Motion in Two Dimensions

4.1 The Position, Velocity, and Acceleration Vectors.

4.2 Two-Dimensional Motion with Constant Acceleration.

4.3 Projectile Motion.

4.4 Uniform Circular Motion.

4.5 Tangential and Radial Acceleration.

Introduction

Kinematics in two dimensions:

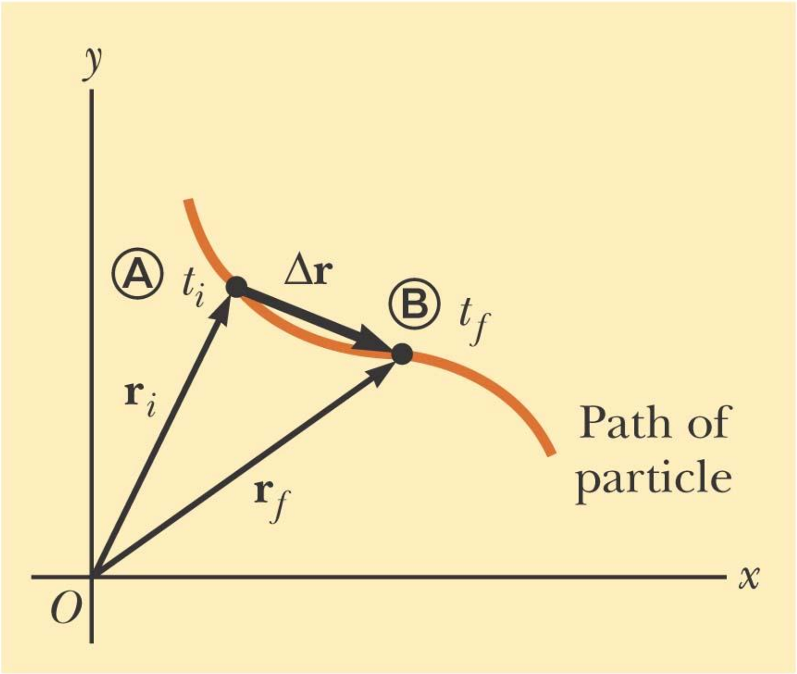
- Describes motion in more than one dimension such as projectile motion and uniform circular motion.
- Vectors can be used to more fully describe motion
- Still interested in displacement, velocity, and acceleration
- Will serve as the basis of multiple types of motion in future chapters

4.1 The Position, Velocity, and Acceleration Vectors

The position of an object is described by its position vector, $\mathbf{r}$

The displacement of the object is defined as the ***change in its position:***

$$\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$$



General Motion Ideas

In two- or three-dimensional kinematics, everything is the same as as in one-dimensional motion except that we must now use full vector notation.

- Positive and negative signs are no longer sufficient to determine the direction

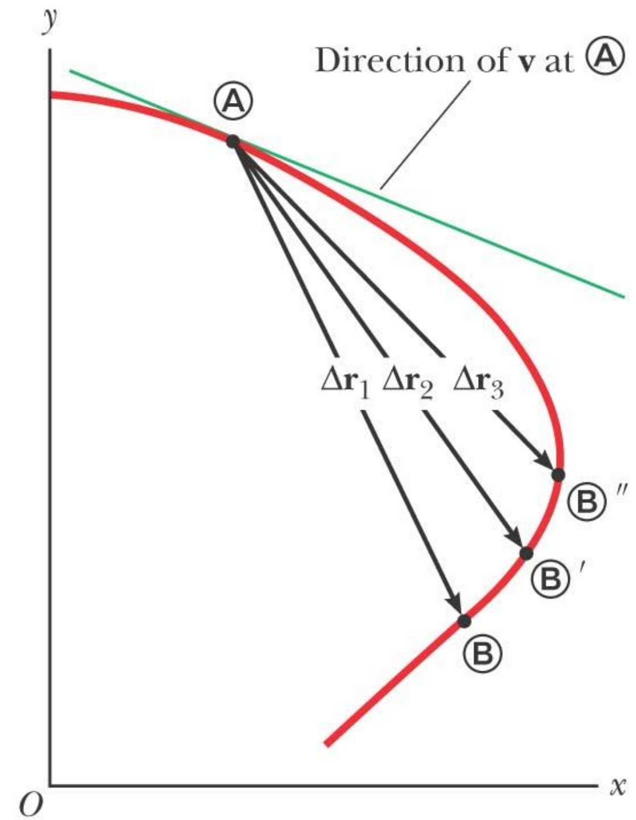
Average Velocity

The average velocity is is the ratio of the displacement $\Delta\vec{\boldsymbol{r}}$ to the time interval Δt for the displacement

$$\vec{\mathbf{v}}_{\text{avg}} = \frac{\Delta\vec{\boldsymbol{r}}}{\Delta t}$$

The direction of $\vec{\mathbf{v}}_{\text{avg}}$ is the same as the direction of the displacement vector $\Delta\vec{\boldsymbol{r}}$.

The average velocity between points is *independent of the path* taken.



Instantaneous Velocity

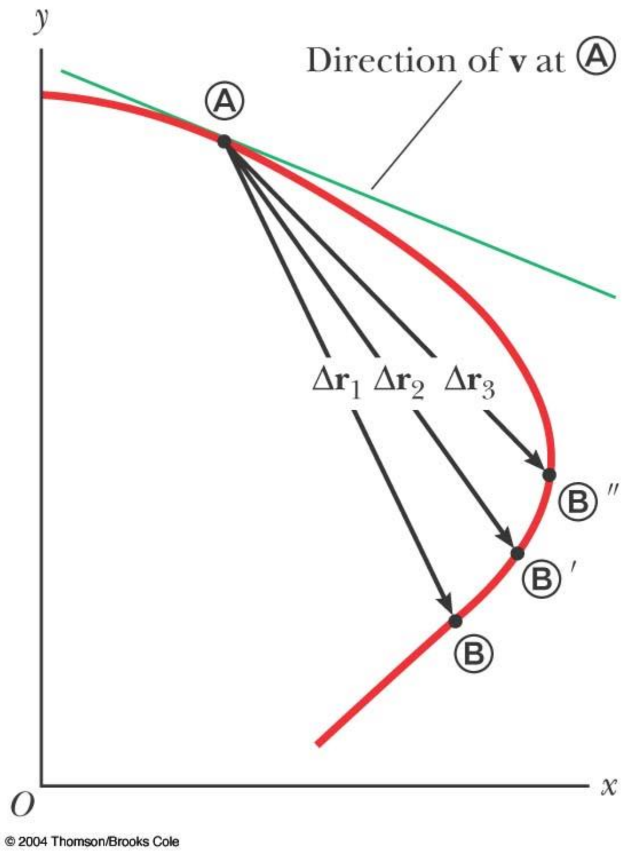
The instantaneous velocity is the limit of the average velocity as the time interval becomes infinitesimally short, or as the time interval approaches zero

$$\mathbf{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

- The instantaneous velocity can be positive, negative, or zero.

Instantaneous Velocity, graph

- The direction of the instantaneous velocity vector at any point in a particle's path is along a line tangent to the path at that point and in the direction of motion
- The magnitude of the instantaneous velocity vector is the speed
  - The speed is a scalar quantity

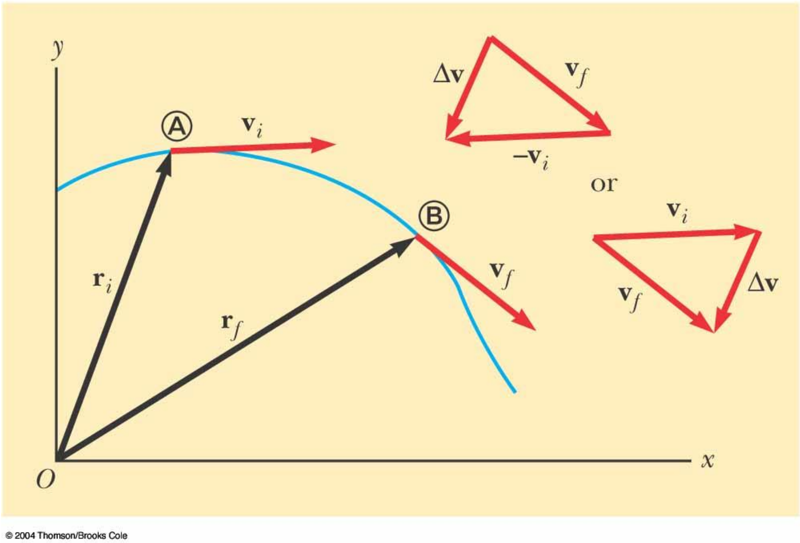


Average Acceleration

- The **average acceleration** $\mathbf{a}_{avg}$ of the particle is defined as the *change* in velocity $\Delta \boldsymbol{v}$ divided by the time interval Δt during which that change occurs:

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_{\mathrm{f}} - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta \mathbf{v}}{\Delta t}$$

- As a particle moves, $\Delta\mathbf{v}$ can be found in different ways
- The $\mathbf{a}_{avg}$ is a vector quantity directed along $\Delta\mathbf{v}$



Instantaneous Acceleration

is defined as the limit of the average acceleration as Δt approaches zero.

$$\mathbf{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

The instantaneous **a** equals the derivative of the velocity vector with respect to time.

Producing An Acceleration

- Various changes in a particle's motion may produce an acceleration
- The magnitude of the velocity vector may change
- The direction of the velocity vector may change
    - Even if the magnitude remains constant
- Both may change simultaneously

4.2 Two-Dimensional Motion with Constant Acceleration

- When the two-dimensional motion has a constant acceleration, a series of equations can be developed that describe the motion
- These equations will be similar to those of one-dimensional kinematics
- Motion in two dimensions can be modeled as two independent motions in each of the two perpendicular directions associated with the *x* and *y* axes.

(2.9)

Kinematic Equations for Two-Dimensional Motion

- Position vector $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$
- Velocity $\vec{\mathbf{v}} = \dfrac{d\vec{\mathbf{r}}}{dt} = \dfrac{dx}{dt}\hat{\mathbf{i}} + \dfrac{dy}{dt}\hat{\mathbf{j}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$
- Velocity vector as a function of time:

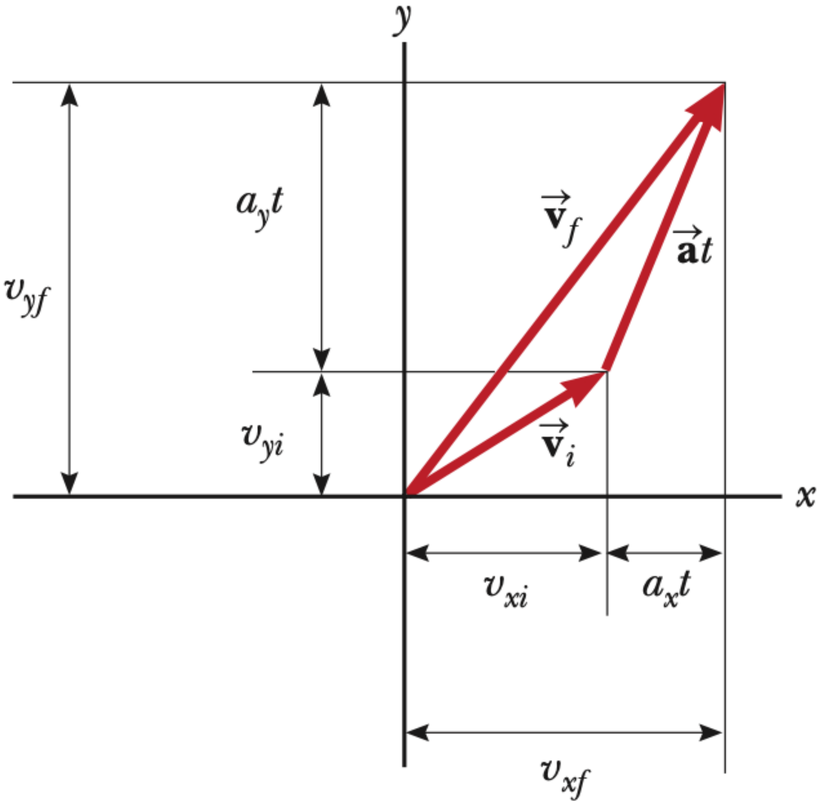
$$\begin{aligned}\mathbf{v}_f &= (v_{xi} + a_x t)\hat{\mathbf{i}} + (v_{yi} + a_y t)\hat{\mathbf{j}} \\ &= (v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}}) + (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}})t\end{aligned}$$

$$\boxed{\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t}$$

Graphical Representation of Final Velocity

- The velocity vector can be represented by its components

v$_f$ is generally not along the direction of either **v**$_i$ or **a**t



Position vector as a function of time

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2 \qquad y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

$$\vec{\mathbf{r}}_f = \left(x_i + v_{xi}t + \tfrac{1}{2}a_x t^2\right)\hat{\mathbf{i}} + \left(y_i + v_{yi}t + \tfrac{1}{2}a_y t^2\right)\hat{\mathbf{j}}$$

$$= \left(x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}}\right) + \left(v_{xi}\hat{\mathbf{i}} + v_{yi}\hat{\mathbf{j}}\right)t + \tfrac{1}{2}\left(a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}}\right)t^2$$

$$\boxed{\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \tfrac{1}{2}\vec{\mathbf{a}}t^2}$$

This indicates that the position vector is the sum of three other vectors:

-The initial position vector

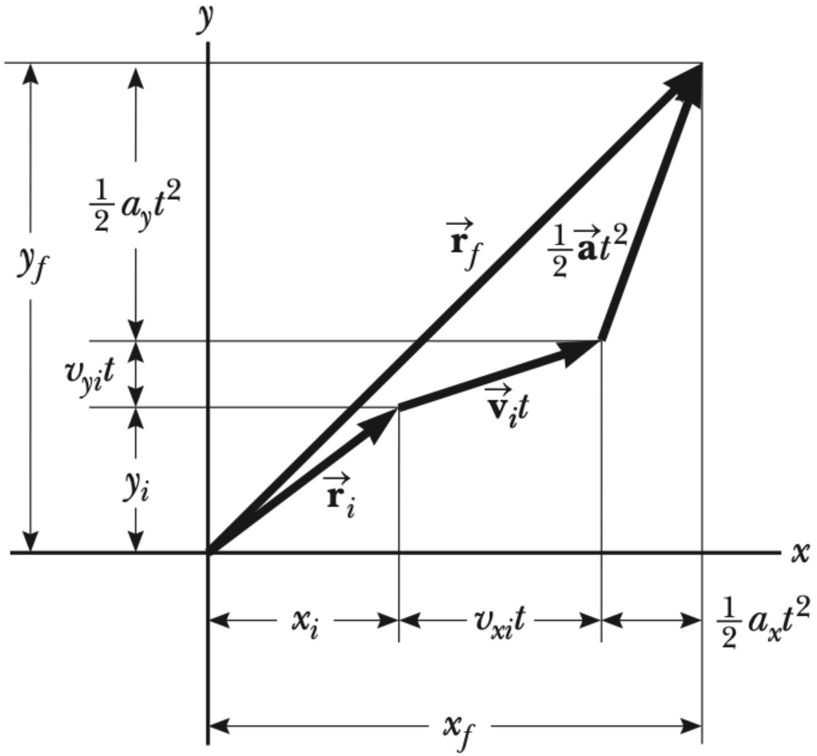
-The displacement resulting from $\mathbf{v}_i\, t$

-The displacement resulting from ½ $\mathbf{a}t^2$

The vector representation of the position vector

$\mathbf{r}_f$ is generally not in the same direction as $\mathbf{v}_i$ or as $\mathbf{a}_i$

$\mathbf{r}_f$ and $\mathbf{v}_f$ are generally not in the same direction



Kinematic Equations, Component Equations

The component form of the equations for $\mathbf{v}_f$ and $\mathbf{r}_f$ in two-dimensional motion at a constant acceleration is equivalent to two independent motions having constant accelerations a_x and a_y.

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t \quad \begin{cases} v_{xf} = v_{xi} + a_x t \\ v_{yf} = v_{yi} + a_y t \end{cases}$$

$$\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \frac{1}{2}\mathbf{a}t^2 \quad \begin{cases} x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2 \\ y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2 \end{cases}$$

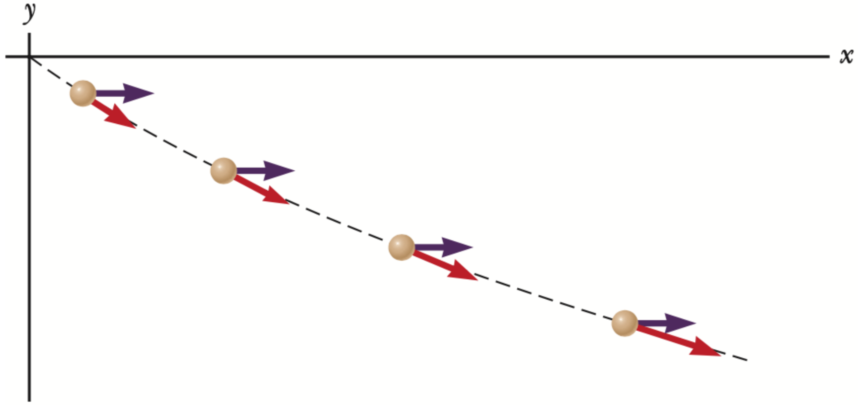
Example 4.1 Motion in a Plane

A particle moves in the *xy* plane, starting from the origin at $t = 0$ with an initial velocity having an *x* component of 20 m/s and a *y* component of -15 m/s. The particle experiences an acceleration in the *x* direction, given by $a_x = 4.0$ m/s^2.

(A) Determine the total velocity vector at any time.

(B) Calculate the velocity and speed of the particle at $t = 5.0$ s and the angle the velocity vector makes with the *x* axis.

(C) Determine the *x* and *y* coordinates of the particle at any time *t* and its position vector at this time.



Example 4.1 Motion in a Plane

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(A) Determine the total velocity vector at any time.

SOLUTION

Conceptualize The components of the initial velocity tell us that the particle starts by moving toward the right and downward. The x component of velocity starts at 20 m/s and increases by 4.0 m/s every second. The y component of velocity never changes from its initial value of −15 m/s. We sketch a motion diagram of the situation in Figure 4.6. Because the particle is accelerating in the $+x$ direction, its velocity component in this direction increases and the path curves as shown in the diagram. Notice that the spacing between successive images increases as time goes on because the speed is increasing. The placement of the acceleration and velocity vectors in Figure 4.6 helps us further conceptualize the situation.

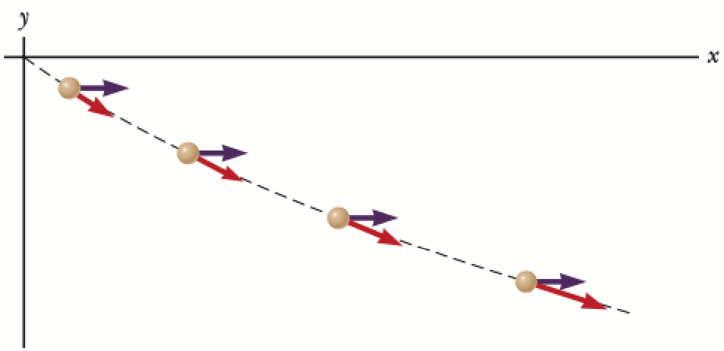


Figure 4.6 (Example 4.1) Motion diagram for the particle.

Categorize Because the initial velocity has components in both the x and y directions, we categorize this problem as one involving a particle moving in two dimensions. Because the particle only has an x component of acceleration, we model it as a *particle under constant acceleration* in the x direction and a *particle under constant velocity* in the y direction.

Analyze To begin the mathematical analysis, we set v_{xi} = 20 m/s, v_{yi} = −15 m/s, a_x = 4.0 m/s^2, and $a_y = 0$.

Use Equation 4.8 for the velocity vector:

$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t = (v_{xi} + a_x t)\hat{\mathbf{i}} + (v_{yi} + a_y t)\hat{\mathbf{j}}$$

Substitute numerical values with the velocity in meters per second and the time in seconds:

$$\vec{\mathbf{v}}_f = [20 + (4.0)t]\hat{\mathbf{i}} + [-15 + (0)t]\hat{\mathbf{j}}$$

$$(1) \quad \vec{\mathbf{v}}_f = \boxed{[(20 + 4.0t)\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]}$$

Finalize Notice that the x component of velocity increases in time while the y component remains constant; this result is consistent with our prediction.

(B) Calculate the velocity and speed of the particle at $t = 5.0$ s and the angle the velocity vector makes with the x axis.

SOLUTION

Analyze

Evaluate the result from Equation (1) at $t = 5.0$ s:

$$\vec{\mathbf{v}}_f = [(20 + 4.0(5.0))\hat{\mathbf{i}} - 15\hat{\mathbf{j}}] = \boxed{(40\hat{\mathbf{i}} - 15\hat{\mathbf{j}})\text{ m/s}}$$

Determine the angle θ that $\vec{\mathbf{v}}_f$ makes with the x axis at $t = 5.0$ s:

$$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(\frac{-15\text{ m/s}}{40\text{ m/s}}\right) = \boxed{-21°}$$

Evaluate the speed of the particle as the magnitude of $\vec{\mathbf{v}}_f$:

$$v_f = |\vec{\mathbf{v}}_f| = \sqrt{{v_{xf}}^2 + {v_{yf}}^2} = \sqrt{(40)^2 + (-15)^2}\text{ m/s} = \boxed{43\text{ m/s}}$$

Finalize The negative sign for the angle θ indicates that the velocity vector is directed at an angle of 21° below the positive x axis. Notice that if we calculate v_i from the x and y components of $\vec{\mathbf{v}}_i$, we find that $v_f > v_i$. Is that consistent with our prediction?

(C) Determine the x and y coordinates of the particle at any time t and its position vector at this time.

SOLUTION

Analyze

Use the components of Equation 4.9 with $x_i = y_i = 0$ at $t = 0$ and with x and y in meters and t in seconds:

$$x_f = v_{xi}t + \tfrac{1}{2}a_x t^2 = \boxed{20t + 2.0t^2}$$

$$y_f = v_{yi}t = \boxed{-15t}$$

Express the position vector of the particle at any time t:

$$\vec{\mathbf{r}}_f = x_f\hat{\mathbf{i}} + y_f\hat{\mathbf{j}} = \boxed{(20t + 2.0t^2)\hat{\mathbf{i}} - 15t\hat{\mathbf{j}}}$$

Finalize Let us now consider a limiting case for very large values of t.

4.3 Projectile Motion

- An object may move in both the x and y directions simultaneously
- The form of two-dimensional motion we will deal with is called **projectile motion**

Assumptions of Projectile Motion

- The free-fall acceleration ***g*** is constant over the range of motion and is directed downward
- The effect of air friction is negligible

With these assumptions, an object in projectile motion will follow a parabolic path

  - This path is called the ***trajectory***

Verifying the Parabolic Trajectory

- Reference frame: the *y* direction is vertical and positive is upward.
- Acceleration components

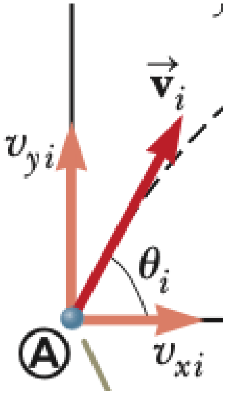
$$a_y = -\,g \text{ (as in one-dimensional free fall)}$$

$$a_x = 0$$

- Initial velocity components

$v_{xi} = v_i \cos\,\theta$

$v_{yi} = v_i \sin\,\theta$



- Displacements

$$x_f = v_{xi}t = (v_i \cos \theta_i)\, t$$

$$y_f = v_{yi}t + \frac{1}{2}a_y t^2 = (v_i \sin \theta_i)\, t - \frac{1}{2}g t^2$$

- Combining the equations gives:

$$y = (\tan \theta_i)\, x - \left(\frac{g}{2{v_i}^2 \cos^2 \theta_i}\right) x^2$$

  - This is in the form of $y = ax - bx^2$ which is the standard form of a parabola

Consider the motion as the superposition of the motions in the *x*- and *y*-directions

- The *x*-direction has **constant velocity**

  $a_x = 0$

- The y-direction is **free fall**

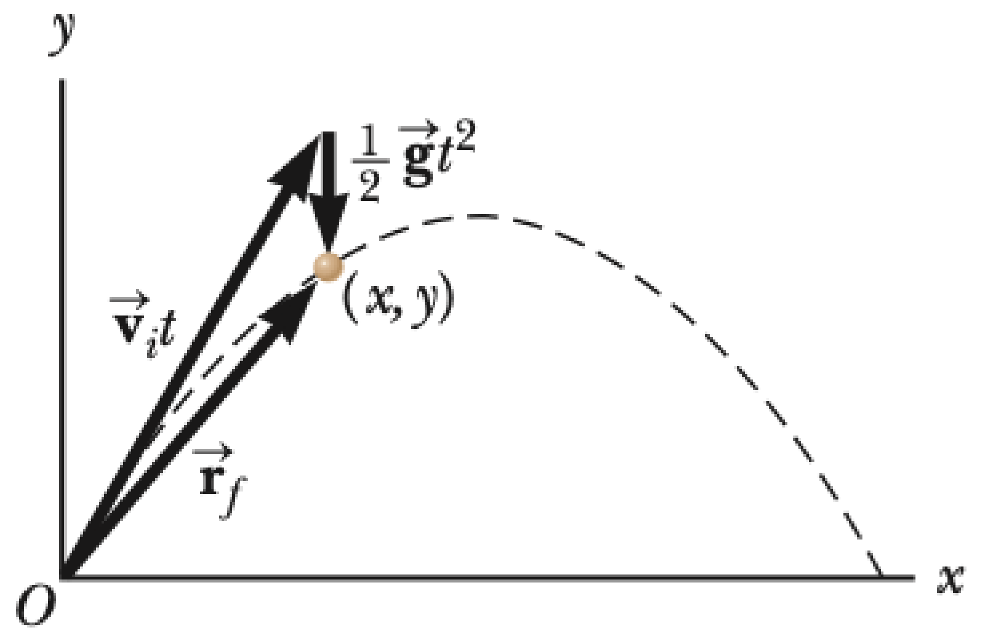
  $a_y = -g$

- The vector expression for the position vector of the projectile as a function of time

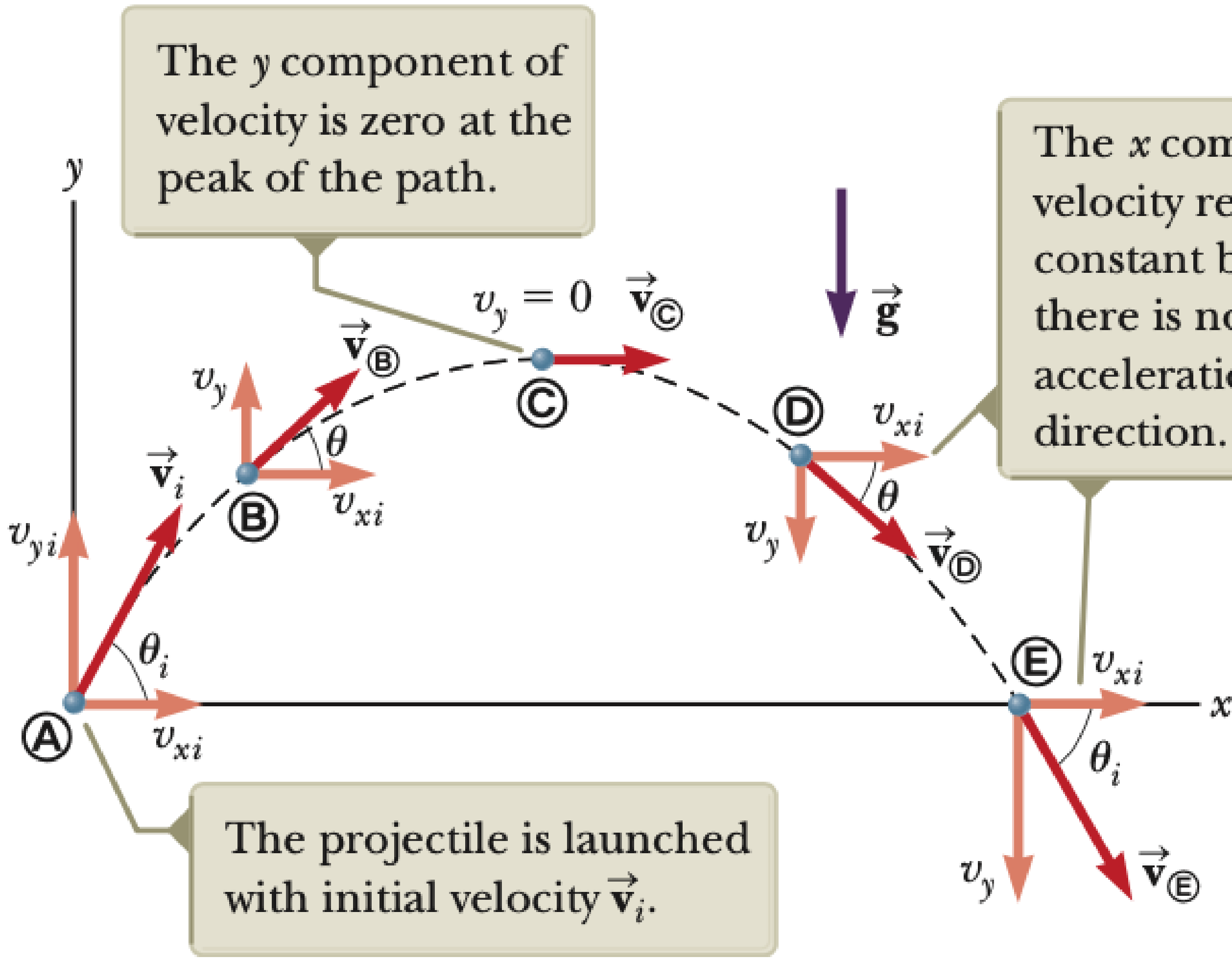
$$\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \tfrac{1}{2}\vec{\mathbf{g}} t^2$$

The final position is the vector sum of the initial position, the position resulting from the initial velocity and the position resulting from the acceleration

$$\vec{\mathbf{r}}_f = \vec{\mathbf{r}}_i + \vec{\mathbf{v}}_i t + \tfrac{1}{2}\vec{\mathbf{g}} t^2$$



Projectile Motion Diagram



Therefore, when solving projectile motion problems, use two analysis models:

1) the particle under constant velocity in the horizontal direction

$$x_f = x_i + v_{xi}t$$

2) the particle under constant acceleration in the vertical direction

$$v_{yf} = v_{yi} - gt$$

$$y_f = y_i + \tfrac{1}{2}(v_{yi} + v_{yf})t$$

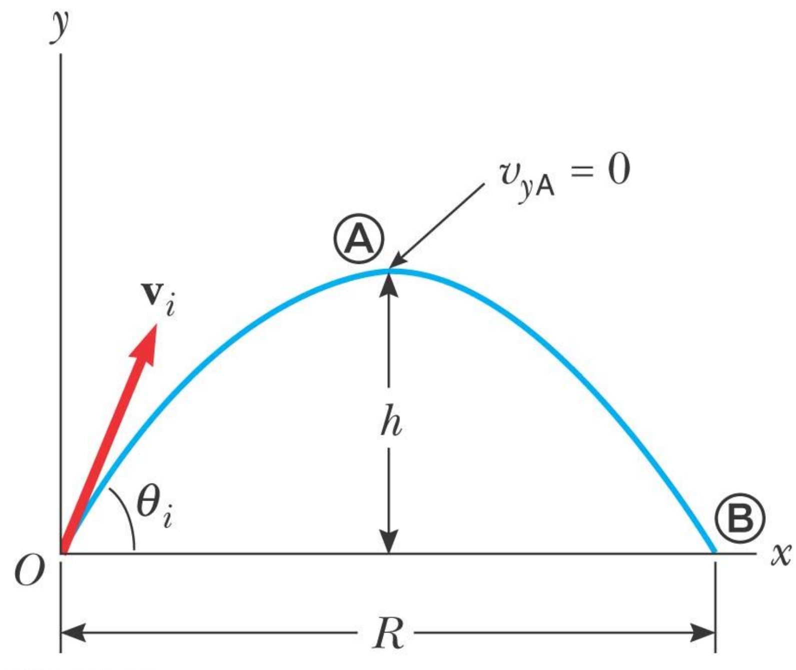
$$y_f = y_i + v_{yi}t - \tfrac{1}{2}gt^2$$

$$v_{yf}^2 = v_{yi}^2 - 2g(y_f - y_i)$$

Horizontal Range and Maximum Height of a Projectile

- When analyzing projectile motion, two characteristics are of special interest
- The range *R*, is the horizontal distance of the projectile
- The maximum height the

projectile reaches is *h*



Height of a Projectile, equation

We can determine h by noting that at the peak, $v_{yA} = 0$. Therefore, we can use Equation 4.8a to determine the time t_A at which the projectile reaches the peak:

$$v_{yf} = v_{yi} + a_y t$$

$$0 = v_i \sin\theta_i - g t_A$$

$$t_A = \frac{v_i \sin\theta_i}{g}$$

Substituting this expression for t_A into the y part of Equation 4.9a and replacing $y = y_A$ with h, we obtain an expression for h in terms of the magnitude and direction of the initial velocity vector:

$$h = (v_i \sin\theta_i)\frac{v_i \sin\theta_i}{g} - \frac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$h = \frac{{v_i}^2 \sin^2\theta_i}{2g} \qquad (4.13)$$

Height of a Projectile, equation

- The maximum height of the projectile can be found in terms of the initial velocity vector:

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$

- This equation is valid only for symmetric motion

Range of a Projectile, equation

The range R is the horizontal position of the projectile at a time that is twice the time at which it reaches its peak, that is, at time $t_B = 2t_A$. Using the x part of Equation 4.9a, noting that $v_{xi} = v_{xB} = v_i \cos\theta_i$ and setting $x_B = R$ at $t = 2t_A$, we find that

$$R = v_{xi}t_B = (v_i \cos\theta_i)2t_A$$

$$= (v_i \cos\theta_i)\frac{2v_i \sin\theta_i}{g} = \frac{2v_i^2 \sin\theta_i \cos\theta_i}{g}$$

Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$ (see Appendix B.4), we write R in the more compact form

$$R = \frac{v_i^2 \sin 2\theta_i}{g} \qquad (4.14)$$

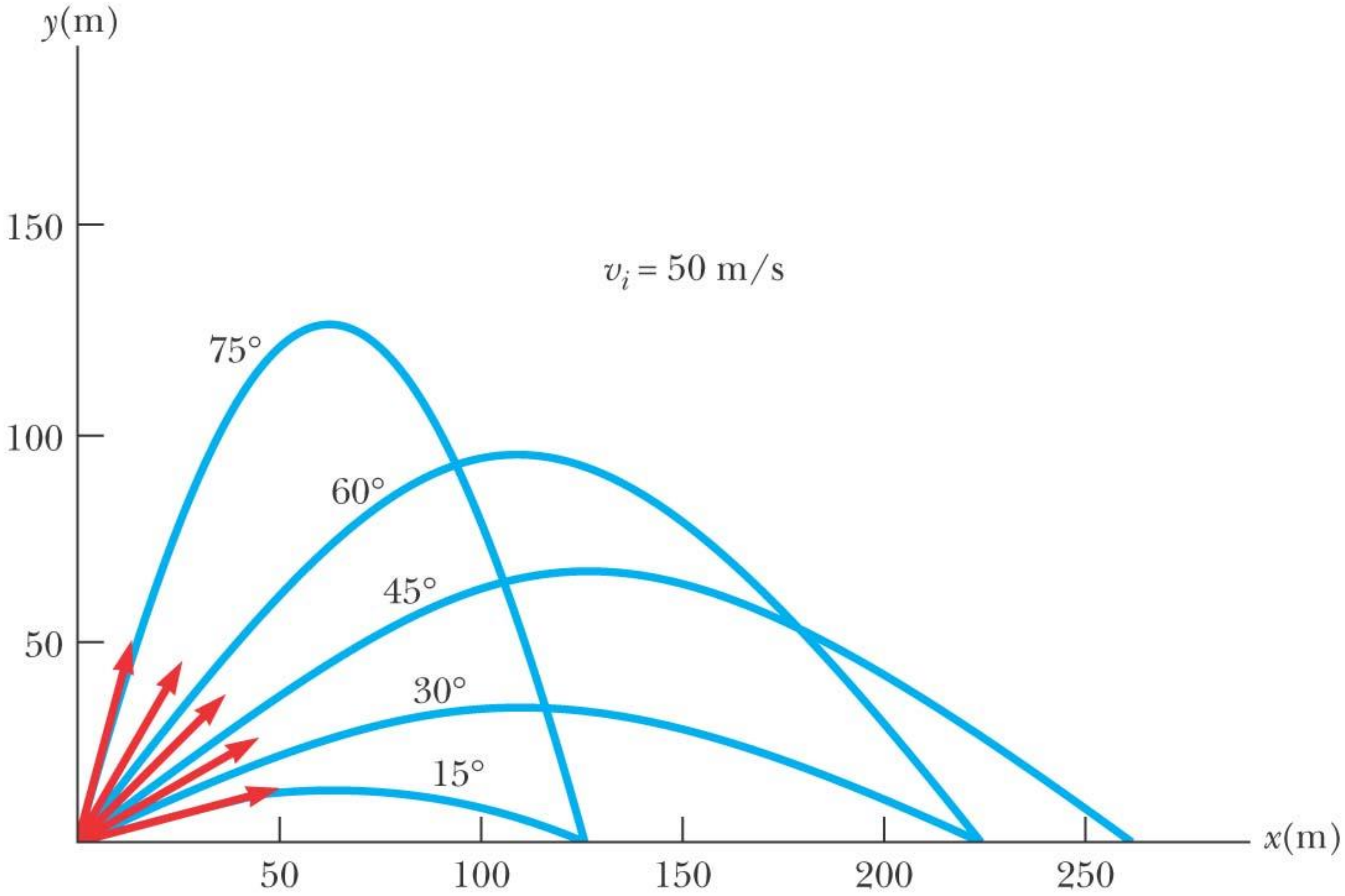
Range of a Projectile, equation

- The range of a projectile can be expressed in terms of the initial velocity vector:

$$R = \frac{{v_i}^2 \sin 2\theta_i}{g}$$

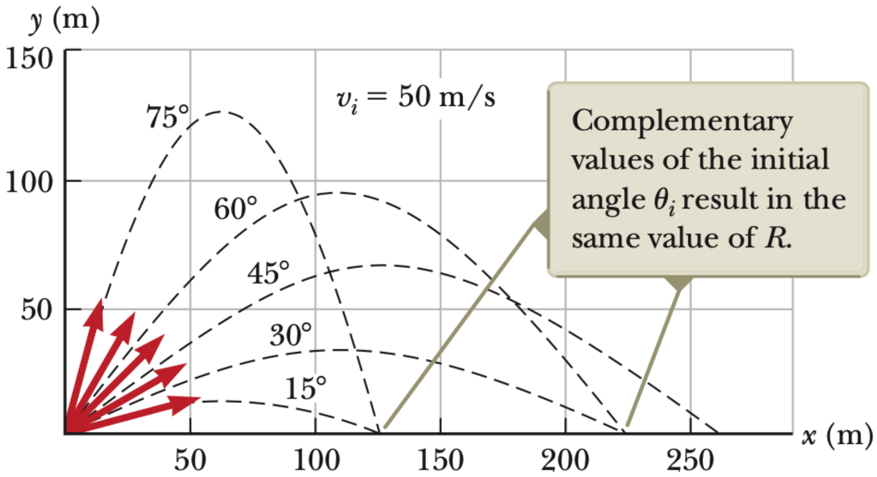
- This is valid only for symmetric trajectory

More About the Range of a Projectile



Range of a Projectile, final

- The maximum range R_{max} occurs at $\theta_i = 45^o$
- Complementary angles will produce the same range
  - The maximum height will be different for the two angles
  - The times of the flight will be different for the two angles



Example 4.3 Long Jump

A long-jumper leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s.

(A) How far does he jump in the horizontal direction? (Assume his motion is equivalent to that of a particle.)

Sipa via AP Images

(A) How far does he jump in the horizontal direction?

The range of the jumper is given by:

$$R = \frac{v_i^2 \sin 2\theta_i}{g} = \frac{(11.0 \text{ m/s})^2 \sin 2(20.0°)}{9.80 \text{ m/s}^2} = \boxed{7.94 \text{ m}}$$

The horizontal motion is described by

$$x_f = x_B = (v_i \cos\theta_i)\,t_B = (11.0\ \text{m/s})(\cos 20.0°)\,t_B$$

The value of x_B can be found if the time of landing t_B is known. We can find t_B by remembering that $a_y = -g$ and by using the y part of Equation 4.8a. We also note that at the top of the jump the vertical component of velocity v_{yA} is zero:

$$v_{yf} = v_{yA} = v_i \sin\theta_i - g t_A$$

$$0 = (11.0\ \text{m/s}) \sin 20.0° - (9.80\ \text{m/s}^2)\,t_A$$

$$t_A = 0.384\ \text{s}$$

This is the time at which the long-jumper is at the *top* of the jump. Because of the symmetry of the vertical motion, another 0.384 s passes before the jumper returns to the ground. Therefore, the time at which the jumper lands is $t_B = 2t_A = 0.768$ s. Substituting this value into the above expression for x_f gives

$$x_f = x_B = (11.0 \text{ m/s})(\cos 20.0°)(0.768 \text{ s}) = \boxed{7.94 \text{ m}}$$

This is a reasonable distance for a world-class athlete.

(B) What is the maximum height reached?

$$h = \frac{v_i^{\,2}\sin^2\theta_i}{2g} = \frac{(11.0\ \mathrm{m/s})^2(\sin 20.0°)^2}{2(9.80\ \mathrm{m/s^2})} = \boxed{0.722\ \mathrm{m}}$$

(B) What is the maximum height reached?

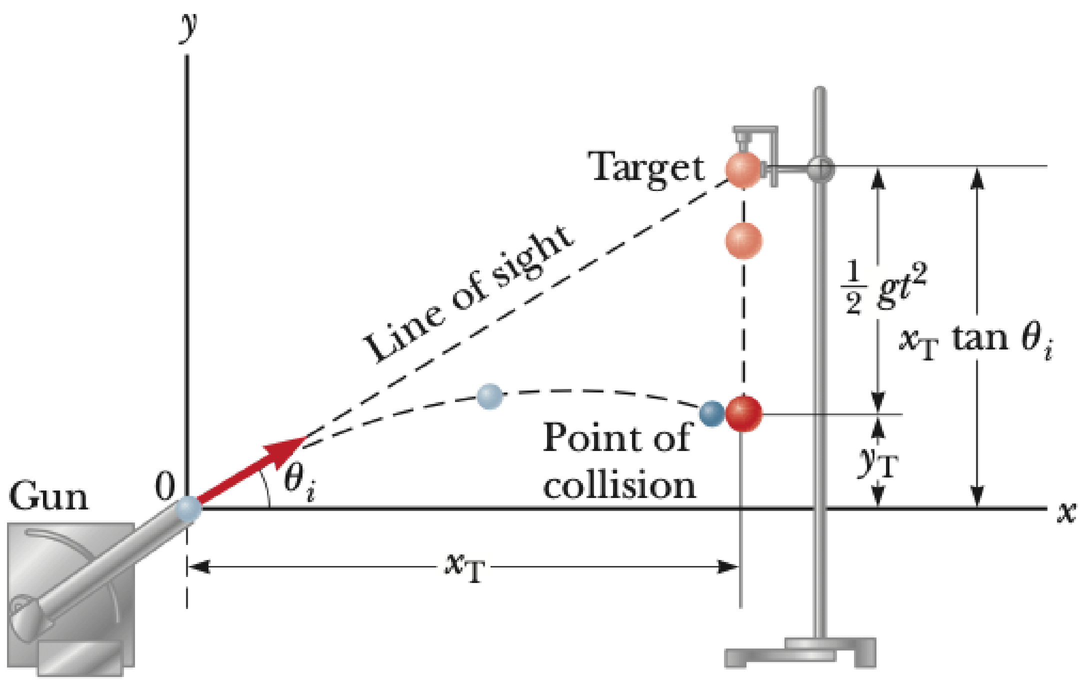
$$y_{\max} = y_A = (v_i \sin \theta_i)\, t_A - \tfrac{1}{2} g {t_A}^2$$

$$= (11.0 \text{ m/s})(\sin 20.0°)(0.384 \text{ s})$$

$$-\tfrac{1}{2}(9.80 \text{ m/s}^2)(0.384 \text{ s})^2 = \boxed{0.722 \text{ m}}$$

Example 4.4 A Bull's-Eye Every Time

In a popular lecture demonstration, a projectile is fired at a target in such a way that the projectile leaves the gun at the same time the target is dropped from rest. Show that if the gun is initially aimed at the stationary target, the projectile hits the falling target as shown in the figure.

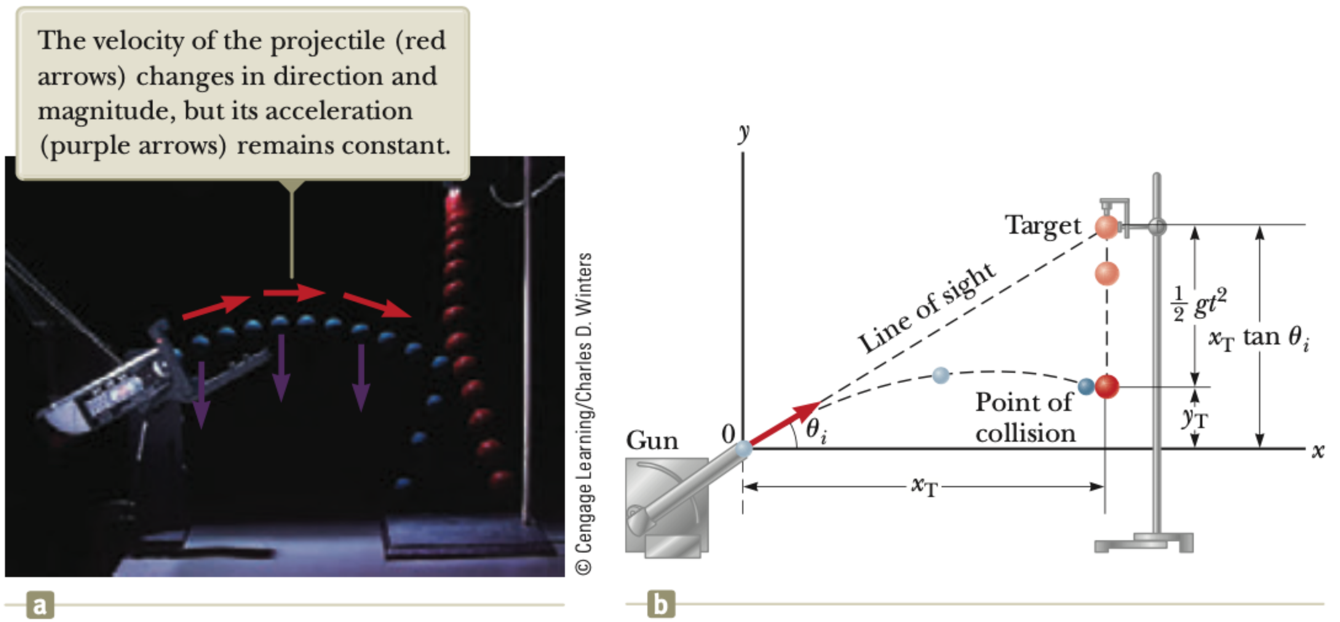


Example 4.4 A Bull's-Eye Every Time

In a popular lecture demonstration, a projectile is fired at a target in such a way that the projectile leaves the gun at the same time the target is dropped from rest. Show that if the gun is initially aimed at the stationary target, the projectile hits the falling target as shown in Figure 4.12a.

SOLUTION

Conceptualize We conceptualize the problem by studying Figure 4.12a. Notice that the problem does not ask for numerical values. The expected result must involve an algebraic argument.



Categorize Because both objects are subject only to gravity, we categorize this problem as one involving two objects in free fall, the target moving in one dimension and the projectile moving in two. The target T is modeled as a *particle under constant acceleration* in one dimension. The projectile P is modeled as a *particle under constant acceleration* in the y direction and a *particle under constant velocity* in the x direction.

Analyze Figure 4.12b shows that the initial y coordinate y_{iT} of the target is $x_T \tan\theta_i$ and its initial velocity is zero. It falls with acceleration $a_y = -g$.

Write an expression for the y coordinate of the target at any moment after release, noting that its initial velocity is zero:

$$(1)\quad y_T = y_{iT} + (0)t - \tfrac{1}{2}gt^2 = x_T \tan\theta_i - \tfrac{1}{2}gt^2$$

Write an expression for the y coordinate of the projectile at any moment:

$$(2)\quad y_P = y_{iP} + v_{yiP}t - \tfrac{1}{2}gt^2 = 0 + (v_{iP}\sin\theta_i)t - \tfrac{1}{2}gt^2 = (v_{iP}\sin\theta_i)t - \tfrac{1}{2}gt^2$$

Write an expression for the x coordinate of the projectile at any moment:

$$x_P = x_{iP} + v_{xiP}t = 0 + (v_{iP}\cos\theta_i)t = (v_{iP}\cos\theta_i)t$$

Solve this expression for time as a function of the horizontal position of the projectile:

$$t = \frac{x_P}{v_{iP}\cos\theta_i}$$

Substitute this expression into Equation (2):

$$(3)\quad y_P = (v_{iP}\sin\theta_i)\left(\frac{x_P}{v_{iP}\cos\theta_i}\right) - \tfrac{1}{2}gt^2 = x_P\tan\theta_i - \tfrac{1}{2}gt^2$$

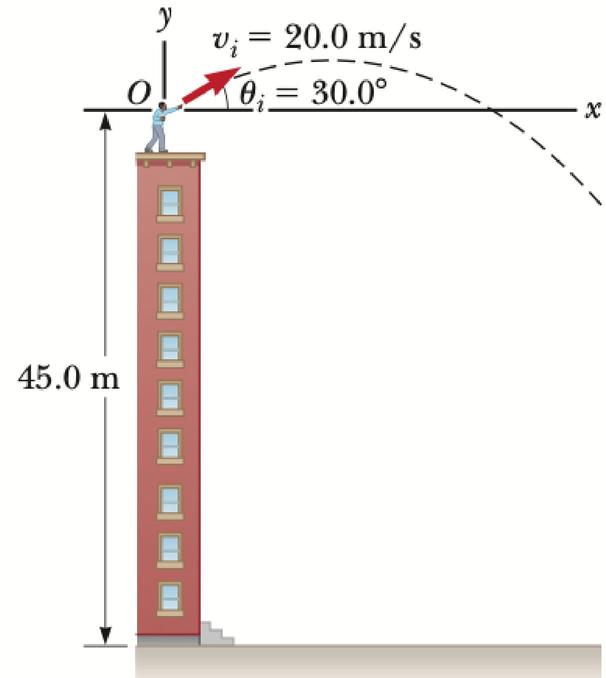
Finalize Compare Equations (1) and (3). We see that when the x coordinates of the projectile and target are the same—that is, when $x_T = x_P$—their y coordinates given by Equations (1) and (3) are the same and a collision results.

Example 4.5 That's Quite an Arm!

A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s, as shown in the figure. If the height of the building is 45.0 m,

(A) how long does it take the stone to reach the ground?

(B) What is the speed of the stone just before it strikes the ground?



Example 4.5 That's Quite an Arm!

A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s as shown in Figure 4.13. The height from which the stone is thrown is 45.0 m above the ground.

(A) How long does it take the stone to reach the ground?

SOLUTION

Conceptualize Study Figure 4.13, in which we have indicated the trajectory and various parameters of the motion of the stone.

Categorize We categorize this problem as a projectile motion problem. The stone is modeled as a *particle under constant acceleration* in the y direction and a *particle under constant velocity* in the x direction.

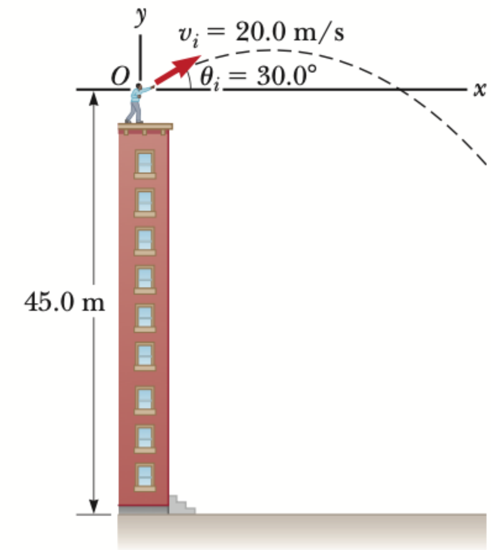


Figure 4.13 (Example 4.4) A stone is thrown from the top of a building.

Analyze We have the information $x_i = y_i = 0$, $y_f = -45.0$ m, $a_y = -g$, and $v_i = 20.0$ m/s (the numerical value of y_f is negative because we have chosen the point of the throw as the origin).

Find the initial x and y components of the stone's velocity:

$$v_{xi} = v_i \cos\theta_i = (20.0 \text{ m/s}) \cos 30.0° = 17.3 \text{ m/s}$$

$$v_{yi} = v_i \sin\theta_i = (20.0 \text{ m/s}) \sin 30.0° = 10.0 \text{ m/s}$$

Express the vertical position of the stone from the particle under constant acceleration model:

$$y_f = y_i + v_{yi}t - \tfrac{1}{2}gt^2$$

Substitute numerical values:

$$-45.0 \text{ m} = 0 + (10.0 \text{ m/s})t + \tfrac{1}{2}(-9.80 \text{ m/s}^2)t^2$$

Solve the quadratic equation for t:

$$t = \boxed{4.22 \text{ s}}$$

(B) What is the speed of the stone just before it strikes the ground?

SOLUTION

Analyze Use the velocity equation in the particle under constant acceleration model to obtain the y component of the velocity of the stone just before it strikes the ground:

$$v_{yf} = v_{yi} - gt$$

Substitute numerical values, using $t = 4.22$ s:

$$v_{yf} = 10.0 \text{ m/s} + (-9.80 \text{ m/s}^2)(4.22 \text{ s}) = -31.3 \text{ m/s}$$

Use this component with the horizontal component $v_{xf} = v_{xi} = 17.3$ m/s to find the speed of the stone at $t = 4.22$ s:

$$v_f = \sqrt{v_{xf}^{\ 2} + v_{yf}^{\ 2}} = \sqrt{(17.3 \text{ m/s})^2 + (-31.3 \text{ m/s})^2} = \boxed{35.8 \text{ m/s}}$$

Finalize Is it reasonable that the y component of the final velocity is negative? Is it reasonable that the final speed is larger than the initial speed of 20.0 m/s?

WHAT IF? What if a horizontal wind is blowing in the same direction as the stone is thrown and it causes the stone to have a horizontal acceleration component $a_x = 0.500$ m/s^2? Which part of this example, (A) or (B), will have a different answer?

Answer Recall that the motions in the x and y directions are independent. Therefore, the horizontal wind cannot affect the vertical motion. The vertical motion determines the time of the projectile in the air, so the answer to part (A) does not change. The wind causes the horizontal velocity component to increase with time, so the final speed will be larger in part (B). Taking $a_x = 0.500$ m/s^2, we find $v_{xf} = 19.4$ m/s and $v_f = 36.9$ m/s.

Example 4.6 The Stranded Explorers

A plane drops a package of supplies to a party of explorers, as shown in Figure 4.15. If the plane is traveling horizontally at 40.0 m/s and is 100 m above the ground, where does the package strike the ground relative to the point at which it is released?

Solution *Conceptualize* what is happening with the assistance of Figure 4.15. The plane is traveling horizontally when it drops the package. Because the package is in free-fall while moving in the horizontal direction, we *categorize* this as a projectile motion problem. To *analyze* the problem, we choose the coordinate system shown in Figure 4.15, in which the origin is at the point of release of the package. Consider first its horizontal motion. The only equation available for finding the position along the horizontal direction is $x_f = x_i + v_{xi}t$ (Eq. 4.9a). The initial x component of the package velocity is the same as that of the plane when the package is released: 40.0 m/s. Thus, we have

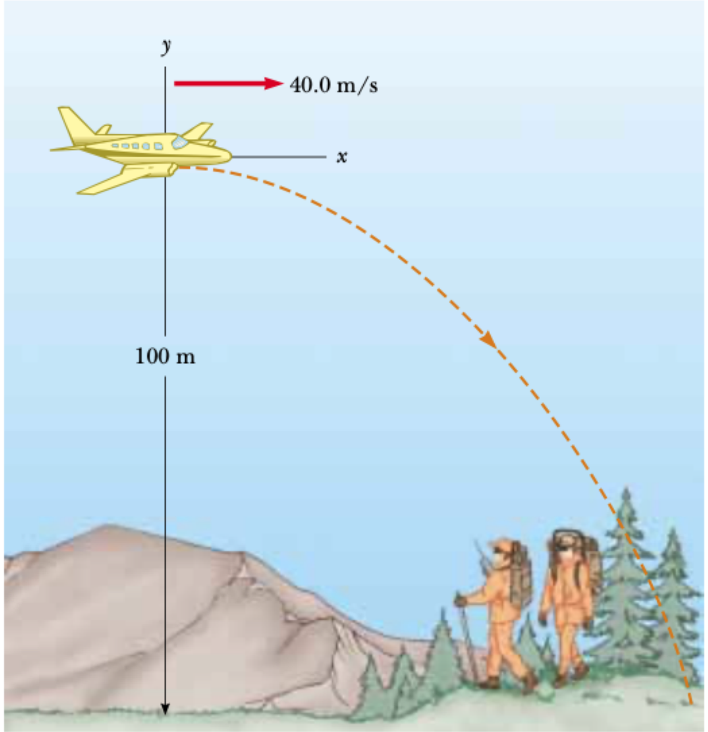


Figure 4.15 (Example 4.6) A package of emergency supplies is dropped from a plane to stranded explorers.

$$x_f = (40.0\ \text{m/s})\,t$$

If we know t, the time at which the package strikes the ground, then we can determine x_f, the distance the package travels in the horizontal direction. To find t, we use the equations that describe the vertical motion of the package. We know that, at the instant the package hits the ground, its y coordinate is $y_f = -100$ m. We also know that the initial vertical component of the package velocity v_{yi} is zero because at the moment of release, the package has only a horizontal component of velocity.

From Equation 4.9a, we have

$$y_f = -\tfrac{1}{2}gt^2$$

$$-100\ \text{m} = -\tfrac{1}{2}(9.80\ \text{m/s}^2)\,t^2$$

$$t = 4.52\,\text{s}$$

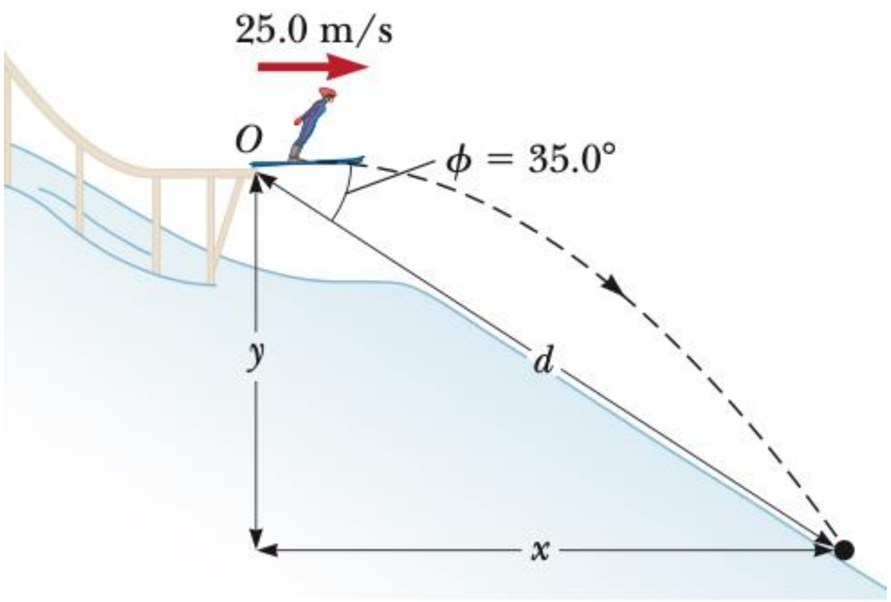
Substitution of this value for the time into the equation for the x coordinate gives

$$x_f = (40.0\ \text{m/s})(4.52\ \text{s}) = \boxed{181\ \text{m}}$$

The package hits the ground 181 m to the right of the drop point. To *finalize* this problem, we learn that an object dropped from a moving airplane does not fall straight down. It hits the ground at a point different from the one right below the plane where it was released. This was an important consideration for free-fall bombs such as those used in World War II.

Example 4.7 The End of the Ski Jump

A ski jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s as shown in Figure. The landing incline below him falls off with a slope of 35.0°. Where does he land on the incline?



Example 4.7 The End of the Ski Jump

A ski-jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s, as shown in Figure 4.16. The landing incline below him falls off with a slope of 35.0°. Where does he land on the incline?

Solution We can *conceptualize* this problem based on observations of winter Olympic ski competitions. We observe the skier to be airborne for perhaps 4 s and go a distance of about 100 m horizontally. We should expect the value of d, the distance traveled along the incline, to be of the same order of magnitude. We *categorize* the problem as that of a particle in projectile motion.

To *analyze* the problem, it is convenient to select the beginning of the jump as the origin. Because $v_{xi} = 25.0$ m/s and $v_{yi} = 0$, the x and y component forms of Equation 4.9a are

$$(1) \qquad x_f = v_{xi}t = (25.0 \text{ m/s})\, t$$

$$(2) \qquad y_f = v_{yi}t + \tfrac{1}{2}a_y t^2 = -\tfrac{1}{2}(9.80 \text{ m/s}^2)\, t^2$$

From the right triangle in Figure 4.16, we see that the jumper's x and y coordinates at the landing point are $x_f = d \cos 35.0°$ and $y_f = -d \sin 35.0°$. Substituting these relationships into (1) and (2), we obtain

$$(3) \qquad d \cos 35.0° = (25.0 \text{ m/s})\, t$$

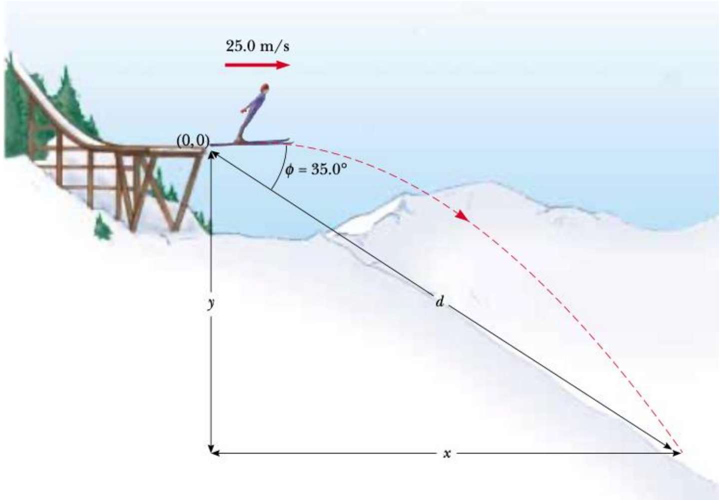
$$(4) \qquad -d \sin 35.0° = -\tfrac{1}{2}(9.80 \text{ m/s}^2)\, t^2$$

Solving (3) for t and substituting the result into (4), we find that $d = 109$ m. Hence, the x and y coordinates of the point at which the skier lands are

$$x_f = d \cos 35.0° = (109 \text{ m}) \cos 35.0° = \boxed{89.3 \text{ m}}$$

$$y_f = -d \sin 35.0° = -(109 \text{ m}) \sin 35.0° = \boxed{-62.5 \text{ m}}$$

To *finalize* the problem, let us compare these results to our expectations. We expected the horizontal distance to be on the order of 100 m, and our result of 89.3 m is indeed on



Example 4.7 The End of the Ski Jump

A ski jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s as shown in Figure 4.14. The landing incline below her falls off with a slope of 35.0°. Where does she land on the incline?

SOLUTION

Conceptualize We can conceptualize this problem based on memories of observing winter Olympic ski competitions. We estimate the skier to be airborne for perhaps 4 s and to travel a distance of about 100 m horizontally. We should expect the value of d, the distance traveled along the incline, to be of the same order of magnitude.

Categorize We categorize the problem as one of a particle in projectile motion. As with other projectile motion problems, we use the *particle under constant velocity* model for the horizontal motion and the *particle under constant acceleration* model for the vertical motion.

Analyze It is convenient to select the beginning of the jump as the origin. The initial velocity components are $v_{xi} = 25.0$ m/s and $v_{yi} = 0$. From the right triangle in Figure 4.14, we see that the jumper's x and y coordinates at the landing point are given by $x_f = d\cos\phi$ and $y_f = -d\sin\phi$.

Express the coordinates of the jumper as a function of time, using the particle under constant velocity model for x and the position equation from the particle under constant acceleration model for y:

$$(1)\quad x_f = v_{xi}t$$

$$(2)\quad y_f = v_{yi}t - \tfrac{1}{2}gt^2$$

$$(3)\quad d\cos\phi = v_{xi}t$$

$$(4)\quad -d\sin\phi = -\tfrac{1}{2}gt^2$$

Solve Equation (3) for t and substitute the result into Equation (4):

$$-d\sin\phi = -\tfrac{1}{2}g\left(\frac{d\cos\phi}{v_{xi}}\right)^2$$

Solve for d and substitute numerical values:

$$d = \frac{2v_{xi}^{\,2}\sin\phi}{g\cos^2\phi} = \frac{2(25.0\text{ m/s})^2\sin 35.0°}{(9.80\text{ m/s}^2)\cos^2 35.0°} = 109\text{ m}$$

Evaluate the x and y coordinates of the point at which the skier lands:

$$x_f = d\cos\phi = (109\text{ m})\cos 35.0° = \boxed{89.3\text{ m}}$$

$$y_f = -d\sin\phi = -(109\text{ m})\sin 35.0° = \boxed{-62.5\text{ m}}$$

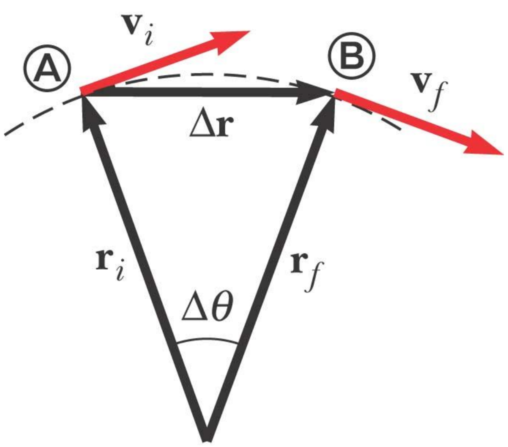
Finalize Let us compare these results with our expectations. We expected the horizontal distance to be on the order of 100 m, and our result of 89.3 m is indeed on this order of magnitude. It might be useful to calculate the time interval that the jumper is in the air and compare it with our estimate of about 4 s.

4.4 Uniform Circular Motion

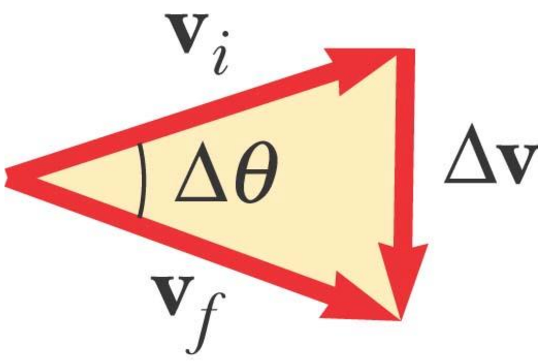
- **Uniform circular motion** occurs when an object moves in a circular path with a constant speed v.
- Even though an object moves at a constant speed in a circular path, it still has an acceleration.
- The acceleration depends on the change in the velocity vector.
- The velocity vector is always tangent to the path of the object.

Changing Velocity in Uniform Circular Motion

- The change in the velocity vector is due to the change in direction



- The vector diagram shows $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$



Centripetal Acceleration

- The acceleration is always
  - perpendicular to the path of the motion
  - points toward the center of the circle of motion
- This acceleration is called the ***centripetal acceleration***

Centripetal Acceleration

- The magnitude of the centripetal acceleration vector is given by

$$a_c = \frac{v^2}{r}$$

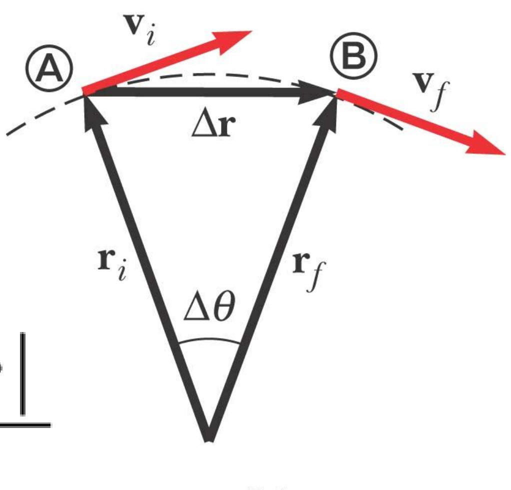
- The direction of the centripetal acceleration vector is always changing, to stay directed toward the center of the circle of motion

Centripetal Acceleration

From the average acceleration:

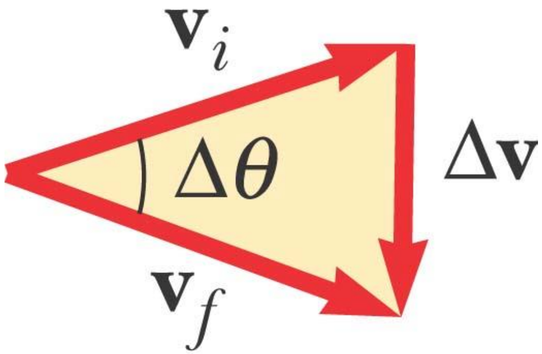
$$\bar{\mathbf{a}} = \frac{\mathbf{v}_f - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta \mathbf{v}}{\Delta t}$$

$\Delta \mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$



$$\frac{|\Delta \mathbf{v}|}{v} = \frac{|\Delta \mathbf{r}|}{r}$$

$$|\bar{\mathbf{a}}| = \frac{|\Delta \mathbf{v}|}{\Delta t} = \frac{v}{r}\frac{|\Delta \mathbf{r}|}{\Delta t}$$



$$a_c = \frac{v^2}{r}$$

Period

- The ***period***, T, is the time required for one complete revolution
- The speed of the particle would be the circumference of the circle of motion divided by the period, or

$$v = \frac{2\pi r}{T}$$

- Therefore, the period is

$$\boxed{T = \frac{2\pi r}{v}}$$

- The period of a particle in uniform circular motion is a measure of the number of seconds for one revolution of the particle around the circle.

Angular Speed

- The inverse of the period is the *rotation rate* and is measured in revolutions per second.
- One full revolution of the particle around the circle corresponds to an angle of 2π radians
- The product of 2π and the rotation rate gives the **angular speed** ω of the particle, measured in radians/s or s^{-1} :

$$\boxed{\omega = \frac{2\pi}{T}}$$

$$\omega = 2\pi\left(\frac{v}{2\pi r}\right) = \frac{v}{r} \quad \rightarrow \quad v = r\omega$$

- We can express the centripetal acceleration of a particle in uniform circular motion in terms of angular speed as

$$a_c = \frac{v^2}{r} \qquad a_c = \frac{(r\omega)^2}{r} \qquad \boxed{a_c = r\omega^2}$$

Example 4.8 The Centripetal Acceleration of the Earth

What is the centripetal acceleration of the Earth as it moves in its orbit around the Sun?

$$a_c = \frac{v^2}{r} = \frac{\left(\dfrac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

$$= \frac{4\pi^2(1.496 \times 10^{11}\,\mathrm{m})}{(1\ \mathrm{yr})^2}\left(\frac{1\ \mathrm{yr}}{3.156 \times 10^7\,\mathrm{s}}\right)^2$$

$$= \boxed{5.93 \times 10^{-3}\,\mathrm{m/s^2}}$$

To *finalize* this problem, note that this acceleration is much smaller than the free-fall acceleration on the surface of the Earth. An important thing we learned here is the technique of replacing the speed v in terms of the period T of the motion.

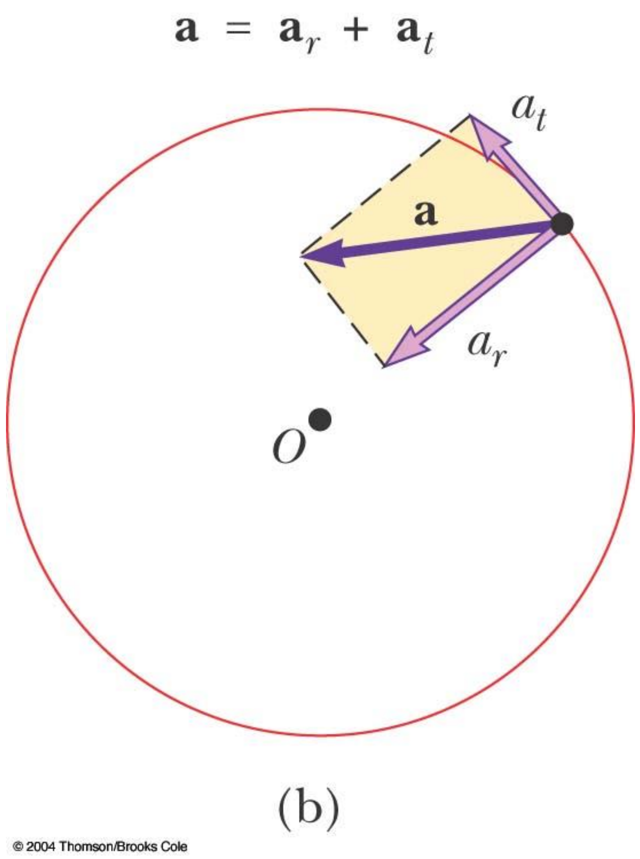
4.5 Tangential and Radial Acceleration

Tangential Acceleration

- The magnitude of the velocity could also be changing
- In this case, there would be a ***tangential acceleration***

Total Acceleration

- The tangential acceleration causes the change in the speed of the particle
- The radial acceleration comes from a change in the direction of the velocity vector



Total Acceleration, equations

- The tangential acceleration: $$a_t = \frac{d\left|\mathbf{v}\right|}{dt}$$
- The radial acceleration: $$a_r = -a_c = -\frac{v^2}{r}$$
- The total acceleration:
  - Magnitude $$a = \sqrt{a_r^2 + a_t^2}$$

Total Acceleration, In Terms of Unit Vectors

- Define the following unit vectors

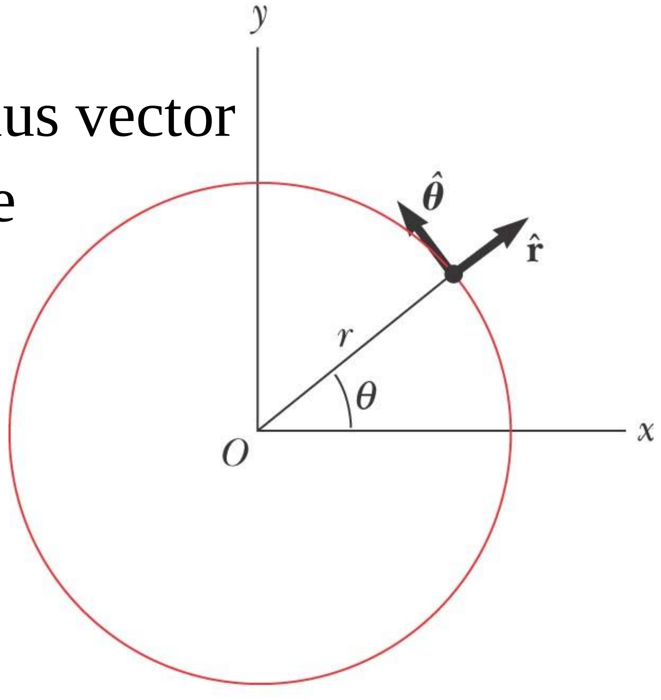
$$\hat{\mathbf{r}} \text{ and } \hat{\theta}$$

$\hat{r}$ is a unit vector lying along the radius vector

$\hat{\theta}$ is a unit vector tangent to the circle

- The total acceleration is

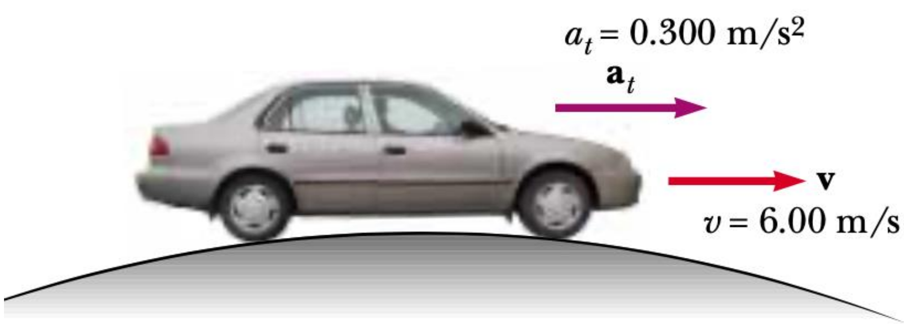
$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r = \frac{d|\mathbf{v}|}{dt}\hat{\boldsymbol{\theta}} - \frac{v^2}{r}\hat{\mathbf{r}}$$

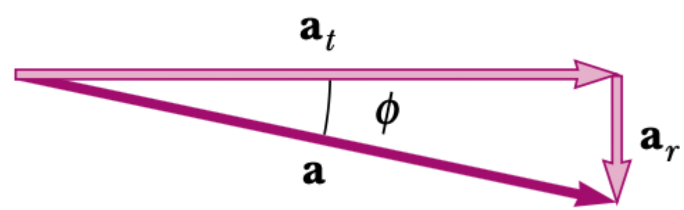


(a)

Example 4.9 Over the Rise

A car exhibits a constant acceleration of 0.300 m/s^2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like a circle of radius 500 m. At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of 6.00 m/s. What is the direction of the total acceleration vector for the car at this instant?





$$a_r = -\frac{v^2}{r} = -\frac{(6.00 \text{ m/s})^2}{500 \text{ m}} = \boxed{-0.0720 \text{ m/s}^2}$$

The radial acceleration vector is directed straight downward while the tangential acceleration vector has magnitude 0.300 m/s^2 and is horizontal. Because $\mathbf{a} = \mathbf{a}_r + \mathbf{a}_t$, the magnitude of $\mathbf{a}$ is

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{(-0.0720)^2 + (0.300)^2} \text{ m/s}^2$$

$$= \boxed{0.309 \text{ m/s}^2}$$

If ϕ is the angle between $\mathbf{a}$ and the horizontal, then

$$\phi = \tan^{-1}\frac{a_r}{a_t} = \tan^{-1}\left(\frac{-0.0720 \text{ m/s}^2}{0.300 \text{ m/s}^2}\right) = \boxed{-13.5^\circ}$$

This angle is measured downward from the horizontal.

Exercises

Problems: 1, 3, 5, 6, 8, 14, 15, 17, 19, 20, 22, 23, 25, 29