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Q1: We have two ways to prove that W={(x+1,0):x∈ℝ} is a subspace of $\mathbb{R}^2$:

The first way:

1- W is not empty since (0+1,0)=(1,0) is in W. Also, you can use (0,0)=(-1+1,0) is in W.
2- Suppose u=(x+1,0) and v=(y+1,0), where x,y∈ℝ. Then u+v=(x+1,0)+ (y+1,0)=((x+y+1)+1,0) ∈W.
3- Suppose u=(x+1,0) and k∈ℝ. Then

   ku=k(x+1,0)=(k(x+1),0)=(kx+k,0)=((kx+k-1)+1,0) ∈W.

   (1), (2) and (3) imply that W is a subspace of $\mathbb{R}^2$.

The second way:

Observe that W={(x+1,0):x∈ℝ}={(y,0):y∈ℝ}, so you can use the three steps above easily.

Q2:

$$\begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 3 \neq 0$$

So the set S={(1,0,0),(1,1,0),(3,3,3)} forms a basis for the vector space $\mathbb{R}^3$.

Q3: B={(1,2,1),(2,0,2),(4,4,4)}. So

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow[-1R_{13}]{-2R_{12}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the leading ones, we deduce that {(1,2,1),(2,0,2)} is a basis of span(B) and hence dim(span(B))=2.

Q4:

$$w(x)=\begin{vmatrix}1 & \sin(x) & e^x\\ 0 & \cos(x) & e^x\\ 0 & -\sin(x) & e^x\end{vmatrix}=e^x\cos(x)+e^x\sin(x)$$

$$w(0)=e^0\cos(0)+e^0\sin(0)=1+0=1\neq 0$$

Q5:

$$[S\,|\,B]=\left[\begin{array}{cc|cc}1 & 2 & 2 & 1\\ 1 & 5 & -1 & 4\end{array}\right]\xrightarrow{-1R_{12}}\left[\begin{array}{cc|cc}1 & 2 & 2 & 1\\ 0 & 3 & -3 & 3\end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2}\left[\begin{array}{cc|cc}1 & 2 & 2 & 1\\ 0 & 1 & -1 & 1\end{array}\right]\xrightarrow{-2R_{21}}\left[\begin{array}{cc|cc}1 & 0 & 4 & -1\\ 0 & 1 & -1 & 1\end{array}\right]$$

$$=[I\,|\,P_{B\to S}]$$

$$P_{B\to S}=\begin{bmatrix}4 & -1\\ -1 & 1\end{bmatrix}$$

Q6: (i)

$$A=\begin{bmatrix}1 & 2 & 1 & 2\\ 3 & 6 & 2 & 4\\ 1 & 2 & 3 & 6\end{bmatrix}\xrightarrow[-1R_{13}]{-3R_{12}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & -1 & -2\\ 0 & 0 & 2 & 4\end{bmatrix}$$

$$\xrightarrow{-1R_2}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 1 & 2\\ 0 & 0 & 2 & 4\end{bmatrix}\xrightarrow{-2R_{23}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Using the leading ones, {$[1\ \ 3\ \ 1]^T$, $[1\ \ 2\ \ 3]^T$} is a basis of Col(A).

(ii) rank(A)+nullity(A^T)=m

So nullity(A^T)=m- rank(A)=3-2=1

Q7: Suppose $v \in V$ has two expressions:

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ and $v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$, so

$0 = (c_1-k_1)v_1 + (c_2-k_2)v_2 + \cdots + (c_n-k_n)v_n$

But $S = \{v_1, v_2, \ldots, v_n\}$ is a basis, so it is linearly independent. Thus,

$c_1-k_1 = c_2-k_2 = \ldots = c_n-k_n = 0$ and hence $c_i = k_i$ for all $i \in \{1,2,\ldots,n\}$ and hence v has exactly one expression.

Q8:

$$\left|[T]\right| = \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = 2 \neq 0$$

So T is 1-1. To find T^{-1} we will use the relation $[T^{-1}] = [T]^{-1}$. So

$$[T^{-1}] = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}$$

So

$$T^{-1}(w_1, w_2) = \begin{bmatrix} 2 & -1 \\ -\frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2w_1 - w_2 \\ -\frac{3}{2}w_1 + w_2 \end{bmatrix}$$

$$= (2w_1 - w_2, -\frac{3}{2}w_1 + w_2)$$