| First Semester 1442 | First Exam | King Saud University |
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| (without calculators) | Time: 8 - 9:30 am | College of Science |
| Wednesday 13-2-1442 | 240 Math | Math. Department |

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Q1: If $A=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, $B=\begin{bmatrix}1 & 0 & 2\\ 2 & 2 & 0\end{bmatrix}$, $C=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ 0 & 1 & 1\end{bmatrix}$ and $P(x)=x^2+x-2$, then find the following:

(a) P(A)=$A^2$+A-2I=

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}+\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}+\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

(b) adj($BB^T$)=

$$adj\left(\begin{bmatrix}1 & 0 & 2\\ 2 & 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 2\\ 2 & 0\end{bmatrix}\right)=adj\left(\begin{bmatrix}5 & 2\\ 2 & 8\end{bmatrix}\right)=\begin{bmatrix}8 & -2\\ -2 & 5\end{bmatrix}$$

$$C=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}$$

Such that

$$C_{11}=(-1)^{1+1}\det[8]=(-1)^2(8)=8$$
$$C_{12}=(-1)^{1+2}\det[2]=(-1)^3(2)=-2$$
$$C_{21}=(-1)^{2+1}\det[2]=(-1)^3(2)=-2$$
$$C_{22}=(-1)^{2+2}\det[5]=(-1)^4(5)=5$$

So

$$C = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$$

and then

$$adj\,(BB^T) = C^T = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$$

(c) the inverse of C

$$[C \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-2)R_{12}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-1)R_{23}]{(-1)R_{21}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] \xrightarrow{(-1)R_{31}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right] = [I \mid C^{-1}]$$

$$C^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

(d) Solution of Bx=0 by Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \xrightarrow{(-2)R_{12}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -4 \end{bmatrix} \xrightarrow{(\frac{1}{2})R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}$$

$$x + 2z = 0 \,\&\, y - 2z = 0$$

$$z = t, x = -2t, y = 2t, t \in \mathbb{R}$$

(e) $T_B(1,2,3)$

$$T_B(1,2,3) = B\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

Q2: Find the determinant of the following matrix, then find the cofactor $C_{12}$:

(5 marks)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 5 & 4 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 5 & 4 \end{vmatrix} \overset{\substack{(-1)R_{12} \\ (-1)R_{13}}}{\underset{(-1)R_{14}}{=}} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{vmatrix} \overset{R_{34}}{=} - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1(1)(2)(1) = -2$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 & 4 \\ 1 & 3 & 5 \\ 1 & 5 & 4 \end{vmatrix} \overset{(-1)R_{12}}{\underset{(-1)R_{13}}{=}} - \begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} \overset{R_{23}}{=} \begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)(2)(1) = 2$$

Q3: (a) If $E = \begin{bmatrix} a & b & a \\ e & -2a & e \\ a & a & a \end{bmatrix}$, then find det(E) and tr(E). Since the first and the third columns are the same, det(E)=0. tr(E)=$a$-2$a$+$a$=0.

(b) Prove that if A is an invertible symmetric matrix, then $A^{-1}$ is symmetric.

$(A^{-1})^T = (A^T)^{-1} = A^{-1}$.

(c) If A is an invertible matrix of size $3 \times 3$ and |A|=2, then find $|2(A^T)^{-1}|$.

$|2(A^T)^{-1}| = 2^3 |(A^T)^{-1}| = 8|(A^{-1})^T| = 8|A^{-1}| = 8(0.5) = 4$.