King Saud University College of Science Mathematical Department Differential equations MATH204	جامعة الملك سعود King Saud University	Mid-Term 1 Summer Semester 1440 Time 1H30min Full Mark:25

Question 1: [4] Find and sketch the largest region of the xy-plane for which the initial value problem

$$\begin{cases} (1+y^3)\dfrac{dy}{dx} = x^2 \\ y(0) = 0 \end{cases}$$

has a unique solution.

Question 2: [4, 4]

a) Solve the initial value problem

$$\begin{cases} (xy^2+4x)dx + (8y-2x^2y)dy = 0, \qquad |x| < 2 \\ y(0) = 0 \end{cases}$$

b) Obtain the general solution of the differential equation

$$2cosx\frac{dy}{dx} + ysinx - (4x+5)^2y^3 = 0, \qquad x \in (\frac{-\pi}{2}, \frac{\pi}{2})$$

Question 3:[4, 4]

a) Solve the differential equation

$$(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$$

b) By using an appropriate substitution, or any other method, find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x}\ln(xy), \qquad x > 0, \qquad y > 0$$

Question 4: [5] A small metal bar, whose initial temperature was 20°C, is dropped into a large container of boiling water. One second later the object's temperature is 22°C. How long will it take the bar to reach 90°C? How long will it take the bar to reach 98°C?

Question 1: Find and sketch the largest region of the xy-plane for which the initial value problem

$$\begin{cases}(1+y^3)\dfrac{dy}{dx} = x^2 \\ y(0)=0\end{cases}$$

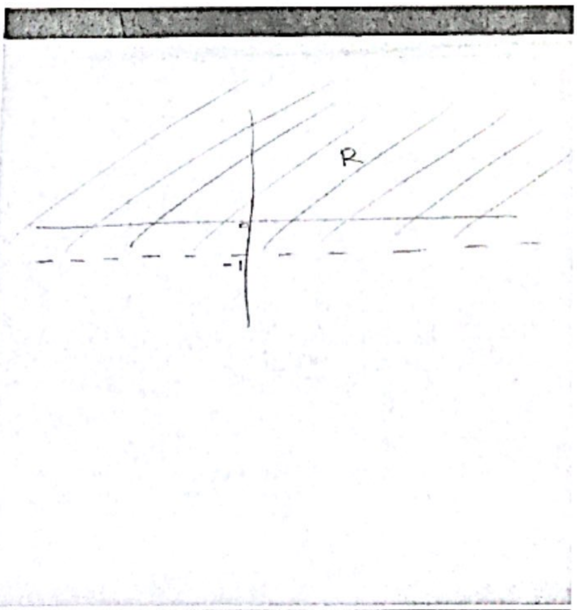
has a unique solution.

Solution:

$$\begin{cases}\dfrac{dy}{dx} = \dfrac{x^2}{(1+y^3)} = f(x,y) \\ y(0)=0\end{cases}$$

f *and* $\frac{df}{dy}$ are continuous on IR x (IR \{-1})

Since $0 > -1$ *then* the answer is IR x (-1,∞)



Question 2:

a) Solve the initial value problem

$$\begin{cases}(xy^2+4x)dx+(8y-2x^2y)dy=0, & |x|<2\\ y(0)=0\end{cases}$$

Solution:

$$x(y^2+4x)dx = 2y(x^2-4)dy$$

$$\frac{x}{x^2-4}dx = \frac{2y}{y^2+4}dy$$

⇦ Separable

$$\frac{1}{2}\int \frac{2x}{x^2-4}dx = \int \frac{2y}{y^2+4}dy$$

$$\frac{1}{2}\ln|x^2-4| = \ln|y^2+4| + c$$

$$at\ (0,0): \qquad \frac{1}{2}\ln|-4| = \ln|4| + c$$

$$c = -\frac{1}{2}\ln(4)$$

$$\frac{1}{2}ln|x^2-4| = ln|y^2+4| - \frac{1}{2}\ln(4)$$

$$\ln\left(\frac{y^2+4}{2}\right) = \ln\left(\sqrt{4-x^2}\right)$$

$$\frac{y^2+4}{2} = \sqrt{4-x^2}$$

$$\therefore\ \mathbf{y^2 = -4 + 2\sqrt{4-x^2}}$$

Question 2:

b) Obtain the general solution of the differential equation

$$2cosx\frac{dy}{dx} + ysinx - (4x+5)^2y^3 = 0, \qquad x \in (\frac{-\pi}{2}, \frac{\pi}{2})$$

Solution:

$$\frac{dy}{dx} + \frac{1}{2}tanx\, y = \frac{(4x+5)^2}{2cosx}\, y^3$$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{2}tanx\,\frac{1}{y^2} = \frac{(4x+5)^2}{2cosx}$$

$$Let\ u = y^{-2}$$

$$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{-2y^{-3}}\frac{du}{dx}$$

$$\therefore\ \frac{1}{y^3}\frac{1}{-2y^{-3}}\frac{du}{dx} + \frac{1}{2}tanx\, u = \frac{(4x+5)^2}{2cosx}$$

$$\frac{du}{dx} + (-tanx)u = \frac{(4x+5)^2}{-cosx}$$ ⬅ Linear

$$\mu(x,y) = e^{\int -tanx\, dx} = e^{\ln(cosx)} = cosx$$

$$cosx\,\frac{du}{dx} + (cosx)(-tanx)u = \frac{(4x+5)^2}{-cosx}(cosx)$$ ⬅ Multiply by (cosx)

$$P(x) = -sinx\ , \qquad Q(x) = (4x+5)^2$$

$$\frac{d}{dx}(\mu.u) = \mu.Q(x) \ \rightarrow\ \mu.u = \int \mu.Q(x)dx$$

$$cosx\,.u = -\int (4x+5)^2\, dx$$

$$= -\frac{1}{4}\int 4(4x+5)^2\, dx$$

$$cosx\,.u = -\frac{1}{4}\frac{(4x+5)^3}{3} + c$$

$$\therefore\ \frac{cosx}{y^2} = -\frac{1}{12}(4x+5)^3 + c$$

Question 3:

a) Solve the differential equation

$$(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$$

Solution:

$$M(x,y) = cosy + y\ cosx\ , \qquad \frac{\partial M}{\partial y} = -siny + cosx$$

$$N(x,y) = sinx - x\ siny\ , \qquad \frac{\partial N}{\partial x} = cosx - siny$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow D.E\ is\ Exact.$$

$$f(x,y) = \int M(x,y)dx = \int (cosy + y\ cosx)dx = x\ cosy + y\ sinx + \varphi(y)$$

$$\frac{\partial f}{\partial y} = -x\ siny + sinx + \varphi'(y)$$

$$\frac{\partial f}{\partial y} = N(x,y) \rightarrow -x\ siny + sinx + \varphi'(y) = sinx - x\ siny$$

$$\varphi'(y) = 0$$

$$\varphi(y) = c$$

$$\therefore f(x,y) \rightarrow \boldsymbol{x\ cosy + y\ sinx = c}$$

Question 3:

b) By using an appropriate substitution, or any other method, find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x}\ln(xy), \qquad x > 0, \qquad y > 0$$

Solution:

$$Let\ u = xy \ \rightarrow y = \frac{u}{x}$$

$$\frac{dy}{dx} = \frac{x\frac{du}{dx} - u}{x^2}$$

$$\frac{x\frac{du}{dx} - u}{x^2} = \frac{u}{x^2}\ln(u)$$

$$x\frac{du}{dx} - u = u\ln(u)$$

$$x\,\frac{du}{dx} = u\ln(u) + u$$

$$\frac{1}{u\ln(u) + u}\ du = \frac{1}{x}\ dx$$

$$\int \frac{1}{u\,ln(u) + u}\ du = \int \frac{1}{x}\ dx$$

$$\int \frac{1}{u\,[ln(u) + 1]}\ du = \int \frac{1}{x}\ dx$$

$$\int \frac{\frac{1}{u}}{ln(u) + 1}\ du = \int \frac{1}{x}\ dx$$

$$ln|\ln(u) + 1| = \ln|x| + c$$

$$\therefore \boldsymbol{D.E\ is\ \rightarrow ln|\ln(xy) + 1| = \ln x + c}$$

Question 4: A small metal bar, whose initial temperature was 20°C, is dropped into a large container of boiling water. One second later the object's temperature is 22°C. How long will it take the bar to reach 90°C? How long will it take the bar to reach 98°C?

Solution:

The temperature of the bar at time t is given by the formula

$$T(t) = T_S + Ce^{kt}$$

But T(0) = 20 , Ts = 100 and T(1) = 22 then

$20 = 100 + C \qquad C = -80$

$$22 = 100 - 80e^{k}$$

$$e^{k} = \frac{78}{80} = \frac{39}{40}$$

The time t taken by the bar to reach 90°C satisfies

$$90 = 100 - 80e^{kt} = 100 - 80\left(\frac{39}{40}\right)^{t}$$

$$\left(\frac{39}{40}\right)^{t} = \frac{1}{8}$$

$$t \approx 82$$

On the other hand, the time t taken by the bar to reach 98°C satisfies

$$98 = 100 - 80e^{kt} = 100 - 80\left(\frac{39}{40}\right)^{t}$$

$$\left(\frac{39}{40}\right)^{t} = \frac{2}{80}$$

$$t \approx 146$$