Phys 103

Chapter 10

Rotation of a Rigid Object About a Fixed Axis

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LECTURE OUTLINE

10.1 Angular Position, Velocity, and Acceleration

10.2 Rotational Kinematics:

Rotational Motion with Constant Angular Acceleration

10.3 Angular and Linear Quantities

10.4 Rotational Kinetic Energy

10.5 Calculation of Moments of Inertia

10.6 Torque

10.7 Relationship Between Torque and Angular Acceleration

10.8 Work, Power, and Energy in Rotational Motion

Introduction

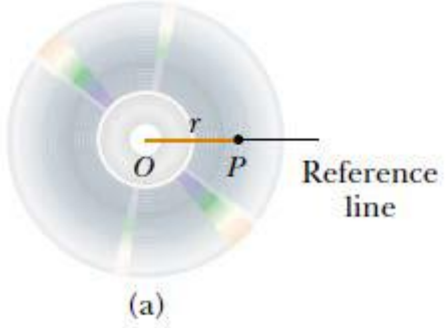
When an extended object such as a wheel rotates about its axis, the motion cannot be analyzed by treating the object as a particle because at any given time different parts of the object have different linear velocities and linear accelerations. We can, however, analyze the motion by considering an extended object to be composed of a collection of particles, each of which has its own linear velocity and linear acceleration.

In dealing with a rotating object, analysis is greatly simplified by assuming that the object is rigid. A rigid object is one that is nondeformable—that is, the relative locations of all particles of which the object is composed remain constant. All real objects are deformable to some extent; however, our rigid-object model is useful in many situations in which deformation is negligible.

10.1 Angular Position, Velocity, and Acceleration

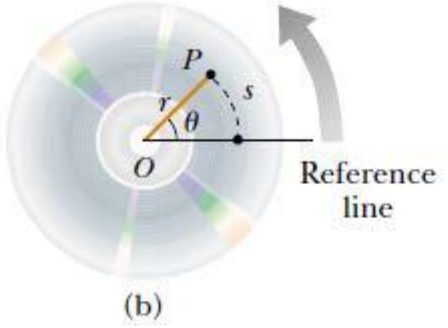
Angular Position

Consider a particle at P is at a fixed distance r from the origin and rotates about it in a circle of radius r.



The particle moves through an arc of length s, as in Figure . The arc length *s* is related to the angle θ through the relationship:

$$s = r\theta \ \rightarrow \theta = \frac{s}{r}$$



Note the dimensions of θ in Equation $\theta = \frac{s}{r}$. Because θ is the ratio of an arc length and the radius of the circle, it is a pure number. However, we commonly give θ the artificial unit radian (rad).

10.1 Angular Position, Velocity, and Acceleration

Angular Speed

As a particle travels from position 1 to position 2 in a time interval Δ, the reference line of length *r* sweeps out an angle Δθ= θf-θi. This quantity Δθis defined as the ***angular displacement*** of the rigid object:

$$\Delta\theta = \theta_f - \theta_i$$

We define the average angular speed as:

$$\bar{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

the instantaneous angular speed is:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Angular speed has units of radians per second (rad/s)

10.1 Angular Position, Velocity, and Acceleration

Angular Acceleration

The average angular acceleration of a rotating rigid object is defined as:

$$\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

the instantaneous angular acceleration is defined as:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Angular acceleration has units of radians per second squared (rad/s2)

When a rigid object is rotating about a *fixed axis*, every particle on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration.

That is, the quantities θ, ω, and αcharacterize the rotational motion of the entire rigid object as well as individual particles in the object.

10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

In our study of linear motion, we found that the simplest form of motion to analyze is motion under constant linear acceleration.

Likewise, for rotational motion about a fixed axis, the simplest motion to analyze is motion under constant angular acceleration.

Rotational Motion about a fixed axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
${\omega_f}^2 = {\omega_i}^2 + 2\alpha(\theta_f - \theta_i)$	${v_f}^2 = {v_i}^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Notice that these expressions are of the same mathematical form as those for linear motion under constant linear acceleration with the change: $x \rightarrow \theta, \omega, a \rightarrow \alpha$

10.3 Angular and Linear Quantities

With $\theta_i = 0, \alpha = constant\ \omega_f = \omega_i + \alpha t, \theta_f = \boldsymbol{\omega_i t} + \frac{1}{2}\alpha \boldsymbol{t^2}\ \ and\ \ {\omega_f}^2 = {\omega_i}^2 + 2\alpha\theta_f$

We shall find relations between linear and angular quantities:

$$\because \omega_f = \omega_i + \alpha t$$

$$\because v = \frac{ds}{dt} = \frac{d(r\theta)}{dt} = r\frac{d\theta}{dt} = r\omega$$

$$\because a_t = \frac{dv}{dt} = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$\because a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2$$

$$\therefore a = \sqrt{{a_t}^2 + {a_c}^2} = \sqrt{(r\alpha)^2 + (r\omega^2)^2} = \sqrt{r^2\alpha^2 + r^2\omega^4}$$

$$\therefore a = r\sqrt{\alpha^2 + \omega^4}$$

a_t: tangential acceleration, a_c: central acceleration, a: total acceleration

10.3 Angular and Linear Quantities

Example 10.1 Rotating Wheel

- A wheel rotates with a constant angular acceleration of 3.50 rad/s^2.

(A) If the angular speed of the wheel is 2.00 rad/s at $t_i = 0$, through what angular displacement does the wheel rotate in 2.00 s?

$$\because \quad \theta_i = 0$$

$$\therefore \Delta\theta = \theta_f = (2)(2) + \frac{1}{2}(3.5)(2)^2 = 11\,rad = 630^o$$

(B) Through how many revolutions has the wheel turned during this time interval?

$$\Delta\theta = 630^o \left(\frac{1 \text{ rev}}{360^o}\right) = 1.75 \text{ rev}$$

(C) What is the angular speed of the wheel at t = 2.00 s?

$$\because \omega_f = \omega_i + \alpha t$$

$$\therefore \omega_f = 2 + (3.5)(2) = 9\,rad/s$$

10.4 Rotational Kinetic Energy

Let us consider an object as a collection of particles and assume that it rotates about a fixed z axis with an angular speed $\boldsymbol{\omega}$ If the mass of the ithparticle is $\boldsymbol{m_i}$ and its tangential speed is $\boldsymbol{v_i}$, its kinetic energy is:

$$K_i = \frac{1}{2} m_i {v_i}^2$$

$$\because v_i = r_i \omega$$

$$\therefore K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i {v_i}^2 = \frac{1}{2} \sum_i m_i {v_i}^2 = \frac{1}{2} \sum_i m_i (r_i \omega)^2$$

$$= \frac{1}{2} \sum_i m_i {r_i}^2 \omega^2 = \frac{1}{2} \left(\sum_i m_i {r_i}^2 \right) \omega^2$$

define the moment of inertia I as:

$$I = \left(\sum_i m_i {r_i}^2 \right), so \quad \therefore K_R = \frac{1}{2} I \, \omega^2$$

10.4 Rotational Kinetic Energy

Example 10.3 The Oxygen Molecule

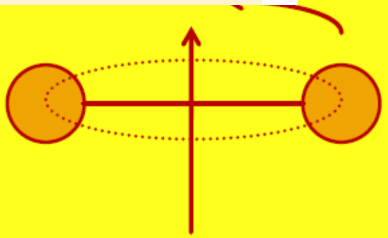
▶Consider an oxygen molecule (O_2) rotating in the xy plane about the z axis. The rotation axis passes through the center of the molecule, perpendicular to its length. The mass of each oxygen atom is $2.66\text{x}10\text{-}^{26}$ kg, and at room temperature the average separation between the two atoms is d = $1.21\text{x}10\text{-}^{10}$ m. (The atoms are modeled as particles.)

(A) *Calculate the moment of inertia of the molecule about the z axis.*

$$I = \sum_i m_i r_i^2 = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2 = \frac{md^2}{2} \qquad (1)$$

$$= \frac{(2.66 \times 10^{-26}\text{ kg})(1.21 \times 10^{-10}\text{ m})^2}{2} = 1.95 \times 10^{-46}\text{ kg}\cdot\text{m}^2$$

This is a very small number, consistent with the Minuscule masses and distances involved



10.4 Rotational Kinetic Energy

Example 10.4 Four Rotating Objects

▶Four tiny spheres are fastened to the ends of two rods of negligible mass lying in the xy plane. We shall assume that the radii of the spheres are small compared with the dimensions of the rods.

▶*(A) If the system rotates about the y axis with an angular speed ω, find the moment of inertia and the rotational kinetic energy about this axis.*

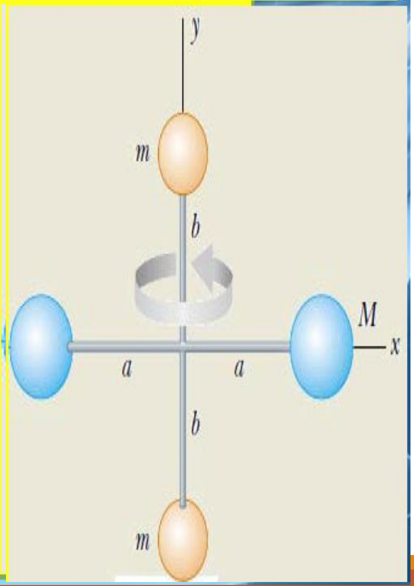
$$\because\ I_y = \sum_i m_i r_i^{\,2} = Ma^2 + Ma^2 + m(0) + m(0) = 2Ma^2 \quad (1)$$

$$\therefore K_R = \frac{1}{2} I_y \omega^2 = \frac{1}{2}(2Ma^2)\omega^2 = Ma^2\omega^2 \quad (2)$$

▶*(B) Same but in the xy plane about the z axis*

$$I_z = \sum_i m_i r_i^{\,2} = Ma^2 + Ma^2 + mb^2 + mb^2 = 2Ma^2 + 2\text{m}b^2$$

$$\therefore K_R = \frac{1}{2} I_z \omega^2 = \frac{1}{2}(2Ma^2 + 2\text{m}b^2)\omega^2 = (Ma^2 + 2\text{m}b^2)\omega^2$$



10.5 Calculation of Moments of Inertia

We can evaluate the moment of inertia of an extended rigid object by imagining the object to be divided into many small volume elements, each of which has mass Δm_i.

We use the definition $I = \sum_i \Delta m_i {r_i}^2$ and take the limit of this sum as $\Delta m_i \rightarrow 0$. In this limit, the sum becomes an integral over the volume of the object:

$$I = \lim_{\Delta m_i \rightarrow 0} \sum_i {r_i}^2 \Delta m_i = \int r^2 dm \quad \because \rho = \frac{m}{V}$$

where ρ is the density of the object and V is its volume. From this equation, the mass of a small element is $\mathrm{dm} = \rho dV \ so \quad I = \int \rho \, r^2 dV$

10.5 Calculation of Moments of Inertia

$$\rho = \frac{m}{V}$$ *volumetric mass density*

$$\sigma = \rho t$$ *surface mass density*

$$\lambda = \rho A$$ *linear mass density*

10.5 Calculation of Moments of Inertia

parallel-axis theorem

The calculation of moments of inertia about an arbitrary axis can be cumbersome, however, even for a highly symmetric object. Fortunately, use of an important theorem, called the parallel-axis theorem, often simplifies the calculation. Suppose the moment of inertia about an axis through the center of mass of an object is I_{CM}. The parallel-axis theorem states that the moment of inertia about any axis parallel to and a distance D away from this axis is

$$I = I_{CM} + MD^2$$

To prove the parallel-axis theorem, uppose that an object rotates in the xy plane about the z axis, as shown in Figure 10.12, and that the coordinates of the center of mass are x_{CM}, y_{CM}. Let the mass element dm have coordinates x, y. Because this element is a distance $r = \sqrt{x^2 + y^2}$ from the z axis, the moment of inertia about the z axis is $I = \int r^2 dm = \int (x^2 + y^2) dm$

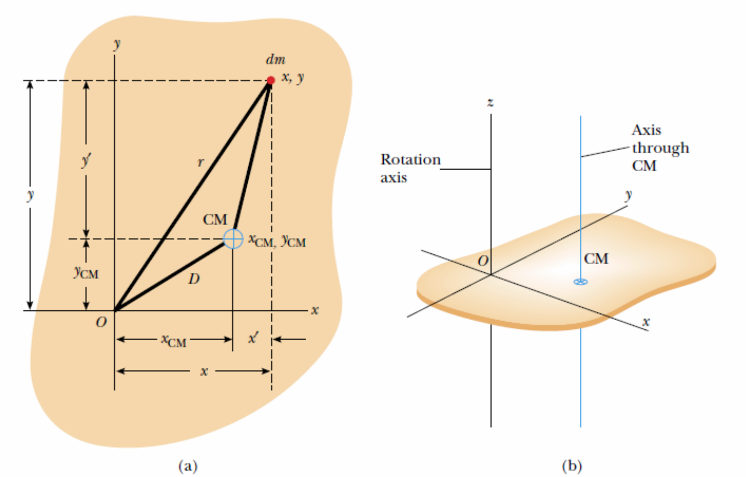


Figure 10.12 (a) The parallel-axis theorem: if the moment of inertia about an axis perpendicular to the figure through the center of mass is I_{CM}, then the moment of inertia about the z axis is $I_z = I_{CM} + MD^2$. (b) Perspective drawing showing the z axis (the axis of rotation) and the parallel axis through the CM.

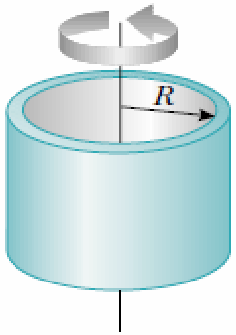
10.5 Calculation of Moments of Inertia

Table 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

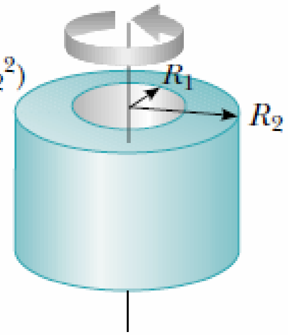
Hoop or thin cylindrical shell

$I_{CM} = MR^2$



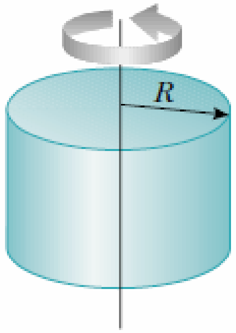
Hollow cylinder

$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$



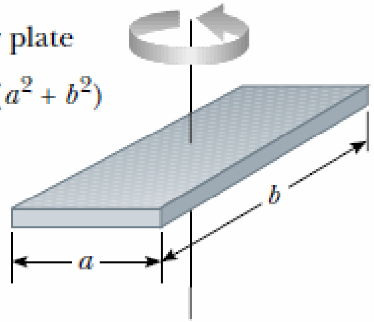
Solid cylinder or disk

$I_{CM} = \frac{1}{2}MR^2$



Rectangular plate

$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



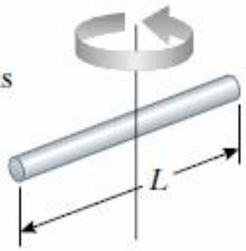
10.5 Calculation of Moments of Inertia

Table 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

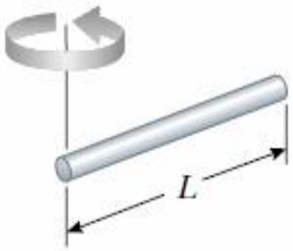
Long thin rod with rotation axis through center

$I_{CM} = \frac{1}{12} ML^2$



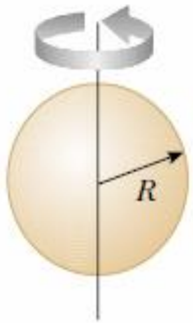
Long thin rod with rotation axis through end

$I = \frac{1}{3} ML^2$



Solid sphere

$I_{CM} = \frac{2}{5} MR^2$



Thin spherical shell

$I_{CM} = \frac{2}{3} MR^2$

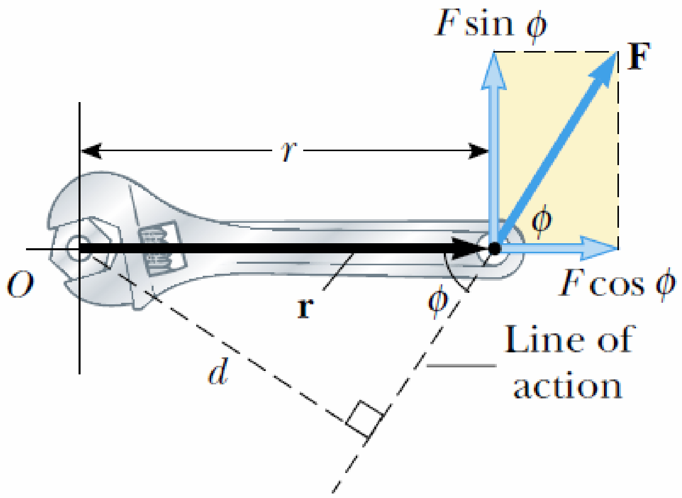
10.6 Torque

The tendency of a force to rotate an object about some axis is measured by a vector quantity called torque τ.

Torque is a vector, but we will consider only its magnitude here and explore its vector nature in Chapter 11.

$$\tau = r \times F = rF \sin\phi = Fd$$

where r is the distance between the pivot point and the point of application of F and d is the perpendicular distance from the pivot point to the line of action of F. (The line of action of a force is an imaginary line extending out both ends of the vector representing the force.



10.7 Relationship Between Torque and Angular Acceleration

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force $\mathbf{F}_t$ and a radial force $\mathbf{F}_r$, as shown in Figure 10.16. The tangential force provides a tangential acceleration $\mathbf{a}_t$, and

$$F_t = ma_t$$

The magnitude of the torque about the center of the circle due to $\mathbf{F}_t$ is

$$\tau = F_t r = (ma_t)r$$

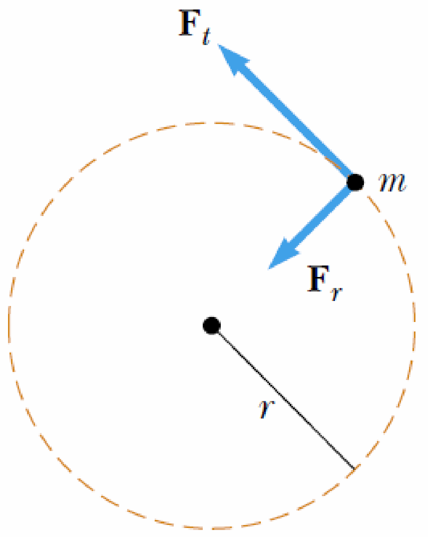
Because the tangential acceleration is related to the angular acceleration through the relationship $a_t = r\alpha$ (see Eq. 10.11), the torque can be expressed as

$$\tau = (mr\alpha)r = (mr^2)\alpha$$

Recall from Equation 10.15 that mr^2 is the moment of inertia of the particle about the z axis passing through the origin, so that

$$\tau = I\alpha \qquad (10.20)$$

That is, **the torque acting on the particle is proportional to its angular acceleration,** and the proportionality constant is the moment of inertia. Note that $\tau = I\alpha$ is the rotational analog of Newton's second law of motion, $F = ma$.



10.7 Relationship Between Torque and Angular Acceleration

Now let us extend this discussion to a rigid object of arbitrary shape rotating about a fixed axis, as in Figure 10.17. The object can be regarded as an infinite number of mass elements dm of infinitesimal size. If we impose a Cartesian coordinate system on the object, then each mass element rotates in a circle about the origin, and each has a tangential acceleration $\mathbf{a}_t$ produced by an external tangential force $d\mathbf{F}_t$. For any given element, we know from Newton's second law that

$$dF_t = (dm)\, a_t$$

The torque $d\tau$ associated with the force $d\mathbf{F}_t$ acts about the origin and is given by

$$d\tau = r\, dF_t = a_t r\, dm$$

Because $a_t = r\alpha$, the expression for $d\tau$ becomes

$$d\tau = \alpha r^2\, dm$$

Although each mass element of the rigid object may have a different linear acceleration $\mathbf{a}_t$, they all have the *same* angular acceleration α. With this in mind, we can integrate the above expression to obtain the net torque $\Sigma\tau$ about O due to the external forces:

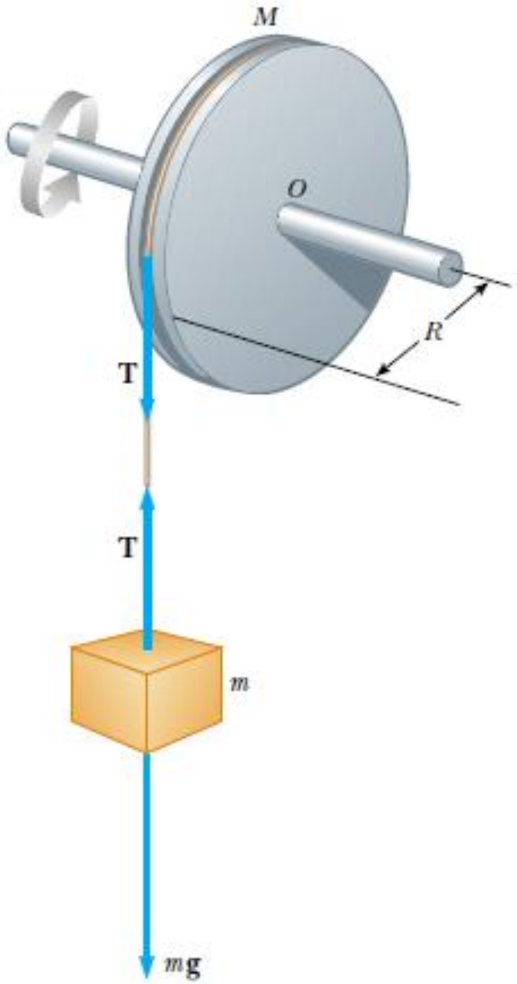
$$\sum \tau = \int \alpha r^2\, dm = \alpha \int r^2\, dm$$

where α can be taken outside the integral because it is common to all mass elements. From Equation 10.17, we know that $\int r^2\, dm$ is the moment of inertia of the object about the rotation axis through O, and so the expression for $\Sigma\tau$ becomes

$$\sum \tau = I\alpha \qquad (10.21)$$

Example 10.12 Angular Acceleration of a Wheel

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless horizontal axle, as in Figure 10.20. A light cord wrapped around the wheel supports an object of mass m. Calculate the angular acceleration of the wheel, the linear acceleration of the object, and the tension in the cord.



$$\sum \tau = I\alpha = TR \qquad \sum F_y = mg - T = ma$$

(1) $$\alpha = \frac{TR}{I}$$

(2) $$a = \frac{mg - T}{m}$$

(3) $$a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m}$$

(5) $$a = \boxed{\frac{g}{1 + (I/mR^2)}}$$

(4) $$T = \boxed{\frac{mg}{1 + (mR^2/I)}}$$

$$\alpha = \frac{a}{R} = \boxed{\frac{g}{R + (I/mR)}}$$

10.8 Work, Power, and Energy in Rotational Motion

The work done by F on the object as it rotates through an infinitesimal distance $ds = r\,d\theta$ is

$$dW = \mathbf{F} \cdot d\mathbf{s} = (F \sin \phi) r\, d\theta$$

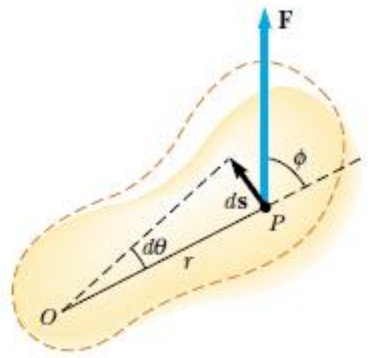


Figure 10.22 A rigid object rotates about an axis through O under the action of an external force $\mathbf{F}$ applied at P.

where F sin ϕ is the tangential component of F, or, in other words, the component of the force along the displacement. Note that the radial component of F does no work because it is perpendicular to the displacement.

Because the magnitude of the torque due to F about *O* is defined as
r F sin ϕ

$$dW = \tau \, d\theta$$

The rate at which work is being done by F as the object rotates about the fixed axis through the angle $d\theta$ in a time interval *dt* is

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

dW / *dt* is the instantaneous power P

$$\mathcal{P} = \frac{dW}{dt} = \tau\omega$$

To show that this is in fact the case, let us begin with $\sum\tau = I\alpha$. Using the chain rule from calculus, we can express the resultant torque as

$$\sum \tau = I\alpha = I\,\frac{d\omega}{dt} = I\,\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\,\frac{d\omega}{d\theta}\,\omega$$

Rearranging this expression and noting that $\sum\tau\, d\theta = dW$, we obtain

$$\sum \tau\, d\theta = dW = I\omega\, d\omega$$

Integrating this expression, we obtain for the total work done by the net external force acting on a rotating system

$$\sum W = \int_{\omega_i}^{\omega_f} I\omega\, d\omega = \tfrac{1}{2}I\omega_f^{\,2} - \tfrac{1}{2}I\omega_i^{\,2} \qquad (10.24)$$

where the angular speed changes from ω_i to ω_f. That is, the **work–kinetic energy theorem for rotational motion** states that

The net work done by external forces in rotating a symmetric rigid object about a fixed axis equals the change in the object's rotational energy.

Table 10.3

Useful Equations in Rotational and Linear Motion

Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Net torque $\Sigma\tau = I\alpha$	Net force $\Sigma F = ma$
If α = constant $\begin{cases} \omega_f = \omega_i + \alpha t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If a = constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$	Work $W = \int_{x_i}^{x_f} F_x \, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

Example 10.14 Rotating Rod Revisited

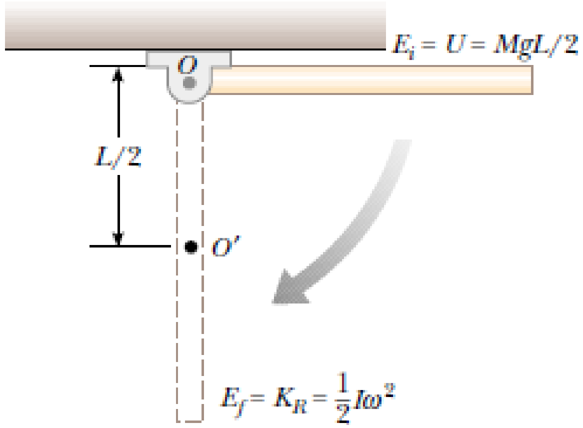
A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end (Fig 10.24). The rod is released from rest in the horizontal position.

(A) What is its angular speed when it reaches its lowest position?

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}I\omega^2 + 0 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2 = 0 + \frac{1}{2}MgL$$

$$\omega = \boxed{\sqrt{\frac{3g}{L}}}$$



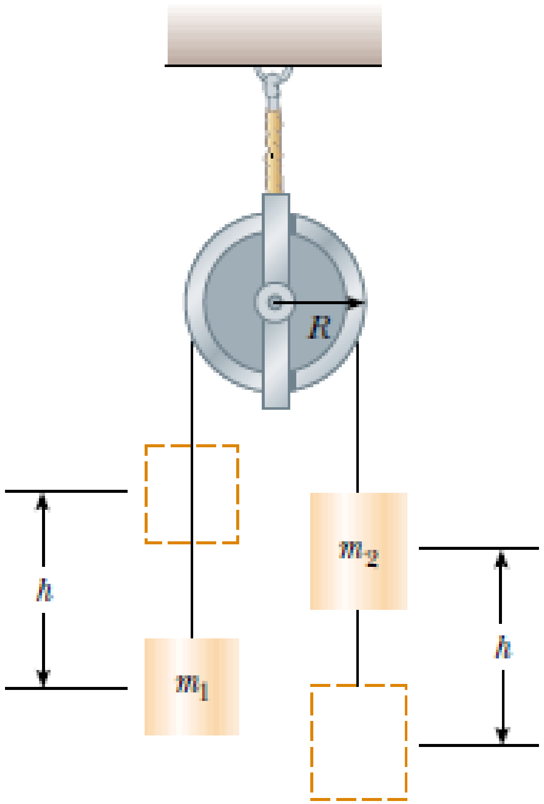
(B) Determine the tangential speed of the center of mass and the tangential speed of the lowest point on the rod when it is in the vertical position.

$$v_{CM} = r\omega = \frac{L}{2}\omega = \boxed{\frac{1}{2}\sqrt{3gL}}$$

$$v = 2v_{CM} = \boxed{\sqrt{3gL}}$$

Example 10.15 Energy and the Atwood Machine

Consider two cylinders having different masses m_1 and m_2, connected by a string passing over a pulley, as shown in Figure 10.25. The pulley has a radius R and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the linear speeds of the cylinders after cylinder 2 descends through a distance h, and the angular speed of the pulley at this time.



potential energy. Because $K_i = 0$ (the system is initially at rest), we have

$$K_f + U_f = K_i + U_i$$

$$\left(\tfrac{1}{2}m_1 v_f^2 + \tfrac{1}{2}m_2 v_f^2 + \tfrac{1}{2}I\omega_f^2\right) + (m_1 gh - m_2 gh) = 0 + 0$$

where v_f is the same for both blocks. Because $v_f = R\omega_f$, this expression becomes

$$\left(\tfrac{1}{2}m_1 v_f^2 + \tfrac{1}{2}m_2 v_f^2 + \tfrac{1}{2}\frac{I}{R^2}v_f^2\right) = (m_2 gh - m_1 gh)$$

$$\tfrac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f^2 = (m_2 gh - m_1 gh)$$

Solving for v_f, we find

$$v_f = \boxed{\left[\frac{2(m_2 - m_1)gh}{[m_1 + m_2 + (I/R^2)]}\right]^{1/2}}$$

The angular speed of the pulley at this instant is

$$\omega_f = \frac{v_f}{R} = \boxed{\frac{1}{R}\left[\frac{2(m_2 - m_1)gh}{(m_1 + m_2 + (I/R^2))}\right]^{1/2}}$$

PROBLEMS

Section 10.1 Angular Position, Velocity, and Acceleration

1. During a certain period of time, the angular position of a swinging door is described by $\boldsymbol{\theta} = 5 + 10t + 2t^2$, where $\boldsymbol{\theta}$ is in radians and t is in seconds. Determine the angular position, angular speed, and angular acceleration of the door (a) at $t = 0$ (b) at $t = 3.00$ s.

SOLUTIONS TO PROBLEM:

(a)

$$\theta\big|_{t=0} = \boxed{5.00 \text{ rad}}$$

$$\omega\big|_{t=0} = \left.\frac{d\theta}{dt}\right|_{t=0} = 10.0 + 4.00t\big|_{t=0} = \boxed{10.0 \text{ rad/s}}$$

$$\alpha_{t=0} = \left.\frac{d\omega}{dt}\right|_{t=0} = \boxed{4.00 \text{ rad/s}^2}$$

(b)

$$\theta\big|_{t=3.00\text{ s}} = 5.00 + 30.0 + 18.0 = \boxed{53.0 \text{ rad}}$$

$$\omega\big|_{t=3.00\text{ s}} = \left.\frac{d\theta}{dt}\right|_{t=3.00\text{ s}} = 10.0 + 4.00t\big|_{t=3.00\text{ s}} = \boxed{22.0 \text{ rad/s}}$$

$$\alpha\big|_{t=3.00\text{ s}} = \left.\frac{d\omega}{dt}\right|_{t=3.00\text{ s}} = \boxed{4.00 \text{ rad/s}^2}$$

PROBLEMS

Section 10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

3. A wheel starts from rest and rotates with constant angular acceleration to reach an angular speed of 12.0 rad/s in 3.00 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle in radians through which it rotates in this time.

SOLUTIONS TO PROBLEM:

(a) $$\alpha = \frac{\omega - \omega_i}{t} = \frac{12.0 \text{ rad/s}}{3.00 \text{ s}} = \boxed{4.00 \text{ rad/s}^2}$$

(b) $$\theta = \omega_i t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\left(4.00 \text{ rad/s}^2\right)(3.00 \text{ s})^2 = \boxed{18.0 \text{ rad}}$$

PROBLEMS

Section 10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

5. An electric motor rotating a grinding wheel at 100 rev/min is switched off. With constant negative angular acceleration of magnitude 2.00 rad/s2, (a) how long does it take the wheel to stop? (b) Through how many radians does it turn while it is slowing down?

SOLUTIONS TO PROBLEM:

$$\omega_i = \frac{100 \text{ rev}}{1.00 \text{ min}}\left(\frac{1 \text{ min}}{60.0 \text{ s}}\right)\left(\frac{2\pi \text{ rad}}{1.00 \text{ rev}}\right) = \frac{10\pi}{3} \text{ rad/s}, \ \omega_f = 0$$

(a) $$t = \frac{\omega_f - \omega_i}{\alpha} = \frac{0 - \frac{10\pi}{3}}{-2.00} \text{ s} = \boxed{5.24 \text{ s}}$$

(b) $$\theta_f = \overline{\omega} t = \left(\frac{\omega_f + \omega_i}{2}\right) t = \left(\frac{10\pi}{6} \text{ rad/s}\right)\left(\frac{10\pi}{6} \text{ s}\right) = \boxed{27.4 \text{ rad}}$$

PROBLEMS

Section 10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

6. A centrifuge in a medical laboratory rotates at an angular speed of 3 600 rev/min. When switched off, it rotates 50.0 times before coming to rest. Find the constant angular acceleration of the centrifuge.

SOLUTIONS TO PROBLEM:

$$\omega_i = 3\,600 \text{ rev/min} = 3.77 \times 10^2 \text{ rad/s}$$

$$\theta = 50.0 \text{ rev} = 3.14 \times 10^2 \text{ rad and } \omega_f = 0$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta$$

$$0 = \left(3.77 \times 10^2 \text{ rad/s}\right)^2 + 2\alpha\left(3.14 \times 10^2 \text{ rad}\right)$$

$$\alpha = \boxed{-2.26 \times 10^2 \text{ rad/s}^2}$$

PROBLEMS

Section 10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

8. A rotating wheel requires 3.00 s to rotate through 37.0 revolutions. Its angular speed at the end of the 3.00-s interval is 98.0 rad/s. What is the constant angular acceleration of the wheel?

SOLUTIONS TO PROBLEM:

$\theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$ and $\omega_f = \omega_i + \alpha t$ are two equations in two unknowns ω_i and α

$\omega_i = \omega_f - \alpha t$: $\qquad \theta_f - \theta_i = (\omega_f - \alpha t)t + \frac{1}{2}\alpha t^2 = \omega_f t - \frac{1}{2}\alpha t^2$

$$37.0 \text{ rev}\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 98.0 \text{ rad/s}(3.00 \text{ s}) - \frac{1}{2}\alpha(3.00 \text{ s})^2$$

$232 \text{ rad} = 294 \text{ rad} - (4.50 \text{ s}^2)\alpha$: $\qquad \alpha = \frac{61.5 \text{ rad}}{4.50 \text{ s}^2} = \boxed{13.7 \text{ rad/s}^2}$

PROBLEMS

Section 10.3 Angular and Linear Quantities

12. A racing car travels on a circular track of radius 250 m. If the car moves with a constant linear speed of 45.0 m/s, find (a) its angular speed and (b) the magnitude and direction of its acceleration.

SOLUTIONS TO PROBLEM:

(a) $$v = r\omega;\ \omega = \frac{v}{r} = \frac{45.0\ \text{m/s}}{250\ \text{m}} = \boxed{0.180\ \text{rad/s}}$$

(b) $$a_r = \frac{v^2}{r} = \frac{(45.0\ \text{m/s})^2}{250\ \text{m}} = \boxed{8.10\ \text{m/s}^2\ \text{toward the center of track}}$$

PROBLEMS

Section

13. A v... consta... $t = 0$, a... angle ... angula... accele...

...es with a ...t rest at ...s an ...(a) the ...tal ...point *P*.

SOLUT

Given $r = 1.00$ m, $\alpha = 4.00\ \text{rad/s}^2$, $\omega_i = 0$ and $\theta_i = 57.3° = 1.00$ rad

(a) $\omega_f = \omega_i + \alpha t = 0 + \alpha t$

At $t = 2.00$ s, $\omega_f = 4.00\ \text{rad/s}^2\,(2.00\ \text{s}) = \boxed{8.00\ \text{rad/s}}$

(b) $v = r\omega = 1.00\ \text{m}(8.00\ \text{rad/s}) = \boxed{8.00\ \text{m/s}}$

$|a_r| = a_c = r\omega^2 = 1.00\ \text{m}(8.00\ \text{rad/s})^2 = 64.0\ \text{m/s}^2$

$a_t = r\alpha = 1.00\ \text{m}\left(4.00\ \text{rad/s}^2\right) = 4.00\ \text{m/s}^2$

The magnitude of the total acceleration is:

$$a = \sqrt{a_r^2 + a_t^2} = \sqrt{\left(64.0\ \text{m/s}^2\right)^2 + \left(4.00\ \text{m/s}^2\right)^2} = \boxed{64.1\ \text{m/s}^2}$$

The direction of the total acceleration vector makes an angle ϕ with respect to the radius to point P:

$$\phi = \tan^{-1}\left(\frac{a_t}{a_c}\right) = \tan^{-1}\left(\frac{4.00}{64.0}\right) = \boxed{3.58°}$$

(c) $\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 = (1.00\ \text{rad}) + \frac{1}{2}\left(4.00\ \text{rad/s}^2\right)(2.00\ \text{s})^2 = \boxed{9.00\ \text{rad}}$

PROBLEMS

Section 10.3 Angular and Linear Quantities

16. A car accelerates uniformly from rest and reaches a speed of 22.0 m/s in 9.00 s. If the diameter of a tire is 58.0 cm, find (a) the number of revolutions the tire makes during this motion, assuming that no slipping occurs. (b) What is the final angular speed of a tire in revolutions per second?

SOLUTIONS TO PROBLEM:

(a) $$s = \bar{v}t = (11.0\ \text{m/s})(9.00\ \text{s}) = 99.0\ \text{m}$$

$$\theta = \frac{s}{r} = \frac{99.0\ \text{m}}{0.290\ \text{m}} = 341\ \text{rad} = \boxed{54.3\ \text{rev}}$$

(b) $$\omega_f = \frac{v_f}{r} = \frac{22.0\ \text{m/s}}{0.290\ \text{m}} = 75.9\ \text{rad/s} = \boxed{12.1\ \text{rev/s}}$$

PROBLEMS

Section 10.3 Angular and Linear Quantities

17. A disk 8.00 cm in radius rotates at a constant rate of 1 200 rev/min about its central axis. Determine (a) its angular speed, (b) the tangential speed at a point 3.00 cm from its center, (c) the radial acceleration of a point on the rim, and (d) the total distance a point on the rim moves in 2.00 s.

SOLUTIONS TO PROBLEM:

(a) $$\omega = 2\pi f = \frac{2\pi\,\text{rad}}{1\text{ rev}}\left(\frac{1\,200\text{ rev}}{60.0\text{ s}}\right) = \boxed{126\ \text{rad/s}}$$

(b) $$v = \omega r = (126\ \text{rad/s})(3.00\times 10^{-2}\ \text{m}) = \boxed{3.77\ \text{m/s}}$$

(c) $$a_c = \omega^2 r = (126)^2(8.00\times 10^{-2}) = 1\,260\ \text{m/s}^2 \text{ so } \mathbf{a}_r = \boxed{1.26\ \text{km/s}^2\ \text{toward the center}}$$

(d) $$s = r\theta = \omega r t = (126\ \text{rad/s})(8.00\times 10^{-2}\ \text{m})(2.00\ \text{s}) = \boxed{20.1\ \text{m}}$$

PROBLEMS

Section 10.3 Angular and Linear Quantities

18. A car traveling on a flat (unbanked) circular track accelerates uniformly from rest with a tangential acceleration of 1.70 m/s2. The car makes it one quarter of the way around the circle before it skids off the track. Determine the coefficient of static friction between the car and track from these data.

SOLUTIONS TO PROBLEM:

The force of static friction must act forward and then more and more inward on the tires, to produce both tangential and centripetal acceleration. Its tangential component is $m(1.70\ \text{m/s}^2)$. Its radially inward component is $\frac{mv^2}{r}$. This takes the maximum value

$$m\omega_f^2 r = mr(\omega_i^2 + 2\alpha\Delta\theta) = mr\left(0 + 2\alpha\frac{\pi}{2}\right) = m\pi r\alpha = m\pi a_t = m\pi(1.70\ \text{m/s}^2).$$

With skidding impending we have $\sum F_y = ma_y$, $+n - mg = 0$, $n = mg$

$$f_s = \mu_s n = \mu_s mg = \sqrt{m^2(1.70\ \text{m/s}^2)^2 + m^2\pi^2(1.70\ \text{m/s}^2)^2}$$

$$\mu_s = \frac{1.70\ \text{m/s}^2}{g}\sqrt{1+\pi^2} = \boxed{0.572}$$

PROBLEMS

Section 10.4 Rotational Kinetic Energy

20. Rigid rods of negligible mass lying along the *y* axis connect three particles (Fig. P10.20). If the system rotates about the *x* axis with an angular speed of 2.00 rad/s, find (a) the moment of inertia about the *x* axis and the total rotational kinetic energy evaluated from $\frac{1}{2}I\,\omega^2$ and (b) the tangential speed of each particle and the total kinetic energy evaluated from $\sum_i \frac{1}{2}m_i v_i{}^2$.

SOLUTIONS TO PROBLEM:

$m_1 = 4.00$ kg, $r_1 = |y_1| = 3.00$ m;

$m_2 = 2.00$ kg, $r_2 = |y_2| = 2.00$ m;

$m_3 = 3.00$ kg, $r_3 = |y_3| = 4.00$ m;

$\omega = 2.00$ rad/s about the x-axis

(a) $I_x = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$

$I_x = 4.00(3.00)^2 + 2.00(2.00)^2 + 3.00(4.00)^2 = \boxed{92.0\ \text{kg}\cdot\text{m}^2}$

$K_R = \frac{1}{2}I_x\omega^2 = \frac{1}{2}(92.0)(2.00)^2 = \boxed{184\ \text{J}}$

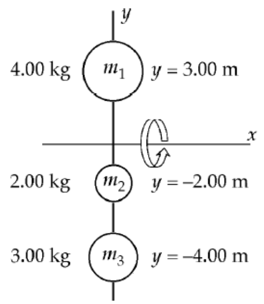


FIG. P10.20

(b) $v_1 = r_1\omega = 3.00(2.00) = \boxed{6.00\ \text{m/s}}$ $\qquad K_1 = \frac{1}{2}m_1 v_1^2 = \frac{1}{2}(4.00)(6.00)^2 = 72.0$ J

$v_2 = r_2\omega = 2.00(2.00) = \boxed{4.00\ \text{m/s}}$ $\qquad K_2 = \frac{1}{2}m_2 v_2^2 = \frac{1}{2}(2.00)(4.00)^2 = 16.0$ J

$v_3 = r_3\omega = 4.00(2.00) = \boxed{8.00\ \text{m/s}}$ $\qquad K_3 = \frac{1}{2}m_3 v_3^2 = \frac{1}{2}(3.00)(8.00)^2 = 96.0$ J

$K = K_1 + K_2 + K_3 = 72.0 + 16.0 + 96.0 = \boxed{184\ \text{J}} = \frac{1}{2}I_x\omega^2$

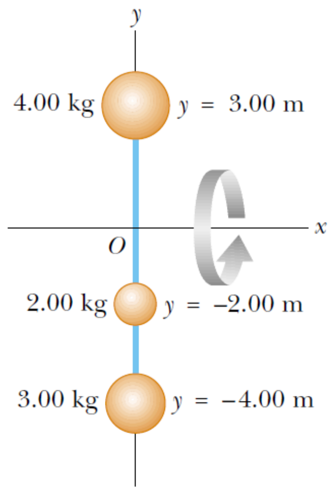


Figure P10.20

PROBLEMS

Section 10.4 Rotational Kinetic Energy

21. The four particles in Figure P10.21 are connected by rigid rods of negligible mass. The origin is at the center of the rectangle. If the system rotates in the *xy* plane about the *z* axis with an angular speed of 6.00 rad/s, calculate (a) the moment of inertia of the system about the *z* axis and (b) the rotational kinetic energy of the system.

SOLUTIONS TO PROBLEM:

(a) $I = \sum_j m_j r_j^2$

In this case,

$$r_1 = r_2 = r_3 = r_4$$

$$r = \sqrt{(3.00\text{ m})^2 + (2.00\text{ m})^2} = \sqrt{13.0}\text{ m}$$

$$I = \left[\sqrt{13.0}\text{ m}\right]^2 [3.00 + 2.00 + 2.00 + 4.00]\text{ kg}$$

$$= \boxed{143\text{ kg}\cdot\text{m}^2}$$

(b) $K_R = \frac{1}{2} I\omega^2 = \frac{1}{2}(143\text{ kg}\cdot\text{m}^2)(6.00\text{ rad/s})^2$

$$= \boxed{2.57 \times 10^3\text{ J}}$$

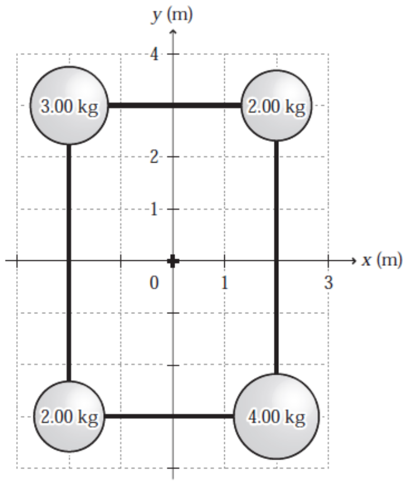


FIG. P10.21

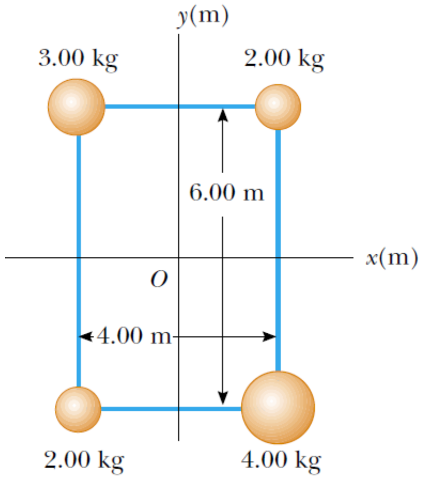


Figure P10.21

PROBLEMS

Section 10.6 Torque

31. Find the net torque on the wheel in Figure P10.31 about the axle through *O* if *a* =10.0 cm and *b*= 25.0 cm.

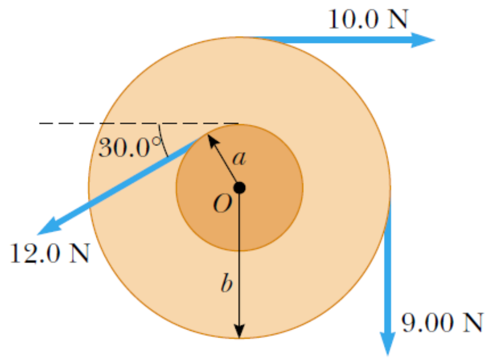


Figure P10.31

SOLUTIONS TO PROBLEM:

$$\sum \tau = 0.100 \text{ m}(12.0 \text{ N}) - 0.250 \text{ m}(9.00 \text{ N}) - 0.250 \text{ m}(10.0 \text{ N}) = \boxed{-3.55 \text{ N}\cdot\text{m}}$$

The thirty-degree angle is unnecessary information.

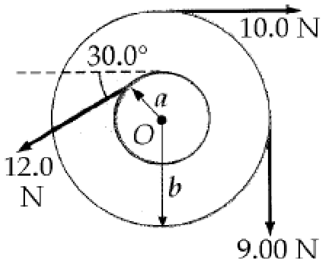


FIG. P10.31

PROBLEMS

Section 10.7 Relationship between Torque and Angular Acceleration

35. A model airplane with mass 0.750 kg is tethered by a wire so that it flies in a circle 30.0 m in radius. The airplane engine provides a net thrust of 0.800 N perpendicular to the tethering wire. (a) Find the torque the net thrust produces about the center of the circle. (b) Find the angular acceleration of the airplane when it is in level flight.

(c) Find the linear acceleration of the airplane tangent to its flight path.

SOLUTIONS TO PROBLEM:

$m = 0.750\ \text{kg}$, $F = 0.800\ \text{N}$

(a) $$\tau = rF = 30.0\ \text{m}(0.800\ \text{N}) = \boxed{24.0\ \text{N}\cdot\text{m}}$$

(b) $$\alpha = \frac{\tau}{I} = \frac{rF}{mr^2} = \frac{24.0}{0.750(30.0)^2} = \boxed{0.035\,6\ \text{rad/s}^2}$$

(c) $$a_t = \alpha r = 0.035\,6(30.0) = \boxed{1.07\ \text{m/s}^2}$$

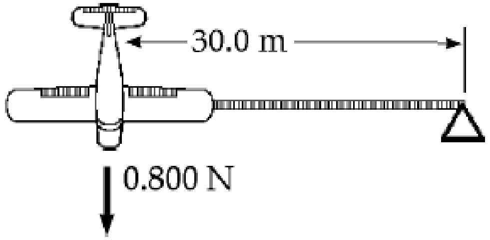


FIG. P10.35

PROBLEMS

Section 10.7 Relationship between Torque and Angular Acceleration

37. A block of mass *m*1=2 kg and a block of mass *m*2= 6 kg are connected by a massless string over a pulley in the shape of a solid disk having radius *R* =0.250 m and mass *M* =10 kg. These blocks are allowed to move on a fixed block-wedge of angle $\boldsymbol{\theta}$=30.0° as in Figure P10.37. The coefficient of kinetic friction is 0.360 for both blocks. Draw free-body diagrams of both blocks and of the pulley. Determine (a) the acceleration of the two blocks and (b) the tensions in the string on both sides of the pulley.

SOLUTIONS TO PROBLEM:

For m_1,

$$\sum F_y = ma_y: \quad +n - m_1 g = 0$$
$$n_1 = m_1 g = 19.6 \text{ N}$$
$$f_{k1} = \mu_k n_1 = 7.06 \text{ N}$$

$$\sum F_x = ma_x: \quad -7.06 \text{ N} + T_1 = (2.00 \text{ kg})a \qquad (1)$$

For the pulley,

$$\sum \tau = I\alpha: \quad -T_1 R + T_2 R = \frac{1}{2} MR^2 \left(\frac{a}{R}\right)$$
$$-T_1 + T_2 = \frac{1}{2}(10.0 \text{ kg})a$$
$$-T_1 + T_2 = (5.00 \text{ kg})a \qquad (2)$$

For m_2, $\quad +n_2 - m_2 g \cos\theta = 0$

$$n_2 = 6.00 \text{ kg}(9.80 \text{ m/s}^2)(\cos 30.0°)$$
$$= 50.9 \text{ N}$$

$$f_{k2} = \mu_k n_2$$
$$= 18.3 \text{ N}: \quad -18.3 \text{ N} - T_2 + m_2 \sin\theta = m_2 a$$
$$-18.3 \text{ N} - T_2 + 29.4 \text{ N} = (6.00 \text{ kg})a \qquad (3)$$

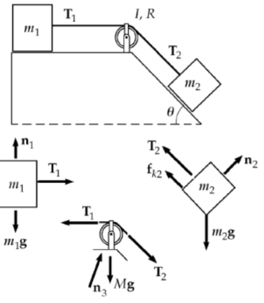


FIG. P10.37

(a) Add equations (1), (2), and (3):

$$-7.06 \text{ N} - 18.3 \text{ N} + 29.4 \text{ N} = (13.0 \text{ kg})a$$
$$a = \frac{4.01 \text{ N}}{13.0 \text{ kg}} = \boxed{0.309 \text{ m/s}^2}$$

(b) $T_1 = 2.00 \text{ kg}(0.309 \text{ m/s}^2) + 7.06 \text{ N} = \boxed{7.67 \text{ N}}$

$T_2 = 7.67 \text{ N} + 5.00 \text{ kg}(0.309 \text{ m/s}^2) = \boxed{9.22 \text{ N}}$

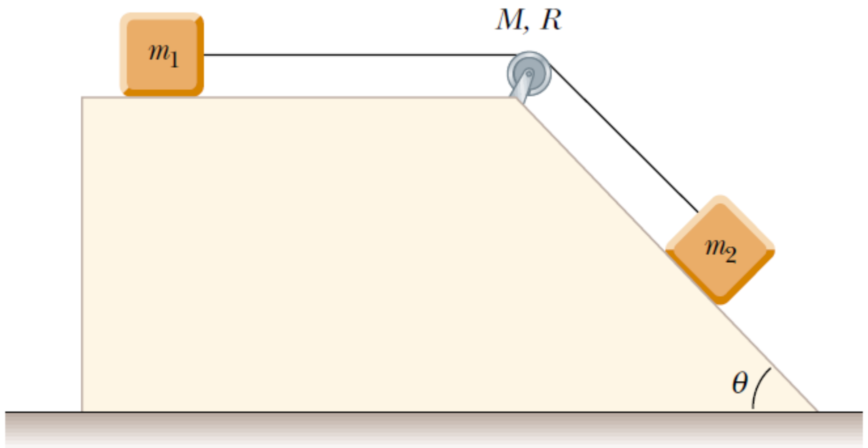


Figure P10.37

PROBLEMS

Section 10.8 Work, Power, and Energy in Rotational Motion

46. A 15.0-kg object and a 10.0-kg object are suspended, joined by a cord that passes over a pulley with a radius of 10.0 cm and a mass of 3.00 kg (Fig. P10.46). The cord has a negligible mass and does not slip on the pulley. The pulley rotates on its axis without friction. The objects start from rest 3.00 m apart. Treat the pulley as a uniform disk, and determine the speeds of the two objects as they pass each other.

SOLUTIONS TO PROBLEM:

Choose the zero gravitational potential energy at the level where the masses pass.

$$K_f + U_{gf} = K_i + U_{gi} + \Delta E$$

$$\frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2 + \frac{1}{2}I\omega^2 = 0 + m_1 g h_{1i} + m_2 g h_{2i} + 0$$

$$\frac{1}{2}(15.0 + 10.0)v^2 + \frac{1}{2}\left[\frac{1}{2}(3.00)R^2\right]\left(\frac{v}{R}\right)^2 = 15.0(9.80)(1.50) + 10.0(9.80)(-1.50)$$

$$\frac{1}{2}(26.5\ \text{kg})v^2 = 73.5\ \text{J} \Rightarrow v = \boxed{2.36\ \text{m/s}}$$

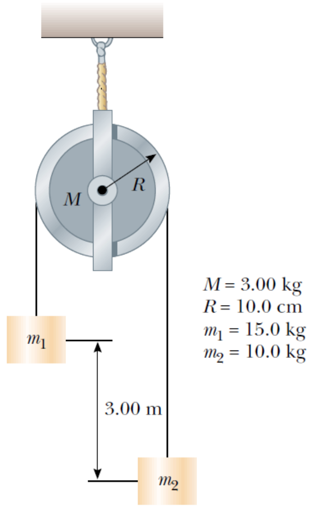


Figure P10.46

PROBLEMS

Additional Problems

70. The reel shown in Figure P10.70 has radius *R* and moment of inertia *I*. One end of the block of mass *m* is connected to a spring of force constant *k*, and the other end is fastened to a cord wrapped around the reel. The reel axle and the incline are frictionless. The reel is wound counterclockwise so that the spring stretches a distance *d* from its unstretched position and is then released from rest. (a) Find the angular speed of the reel when the spring is again unstretched. (b) Evaluate the angular speed numerically at this point if *I* =1 kg ·m2, *R* =0.3 m, *k* =50 N/m, *m* =0.5 kg, *d* =0.2 m, and ***θ***=37.0°.

SOLUTIONS

(a)

$$W = \Delta K + \Delta U$$

$$W = K_f - K_i + U_f - U_i$$

$$0 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 - mgd\sin\theta - \frac{1}{2}kd^2$$

$$\frac{1}{2}\omega^2\left(I + mR^2\right) = mgd\sin\theta + \frac{1}{2}kd^2$$

$$\boxed{\omega = \sqrt{\frac{2mgd\sin\theta + kd^2}{I + mR^2}}}$$

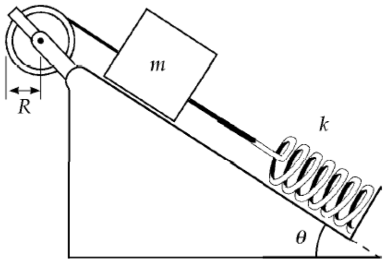


FIG. P10.70

(b)

$$\omega = \sqrt{\frac{2(0.500\text{ kg})(9.80\text{ m/s}^2)(0.200\text{ m})(\sin 37.0°) + 50.0\text{ N/m}(0.200\text{ m})^2}{1.00\text{ kg}\cdot\text{m}^2 + 0.500\text{ kg}(0.300\text{ m})^2}}$$

$$\omega = \sqrt{\frac{1.18 + 2.00}{1.05}} = \sqrt{3.04} = \boxed{1.74\text{ rad/s}}$$

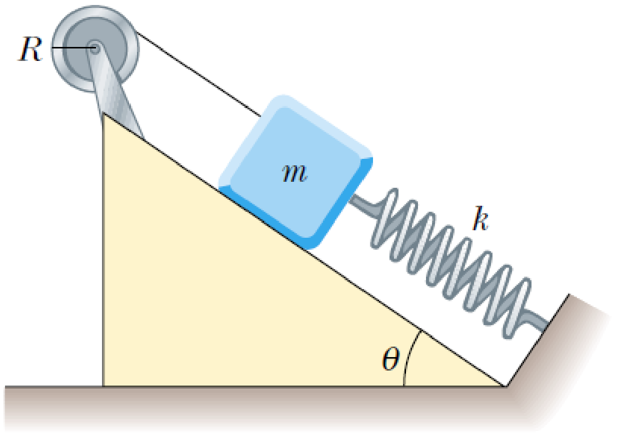


Figure P10.70

PROBLEMS

Additional Problems

71. Two blocks, as shown in Figure P10.71, are connected by a string of negligible mass passing over a pulley of radius 0.250 m and moment of inertia *I*. The block on the frictionless incline is moving up with a constant acceleration of 2.00 m/s2. (a) Determine *T*1 and *T*2, the tensions in the two parts of the string. (b) Find the moment of inertia of the pulley.

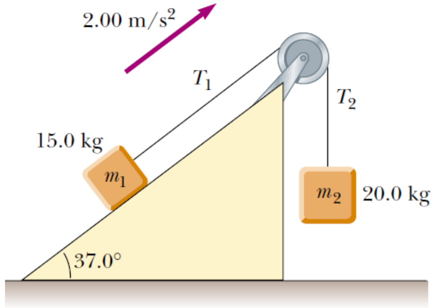


Figure P10.71

SOLUTIONS TO PROBLEM:

(a) $m_2 g - T_2 = m_2 a$

$T_2 = m_2(g-a) = 20.0\text{ kg}\left(9.80\text{ m/s}^2 - 2.00\text{ m/s}^2\right) = \boxed{156\text{ N}}$

$T_1 - m_1 g\sin 37.0° = m_1 a$

$T_1 = (15.0\text{ kg})(9.80\sin 37.0° + 2.00)\text{ m/s}^2 = \boxed{118\text{ N}}$

(b) $(T_2 - T_1)R = I\alpha = I\left(\dfrac{a}{R}\right)$

$$I = \frac{(T_2 - T_1)R^2}{a} = \frac{(156\text{ N} - 118\text{ N})(0.250\text{ m})^2}{2.00\text{ m/s}^2} = \boxed{1.17\text{ kg}\cdot\text{m}^2}$$

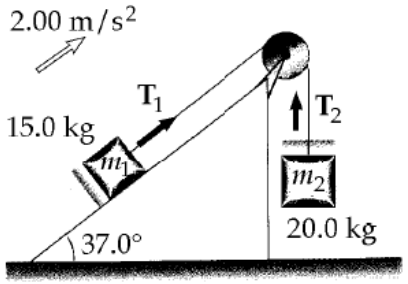


FIG. P10.71