$. Hence, $x = \frac{|A_x|}{|\mathbf{A}|} = 1$. Similarly, $y = -1$ and $z = 0$.

b) Find the value of δ for which the following system is inconsistent:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + \delta x_2 + x_3 &= 2 \\ 3x_1 + 3x_2 + \delta x_3 &= 3. \end{aligned}$$

Solution: $[\boldsymbol{A}|\boldsymbol{B}] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \delta - 1 & 0 & 1 \\ 0 & 0 & \delta - 3 & 0 \end{array}\right] \Rightarrow \delta = 1$ for inconsistency of the given system.

c) Find a condition on α and β sufficient for the following system to be consistent:

$$\begin{aligned} 3w + x + 2y + z &= \alpha \\ 2w + 2x + y + 3z &= \beta \\ 9w - x + 7y - 4z &= 1. \end{aligned}$$

Solution: $[\boldsymbol{A}|\boldsymbol{B}] \sim \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & \alpha \\ 0 & -3 & 1 & -4 & \boldsymbol{\beta} - 2\alpha \\ 0 & 0 & 0 & 0 & 3\boldsymbol{\beta} - 5\alpha + 1 \end{array}\right]$ So, for consistency of the given system $3\boldsymbol{\beta} - 5\alpha + 1=0$.

Question 3: [Marks: 3+4+5]

a) Show that $\{\begin{bmatrix} x & y \\ 0 & 2x-y \end{bmatrix} : x, y \in \mathbb{R}\}$ is a 2-dimensional vector subspace of $\boldsymbol{M}_2(\mathbb{R})$.

Solution: The given set is a vector subspace of $\boldsymbol{M}_2(\mathbb{R})$ because $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2(0)-0 \end{bmatrix}$ and $\boldsymbol{\alpha}\begin{bmatrix} a & b \\ 0 & 2a-b \end{bmatrix} + \begin{bmatrix} c & d \\ 0 & 2c-d \end{bmatrix} = \begin{bmatrix} \alpha a + c & \alpha b + d \\ 0 & 2(\alpha a + c) - (\alpha b + d) \end{bmatrix}$.Further, it has a basis $\{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\}$, and so it's dimension is 2.

b) Let $\boldsymbol{P_2}$ denote the vector space of all real polynomials in x with degree ≤ 2 under usual addition and scalar multiplication**.** Show that $\boldsymbol{B} = \{1 + x + x^2, 1 - x, 1 - x^2\}$ is a basis of $\boldsymbol{P_2}$. Also find the coordinate vector $[x^2 - x]_{\boldsymbol{B}}$.

Solution: Clearly, $\alpha(1 + x + x^2) + \beta(1 - x) + \gamma(1 - x^2) = 0 \Rightarrow \alpha = \beta = \gamma = 0$. So, $\boldsymbol{B}$ is linearly independent. However, $dim(\boldsymbol{P_2}) = 3$ Therefore, $\boldsymbol{B}$ is a basis of $\boldsymbol{P_2}$**.** Further**,** $x^2 - x = 0(1 + x + x^2) + 1(1 - x) - 1(1 - x^2)$ gives $[x^2 - x]_{\boldsymbol{B}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

c) Find a basis of *col*(A), rank and nullity of the matrix $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -2 & 0 & -1 \\ 1 & 2 & -1 & 0 \end{bmatrix}$.

Solution: Since $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -2 & 0 & -1 \\ 1 & 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (REF), {(1,0,-1,1),(-1,1,0,-1)} is a basis of *col*(A) and so *rank*(A)= *dim*(*col*(A))=2. Hence, *nullity*(A) = 4 – *rank*(A) = 2.

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