3. Vectors

3.1 Coordinate Systems

Many aspects of physics involve a description of a location in space.

This is implemented using "Coordinate systems"

- Cartesian coordinate system
- Polar coordinate system

Coordinate system consists of

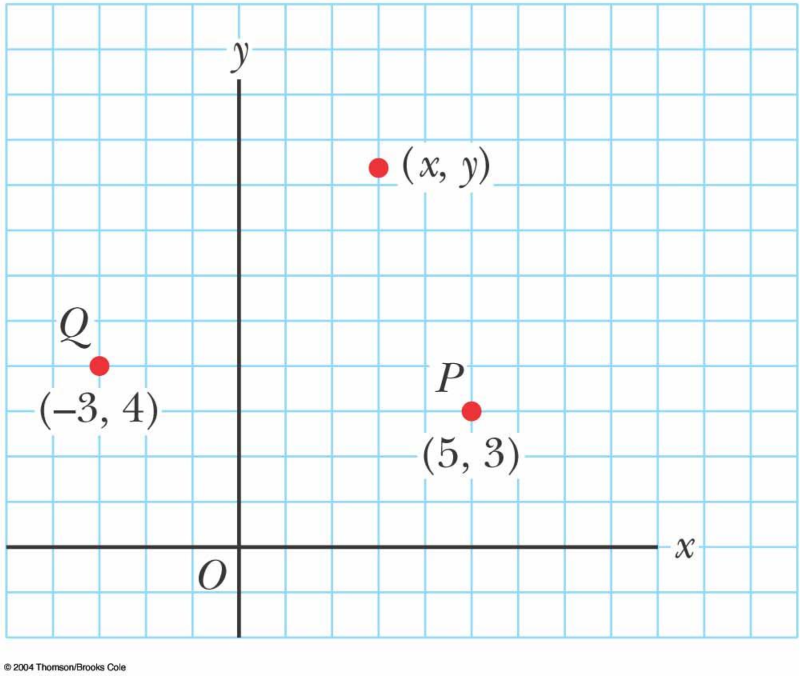
- a fixed reference point called the origin
- specific axes with scales and labels
- instructions on how to label a point relative to the origin and the axes

Cartesian coordinate system:

In two dimensions, perpendicular axes (horizontal and vertical axes) intersect at a point defined as the origin *O*.

Every point is labeled with coordinates (x,y).

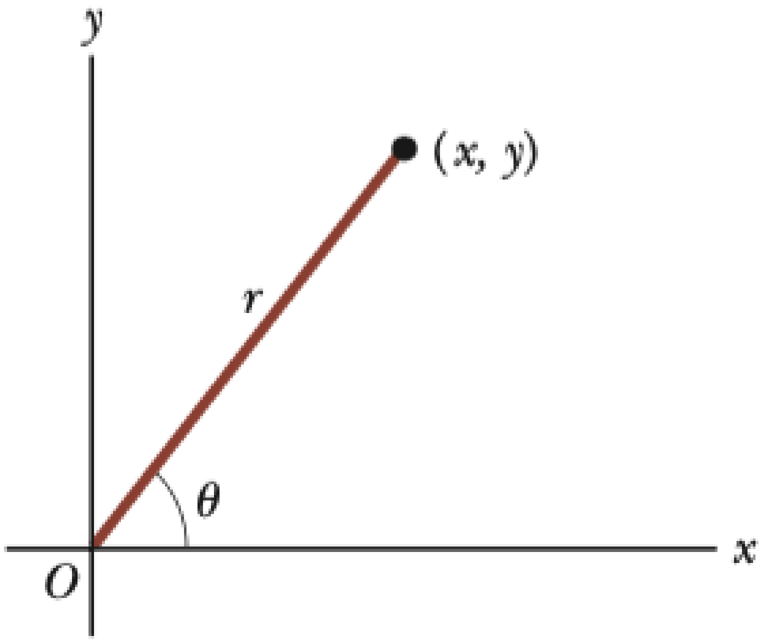
It is also called rectangular coordinate system.



Polar coordinate system

- r is the distance from the origin to the point having Cartesian coordinates (x, y)
- θ is the angle between a fixed axis and a line drawn from the origin to the point
- The fixed axis is often the positive x axis, and θ is usually measured

counterclockwise from it.



- Points are labeled (r, θ)

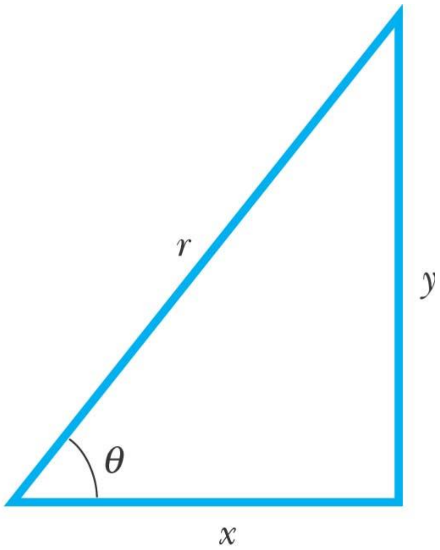
Polar to Cartesian Coordinates

From the right triangle, we find

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$



We can obtain the Cartesian coordinates from Polar coordinates by using the equations:

Cartesian coordinates in terms of polar coordinates ▶

$$x = r\cos\theta$$

$$y = r\sin\theta$$

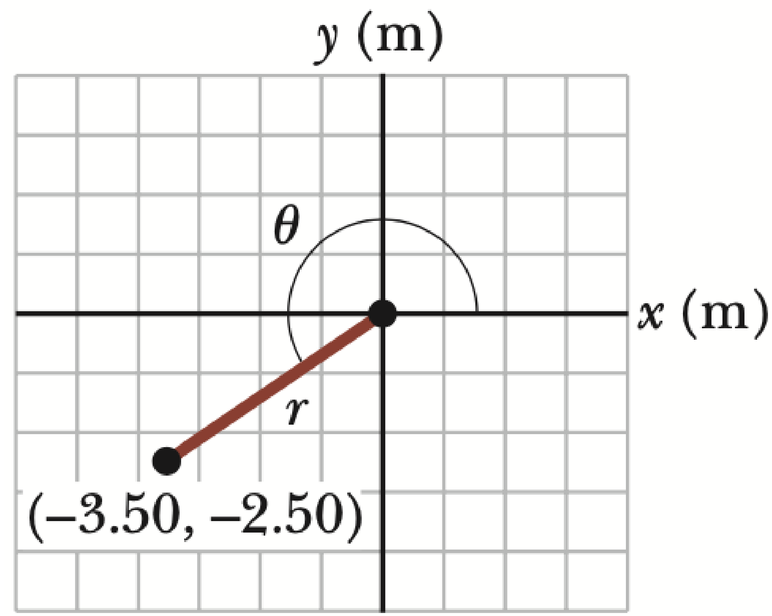
Polar coordinates in terms of Cartesian coordinates ▶

$$\tan\theta = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2}$$

Example 3.1 Polar Coordinates

The Cartesian coordinates of a point in the xy plane are $(x, y) = (-3.50, -2.50)$ m as shown in the Figure. Find the polar coordinates of this point.

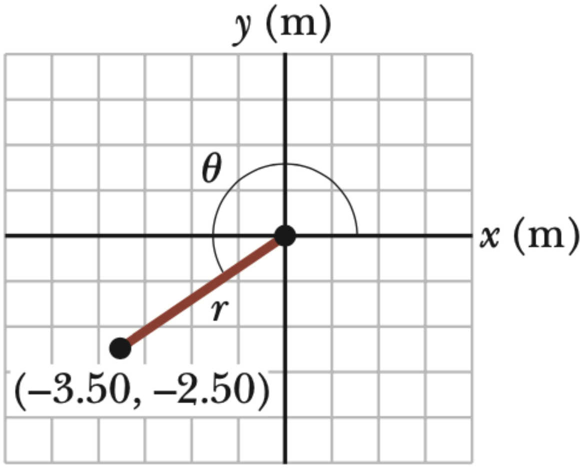


Example 3.1 Polar Coordinates

The Cartesian coordinates of a point in the xy plane are $(x, y) = (-3.50, -2.50)$ m as shown in the Figure. Find the polar coordinates of this point.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50\text{ m})^2 + (-2.50\text{ m})^2}$$

$$= \boxed{4.30\text{ m}}$$



$$\tan\theta = \frac{y}{x} = \frac{-2.50\text{ m}}{-3.50\text{ m}} = 0.714$$

$$\theta = \boxed{216°}$$

3.2 Vector and Scalar Quantities

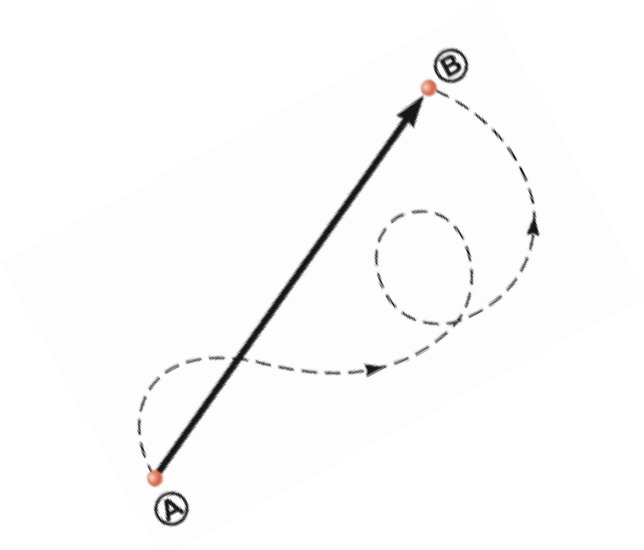
- **A scalar quantity** is completely specified by a single value with an appropriate unit and **has no direction**.

- **A vector quantity** is completely specified by a number with an appropriate unit (the *magnitude* of the vector) **plus a direction**.

- A boldface letter with an arrow over the letter, such as $\vec{\mathbf{A}}$ or **A**, is used to represent a vector.
- The magnitude of the vector $\vec{\mathbf{A}}$ is written either A or $|\vec{\mathbf{A}}|$

Vector Example

- A particle travels from A to B along the path shown by the broken line. This is the *distance* traveled and is a **scalar**.
- The ***displacement*** is the solid line from A to B
  - The displacement is independent of the path taken between the two points.
  - The displacement is a **vector**.



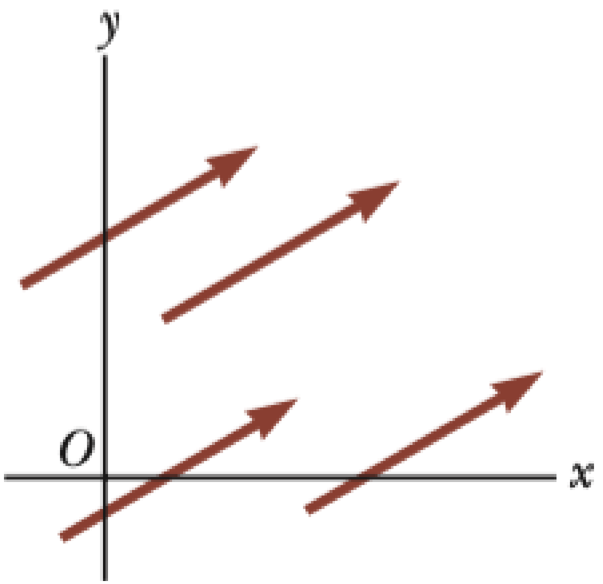
The direction of the arrowhead represents the direction of the displacement, and the length of the arrow represents the magnitude of the displacement.

3.3 Some Properties of Vectors

Equality of Two Vectors:

Two vectors are equal if they have the same magnitude and the same direction.

$\vec{\mathbf{A}} = \vec{\mathbf{B}}$ **if** $A = B$ and $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ point in the same direction along parallel lines.



These vectors are equal because they have equal lengths and point in the same direction.

Adding Vectors:

To add vector $\vec{\mathbf{B}}$ to vector $\vec{\mathbf{A}}$, first draw vector $\vec{\mathbf{A}}$ on graph paper, and then draw vector $\vec{\mathbf{B}}$ to **the same scale** with its tail starting from the tip of $\vec{\mathbf{A}}$, as shown in Figure.

The resultant vector:

$\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is the vector drawn from the tail of $\vec{\mathbf{A}}$ to the tip of $\vec{\mathbf{B}}$.

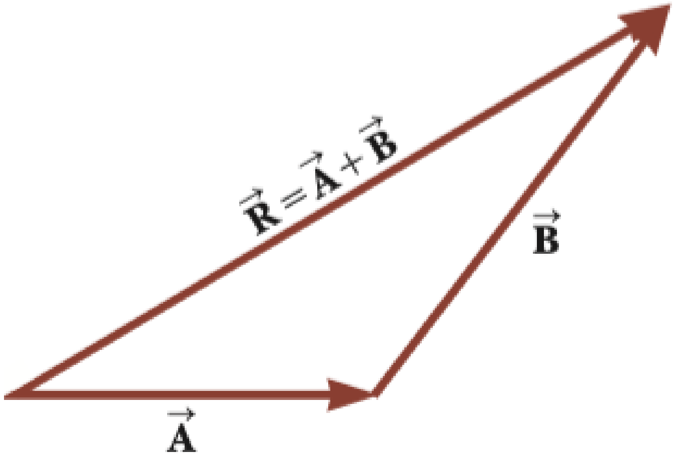


Figure 3.6 When vector $\vec{\mathbf{B}}$ is added to vector $\vec{\mathbf{A}}$, the resultant $\vec{\mathbf{R}}$ is the vector that runs from the tail of $\vec{\mathbf{A}}$ to the tip of $\vec{\mathbf{B}}$.

Commutative law of addition :

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$$

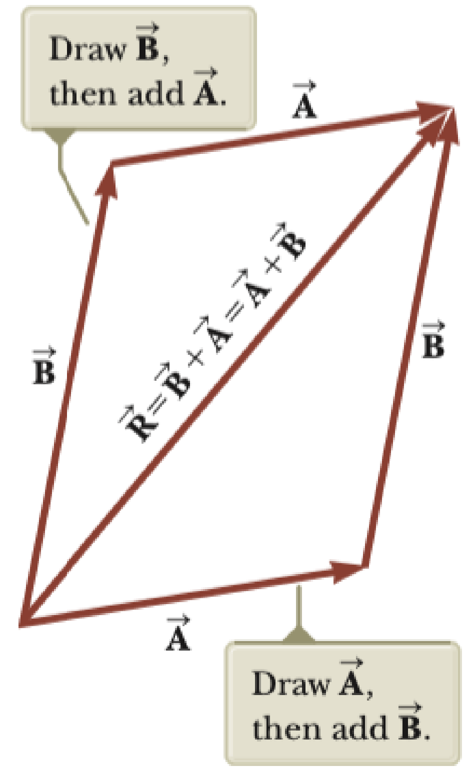
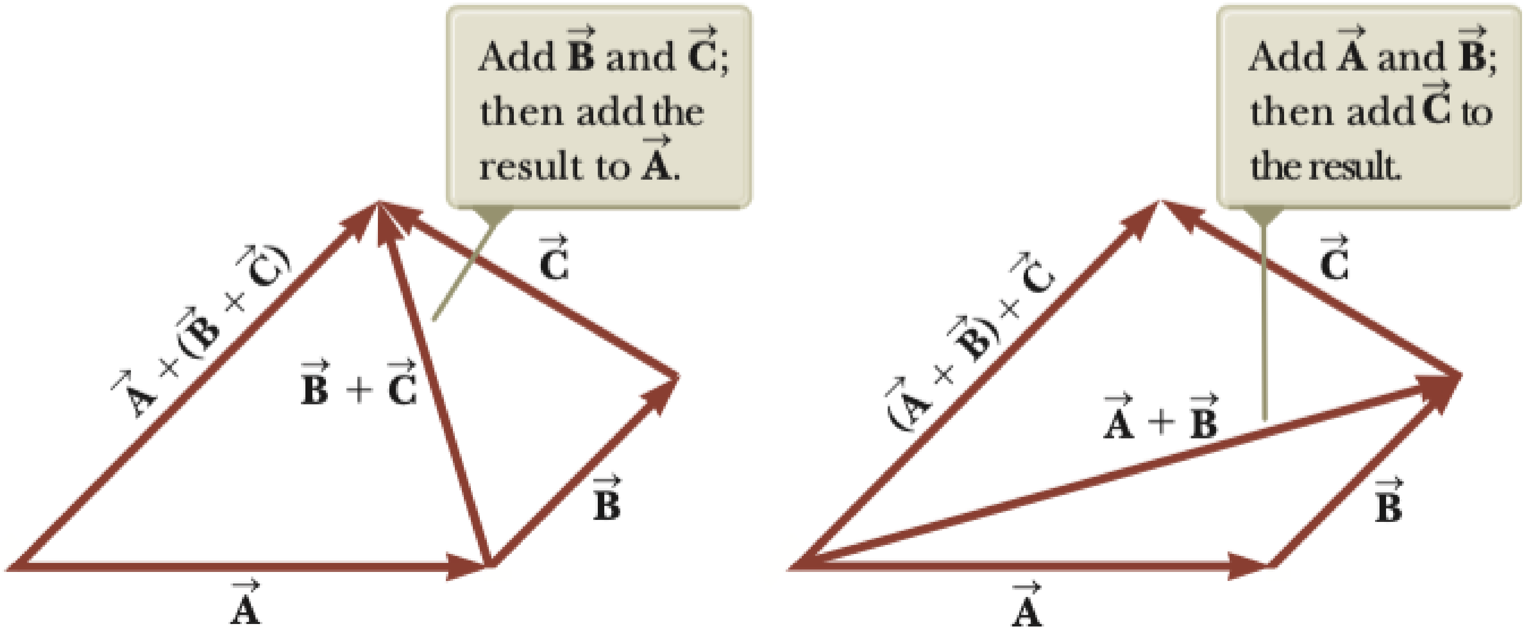


Figure 3.8 This construction shows that $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$ or, in other words, that vector addition is commutative.

Associative law of addition

$$\vec{\mathbf{A}} + \left(\vec{\mathbf{B}} + \vec{\mathbf{C}}\right) = \left(\vec{\mathbf{A}} + \vec{\mathbf{B}}\right) + \vec{\mathbf{C}}$$



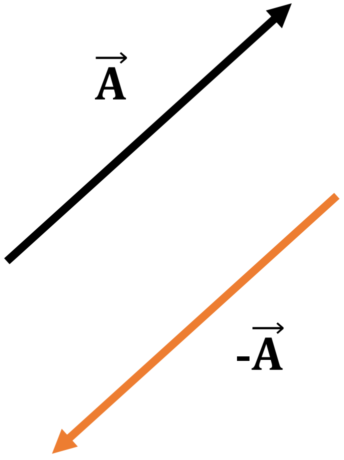
Negative of a Vector:

The negative of the vector $\vec{\mathbf{A}}$ is defined as the vector that when added to $\vec{\mathbf{A}}$ gives zero for the vector sum. That is:

$\vec{\mathbf{A}} + (-\,\vec{\mathbf{A}}) = 0.$

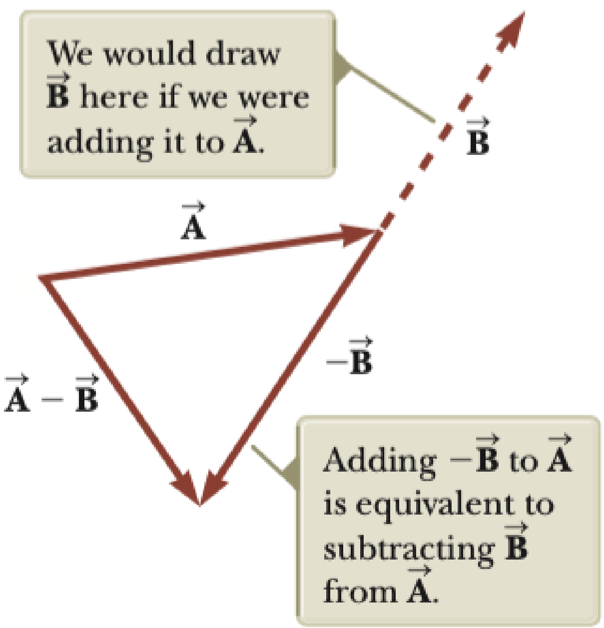
The vectors $\vec{\mathbf{A}}$ and - $\vec{\mathbf{A}}$ have the same magnitude but point in *opposite directions*

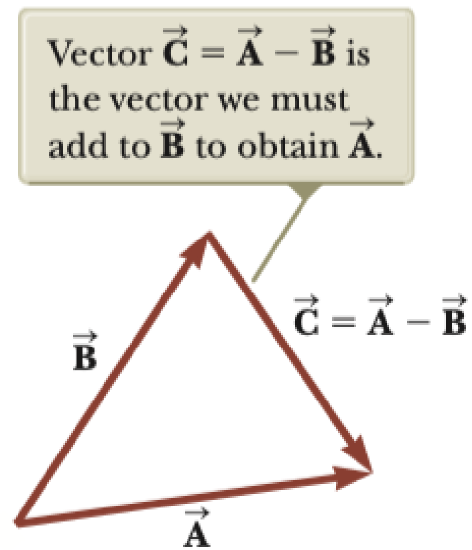
$\vec{\mathbf{B}} = -\vec{\mathbf{A}}$ **and** $|\vec{\mathbf{A}}| = |\vec{\mathbf{B}}|$



Subtracting Vectors:

$$\vec{\mathbf{A}} - \vec{\mathbf{B}} = \vec{\mathbf{A}} + (-\vec{\mathbf{B}})$$





Multiplying a Vector by a Scalar

If a vector $\vec{\mathbf{A}}$ is multiplied by a positive scalar quantity m, the product m$\vec{\mathbf{A}}$ is a vector that has *same* direction of $\vec{\mathbf{A}}$ and magnitude mA.

For example, the vector 5 $\vec{\mathbf{A}}$ is five times as long as $\vec{\mathbf{A}}$ and points in the same direction as $\vec{\mathbf{A}}$.

The vector $-\frac{1}{3}\vec{\mathbf{A}}$ is one-third the length of $\vec{\mathbf{A}}$ and points in the opposite direction of $\vec{\mathbf{A}}$.

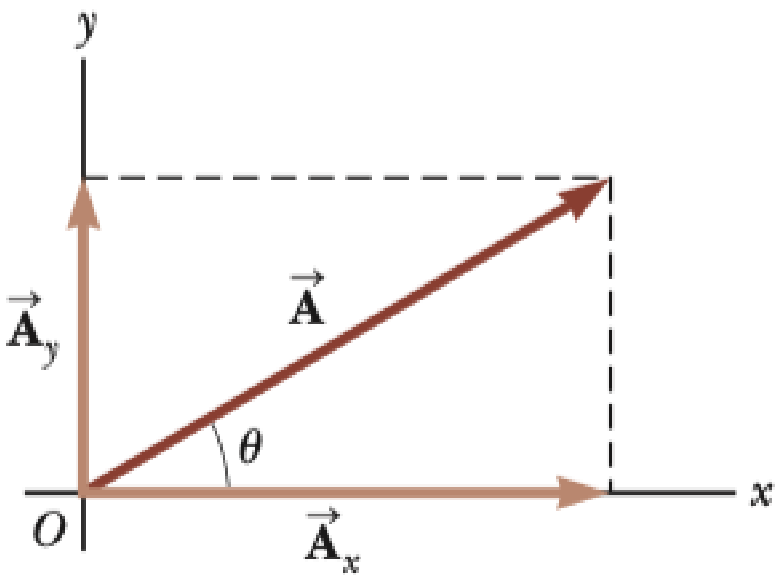
3.4 Vectors Components of a Vector and Unit Vectors

Components of a Vector

A component is a projection of a vector along an axis.

Any vector can be completely described by its components.

These are the projections of the vector along the x- and y-axes.

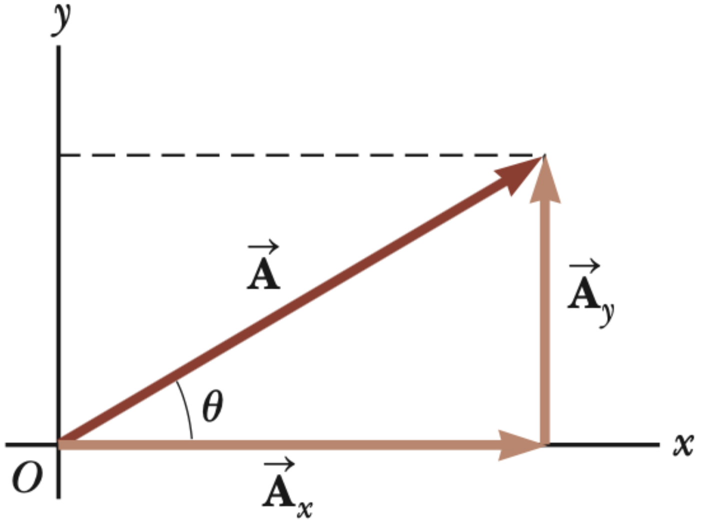


A vector $\vec{\mathbf{A}}$ can be expressed as the sum of two other *component vectors*

$\vec{\mathbf{A}}_x$, which is parallel to the *x* axis,

$\vec{\mathbf{A}}_y$, which is parallel to the *y* axis.

$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$$



The components of $\vec{\mathbf{A}}$ are

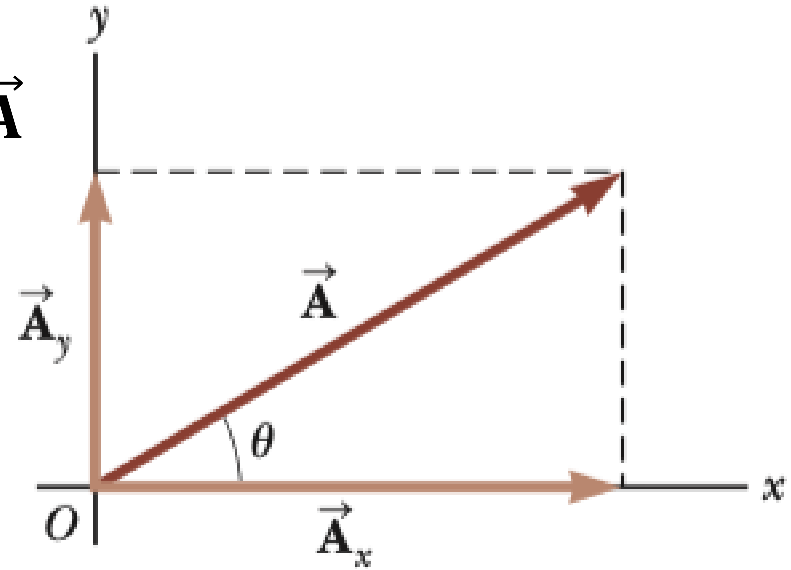
$$A_x = A\cos\theta$$ represents the projection of $\vec{\mathbf{A}}$ along the *x* axis

$$A_y = A\sin\theta$$ represents the projection of $\vec{\mathbf{A}}$ along the *y* axis

The magnitude and direction of $\vec{\mathbf{A}}$

$$A = \sqrt{{A_x}^2 + {A_y}^2}$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$



The signs of the components A_x and A_y depend on the angle θ.

The components have the same units as the original vector.

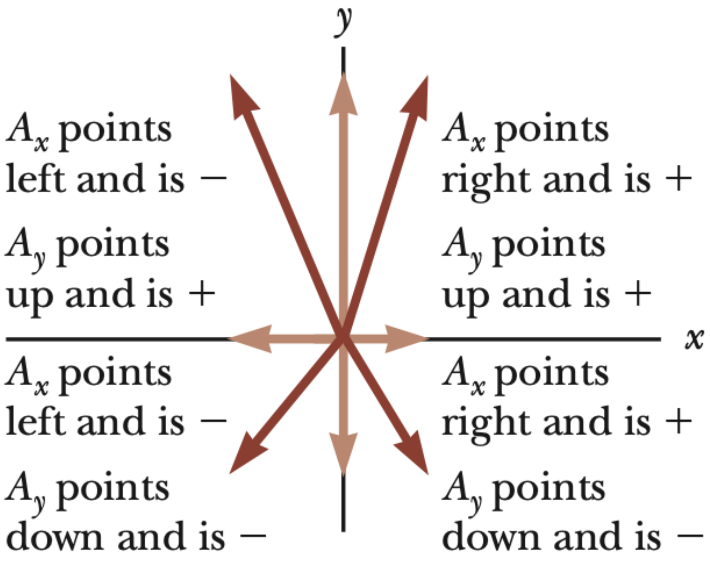


Figure 3.13 The signs of the components of a vector $\vec{\mathbf{A}}$ depend on the quadrant in which the vector is located.

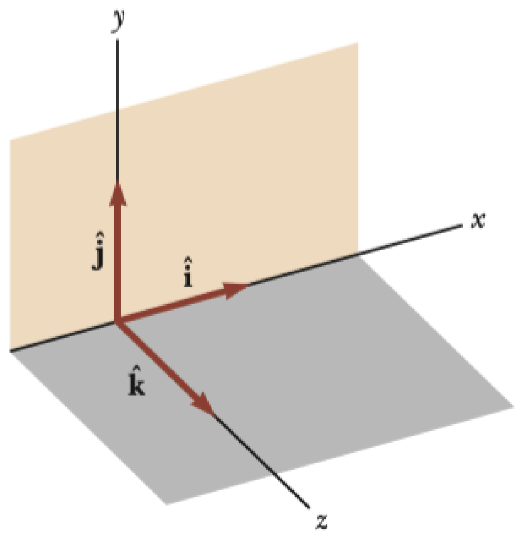
3.4 Vectors Components of a Vector and Unit Vectors

Unit Vectors

A **unit vector** is a dimensionless vector having a magnitude of exactly 1.

Unit vectors are used to specify a given direction.

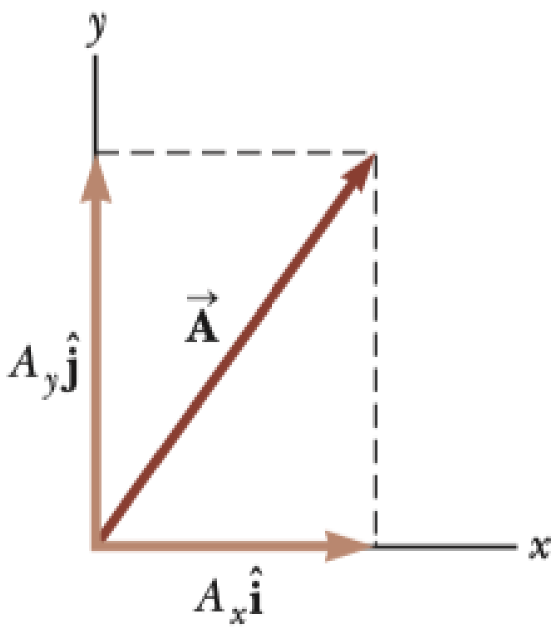
The symbols $\hat{i}, \hat{j}, \hat{k}$ represent unit vectors pointing in the positive *x*, *y*, and *z* directions, respectively.



The product of the component A_x and the unit vector $\hat{\mathbf{i}}$ is the component vector

$\vec{\mathbf{A}}_x = A_x\hat{\mathbf{i}}$

Likewise, $\vec{\mathbf{A}}_y = A_y\hat{\mathbf{j}}$

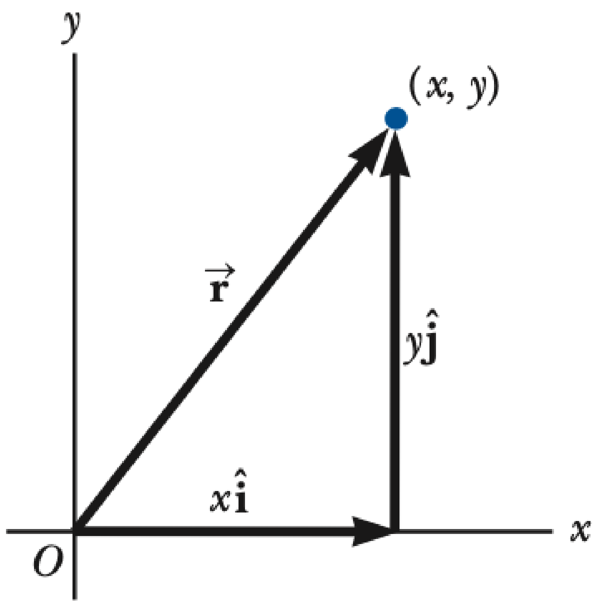


The unit-vector notation for the vector $\vec{\mathbf{A}}$ is

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$

A point (x,y) can be specified by the **position vector** $\vec{\mathbf{r}}$, which in unit-vector form is given by

$$\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

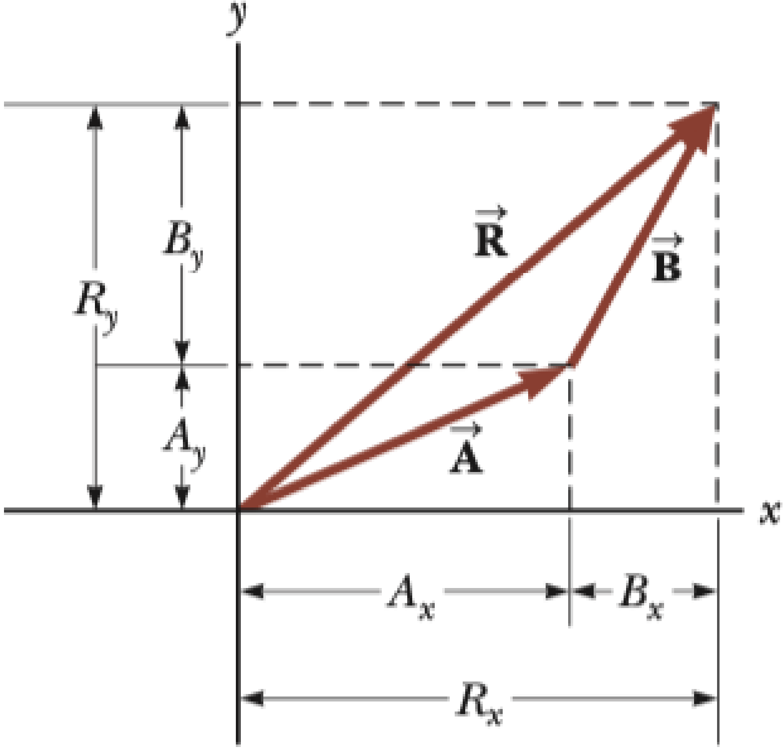


Adding vectors using the components of the individual vectors

Suppose we wish to add vector $\vec{\mathbf{B}}$ to vector $\vec{\mathbf{A}}$

$$\vec{\mathbf{A}} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}$$

$$\vec{\mathbf{B}} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}}$$



The resultant vector $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is

$$\vec{\mathbf{R}} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}}$$

$$\vec{\mathbf{R}} = R_x\,\hat{\mathbf{i}} + R_y\,\hat{\mathbf{j}}$$

The components of the resultant vector are

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$$

$$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}$$

$$\vec{\mathbf{R}} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}}$$

$$\vec{\mathbf{R}} = R_x\hat{\mathbf{i}} + R_y\hat{\mathbf{j}}$$

The magnitude of $\vec{\mathbf{R}}$ and the angle it makes with the *x* axis are obtained from its components using the relationships

$$R = \sqrt{R_x{}^{\mathbf{2}} + R_y{}^{\mathbf{2}}}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{\mathrm{A}_y + \mathrm{By}}{\mathrm{A}_x + \mathrm{B}_x}, \qquad \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

Pitfall Prevention 3.3

Tangents on Calculators Equation 3.17 involves the calculation of an angle by means of a tangent function. Generally, the inverse tangent function on calculators provides an angle between $-90°$ and $+90°$. As a consequence, if the vector you are studying lies in the second or third quadrant, the angle measured from the positive x axis will be the angle your calculator returns plus 180°.

In three component directions

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{R}} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}}$$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

The extension of our method to adding more than two vectors is also straightforward

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} = (A_x + B_x + C_x)\,\hat{\mathbf{i}} + (A_y + By + Cy)\,\hat{\mathbf{j}} + (A_z + Bz + C_z)\,\hat{\mathbf{k}}$$

Example 3.3 The Sum of Two Vectors

Find the sum of two displacement vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ lying in the *xy* plane and given by

$\vec{\mathbf{A}} = (2.0\,\mathbf{i} + 2.0\,\mathbf{j})$ m and $\vec{\mathbf{B}} = (2.0\,\mathbf{i} - 4.0\,\mathbf{j})$ m.

Example 3.3 The Sum of Two Vectors

Find the sum of two displacement vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ lying in the *xy* plane and given by

$\vec{\mathbf{A}} = (2.0\,\mathbf{i} + 2.0\,\mathbf{j})$ m and $\vec{\mathbf{B}} = (2.0\,\mathbf{i} - 4.0\,\mathbf{j})$ m.

$$\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} = (2.0 + 2.0)\hat{\mathbf{i}}\text{ m} + (2.0 - 4.0)\hat{\mathbf{j}}\text{ m}$$

$$R_x = 4.0\text{ m} \qquad R_y = -2.0\text{ m}$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(4.0\text{ m})^2 + (-2.0\text{ m})^2} = \sqrt{20}\text{ m} = \boxed{4.5\text{ m}}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2.0\text{ m}}{4.0\text{ m}} = -0.50$$

Your calculator likely gives the answer $-27°$ for $\theta = \tan^{-1}(-0.50)$. This answer is correct if we interpret it to mean 27° clockwise from the x axis. Our standard form has been to quote the angles measured counterclockwise from the $+x$ axis, and that angle for this vector is $\boxed{\theta = 333°}$.

Example 3.4 The Resultant Displacement

A particle undergoes three consecutive displacements:

$$\Delta\vec{\mathbf{r}}_1 = (15\hat{\mathbf{i}} + 30\hat{\mathbf{j}} + 12\hat{\mathbf{k}})\text{ cm}$$

$$\Delta\vec{\mathbf{r}}_2 = (23\hat{\mathbf{i}} - 14\hat{\mathbf{j}} - 5.0\hat{\mathbf{k}})\text{ cm}$$

$$\Delta\vec{\mathbf{r}}_3 = (-13\hat{\mathbf{i}} + 15\hat{\mathbf{j}})\text{ cm}$$

Find unit-vector notation for the resultant displacement and its magnitude.

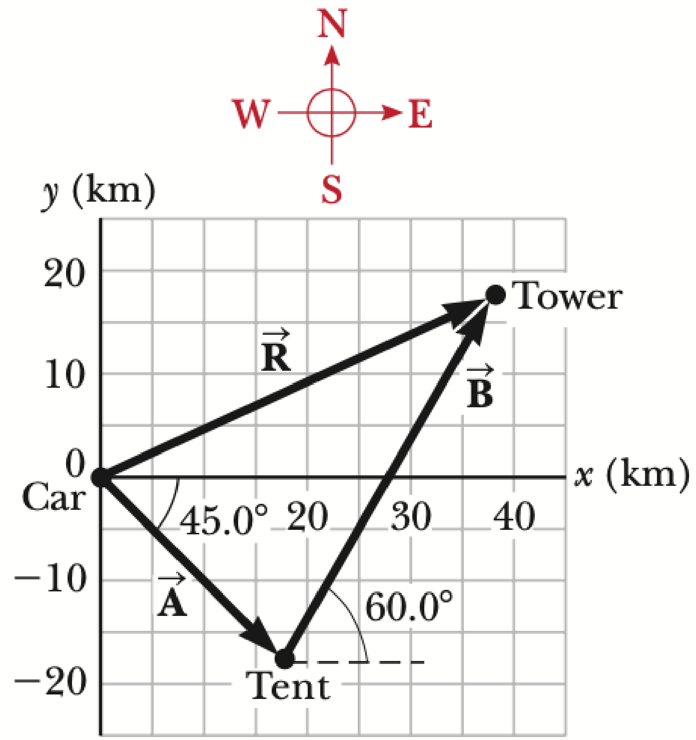
$$
\begin{aligned}
\Delta\vec{\mathbf{r}} &= \Delta\vec{\mathbf{r}}_1 + \Delta\vec{\mathbf{r}}_2 + \Delta\vec{\mathbf{r}}_3 \\
&= (15 + 23 - 13)\,\hat{\mathbf{i}}\text{ cm} + (30 - 14 + 15)\,\hat{\mathbf{j}}\text{ cm} + (12 - 5.0 + 0)\,\hat{\mathbf{k}}\text{ cm} \\
&= \boxed{(25\,\hat{\mathbf{i}} + 31\,\hat{\mathbf{j}} + 7.0\,\hat{\mathbf{k}})\text{ cm}}
\end{aligned}
$$

$$
\begin{aligned}
R &= \sqrt{R_x{}^2 + R_y{}^2 + R_z{}^2} \\
&= \sqrt{(25\text{ cm})^2 + (31\text{ cm})^2 + (7.0\text{ cm})^2} = \boxed{40\text{ cm}}
\end{aligned}
$$

Example 3.5 Taking a Hike

A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower.

(A) Determine the components of the hiker's displacement for each day.



Example 3.5 Taking a Hike

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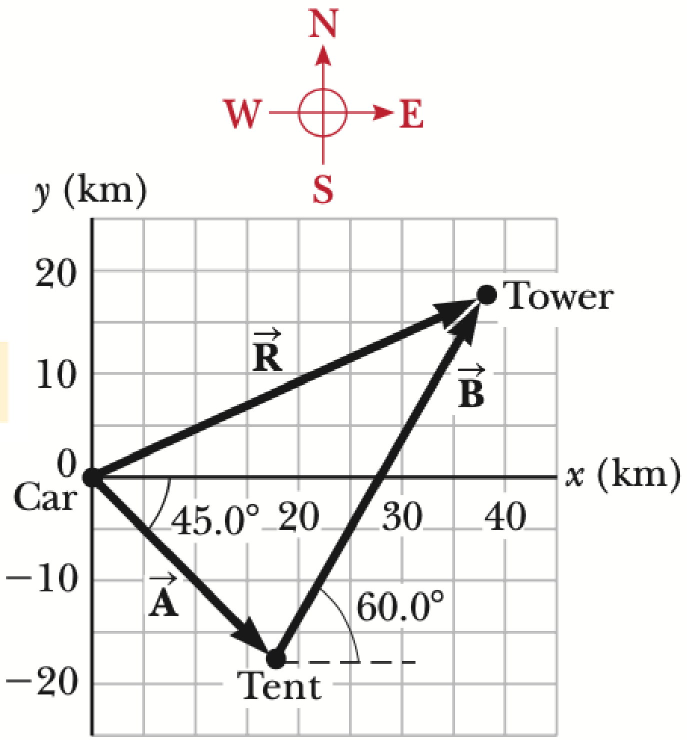
(A) Determine the components of the hiker's displacement for each day.

$$A_x = A\cos(-45.0°) = (25.0\text{ km})(0.707) = \boxed{17.7\text{ km}}$$

$$A_y = A\sin(-45.0°) = (25.0\text{ km})(-0.707) = \boxed{-17.7\text{ km}}$$

$$B_x = B\cos 60.0° = (40.0\text{ km})(0.500) = \boxed{20.0\text{ km}}$$

$$B_y = B\sin 60.0° = (40.0\text{ km})(0.866) = \boxed{34.6\text{ km}}$$



Example 3.5 Taking a Hike

(B) Determine the components of the hiker's resultant displacement $\vec{\boldsymbol{R}}$ for the trip. Find an expression for $\vec{\mathbf{R}}$ in terms of unit vectors.

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(B) Determine the components of the hiker's resultant displacement $\vec{\boldsymbol{R}}$ for the trip. Find an expression for $\vec{\mathbf{R}}$ in terms of unit vectors.

$$R_x = A_x + B_x = 17.7 \text{ km} + 20.0 \text{ km} = \boxed{37.7 \text{ km}}$$

$$R_y = A_y + B_y = -17.7 \text{ km} + 34.6 \text{ km} = \boxed{16.9 \text{ km}}$$

In unit–vector form, we can write the total displacement as

$$\mathbf{R} = \boxed{(37.7\hat{\mathbf{i}} + 16.9\hat{\mathbf{j}}) \text{ km}}$$

WHAT IF? After reaching the tower, the hiker wishes to return to her car along a single straight line. What are the components of the vector representing this hike? What should the direction of the hike be?

Answer The desired vector $\vec{\mathbf{R}}_{\text{car}}$ is the negative of vector $\vec{\mathbf{R}}$:

$$\vec{\mathbf{R}}_{\text{car}} = -\vec{\mathbf{R}} = (-37.7\hat{\mathbf{i}} - 17.0\hat{\mathbf{j}}) \text{ km}$$

The direction is found by calculating the angle that the vector makes with the x axis:

$$\tan\theta = \frac{R_{\text{car},y}}{R_{\text{car},x}} = \frac{-17.0 \text{ km}}{-37.7 \text{ km}} = 0.450$$

which gives an angle of $\theta = 204.2°$, or $24.2°$ south of west.

Suggested Problems

Chapter 3: 1, 4, 19, 21, 27, 30, 31, 33, 39, 49, 50