c) **Show** that $\boldsymbol{B} \coloneqq \{\boldsymbol{t^2+2, -t+1, 2t-1}\}$ is a *basis* of the *real vector space* $\boldsymbol{P_2(t)}$ of all *polynomials* in *real variable* $\boldsymbol{t}$ having *degree* $\boldsymbol{\leq 2}$. Then **find** the *coordinate vector* of the *polynomial* $\boldsymbol{t^2+3t+3}$ *with respect to* the *basis* $\boldsymbol{B}$.

(b) Let $A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 2 & 4 & -1 & 1 & 0 & 2 \end{pmatrix}$.

(i) Find a basis B for the column space of the matrix A.

(ii) Show that B is a basis for $\mathbb{R}^3$.

Question 3 : [7 pts]

Consider the following inner product on $\mathbb{R}^3$:

$$\langle (\mathrm{x},\mathrm{y},\mathrm{z}),(\mathrm{x}',\mathrm{y}',\mathrm{z}')\rangle = 2\mathrm{xx}' + \mathrm{yy}' + \mathrm{zz}' + \mathrm{xy}' + \mathrm{x}'\mathrm{y}.$$

Let $u_1 = (-1,1,x)$, $u_2 = (-1,y,2)$ and $u_3 = (z,1,-2)$.

(a) Find the values of x so that $\|u_1\| = 1$.

(b) Find the values of x, y so that $\cos(\theta) = 0$, where θ the angle between u_1 and u_2.

(c) Find the values of x, y, z so that the set $K = \{u_1, u_2, u_3\}$ is orthogonal.

Question 5 : [5pts]

1. Let F be the subspace of the Euclidean inner product space $\mathbb{R}^3$ spanned by $\{v_1 = (1, 1, 0), v_2 = (1, 1, 1)\}$.
   Use Gram-Schmidt process to get an orthonormal basis of F.

2. Let $\mathbb{R}^3$ be the Euclidean inner product space and $u = (1, -1, 1), v = (2, 0, -2)$ in $\mathbb{R}^3$.

   (a) Find $\|u + v\|^2$.

   (b) Find $\cos\theta$, if θ is the angle between the vectors u and v.

7) If $S = \{1 + x, 2 + x, x^2\}$ is a basis for $\mathcal{P}_2$ and the coordinate vector of $p(x) \in \mathcal{P}_2$ is $(p)_S = (1, 2, 3)$, then $p(x)$ is

(a) $1 + 2x + 3x^2$	(b) $3 + 2x + 3x^2$	(c) $5 + 3x + 3x^2$	(d) None of the previous

8) If B is a 5×7 matrix and null $(B) = 3$, then null (B^T) equals

(a) 2	(b) 5	(c) 3	(d) 1

9) If $v_1 = (a, 1, 2, 6)$ and $v_2 = (2, 2a, 1, -1)$ are two orthogonal vectors, then

(a) $a = 1$	(b) $a = -1$	(c) $a = 0$	(d) None of the previous

10) If B is a 3×3 matrix with $\det B = 2$, then

(a) nullity $(B) = 2$, rank $(B) = 1$	(b) nullity $(B) = 0$, rank $(B) = 3$	(c) nullity $(B) = 3$, rank $(B) = 3$	(d) None of the previous

II) A) Let $S = \{v_1 = (1, 2, 2, 1), v_2 = (3, 6, 6, 3), v_3 = (4, 9, 9, 4), v_4 = (5, 8, 9, 5)\}$.

i) Find a subset of S that forms a basis for span(S).

ii) What is the dimension of span(S)?

B) Let $B = \{v_1 = (1,0,0), v_2 = (1,1,0), v_3 = (1,1,1,)\}$

i) Prove that B is a basis of $\mathbb{R}^3$.

ii) If $v = (0,-1,-1) \in \mathbb{R}^3$, find the coordinate vector $(v)_B$.

iii) Find the vector $w \in \mathbb{R}^3$, if its coordinate vector is $(w)_B = (2,1,-2)$.

[V] **A.** For $A = \begin{bmatrix} -1 & 1 & -2 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$, evaluate

(i) $rank(A)$

(ii) $nullity(A^T)$

Question 2 [Marks: 2.5]**:**

Let $\boldsymbol{B} = \{(\mathbf{1}, \mathbf{0}, \mathbf{0}), (\mathbf{0}, \mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{0}, \mathbf{1})\}$ and $\boldsymbol{C} = \{(\mathbf{1}, \mathbf{0}, \mathbf{0}), (\mathbf{0}, \mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{0}, \mathbf{1})\}$ be *bases* of *Euclidean space* $\mathbb{R}^3$ and $\boldsymbol{u} = (\mathbf{3}, \mathbf{2}, \mathbf{1})$. **Find** the *transition matrix* ${}_{\boldsymbol{C}}\boldsymbol{P}_{\boldsymbol{B}}$ and the *coordinate vector* $[\boldsymbol{u}]_{\boldsymbol{C}}$.

Question 3 [Marks: 2]**:**

Let $\boldsymbol{A}$ be 4×3 *matrix* with *rank* $(\boldsymbol{A}) = \mathbf{3}$. **Find** *nullity* $(\boldsymbol{A}^{\boldsymbol{T}})$**.**

Question 4 [Marks: 2]**:**

Explain! **why** the *function* $< (\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2, \boldsymbol{z}_2) > = \mathbf{2}\boldsymbol{x}_1\boldsymbol{y}_1 + \boldsymbol{y}_2 + \mathbf{2}\boldsymbol{z}_1\boldsymbol{z}_2$ is not an *inner product* on $\mathbb{R}^3$.

Which *one* of the following *vectors* in *Euclidean space* $\mathbb{R}^3$:

$$\boldsymbol{u_1} = (\mathbf{0}, \tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}),\ \boldsymbol{u_2} = (\mathbf{0}, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}),\ \boldsymbol{u_3} = (\mathbf{0}, \tfrac{1}{\sqrt{3}}, \tfrac{-1}{\sqrt{3}}) \text{ and } \boldsymbol{u_4} = (\mathbf{0}, \tfrac{-1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}})$$

is *orthogonal* to *both vectors* $\boldsymbol{v_1} = (\mathbf{1}, -\mathbf{1}, \mathbf{1})$ and $\boldsymbol{v_2} = (\mathbf{1}, \mathbf{0}, \mathbf{0})$?

(iii) If θ is the angle between the matrices $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$ with respect to the inner product $< A, B > = trace\ (AB^{\mathrm{T}})$, then $\cos\theta$ is:

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2}$ c) $\frac{15}{2\sqrt{30}}$ d) 0.

(iv) The value of k for which the vectors $\boldsymbol{u} := (u_1 = 2, u_2 = -4)$ and $\boldsymbol{v} := (v_1 = 1, v_2 = 3)$ in $\mathbb{R}^2$ are orthogonal with respect to the inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 2u_1v_1 + ku_2v_2$ is:

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2}$ c) $\frac{15}{2\sqrt{30}}$ d) $\frac{1}{3}$.

(v) If $B = \{(2,1), (-3,4)\}$ and $C = \{(1,1), (0,3)\}$ are bases of $\mathbb{R}^2$, then the transition matrix ${}_B\boldsymbol{P}_C$ from C to B is:

a) $\begin{bmatrix} 7/11 & 1/11 \\ 9/11 & 6/11 \end{bmatrix}$ b) $\begin{bmatrix} 7/11 & 9/11 \\ 1/11 & 6/11 \end{bmatrix}$ c) $\begin{bmatrix} 7/11 & 9/11 \\ 6/11 & 1/11 \end{bmatrix}$ d) $\begin{bmatrix} 9/11 & 7/11 \\ 1/11 & 6/11 \end{bmatrix}$.

II. Determine whether the following statements are true or false; justify your answer.

(iv) If u, v and w are vectors in an inner product space such that $\langle u,v \rangle = 3$, $\langle v,w \rangle = -5$, $\langle u,w \rangle = -1$ and $\|u\| = 2$, then $\langle u - 2w, 3u + v \rangle = 25$.

Question 2 [Marks: 2+2+2]**:** Consider the matrices $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & -2 \end{bmatrix}$. Then:

b) Show that $nullity(A) \neq nullity(B)$.

c) Find a basis for the null space $N(B)$.

Question 4: [Marks: 2+4]

a) Let u and v be any two vectors in an inner product space. Show that:

$$2(\|u\|^2 + \|v\|^2) = \|u+v\|^2 + \|u-v\|^2.$$

b) Let the set $B := \{\boldsymbol{u}_1 = (1,0,0), \boldsymbol{u}_2 = (3,1,-1), \boldsymbol{u}_3 = (0,3,1)\}$ be linearly independent in the Euclidean inner product space $\mathbb{R}^3$. Construct an orthonormal basis for $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on B.