King Saud University
College of Sciences
Department of Mathematics

Second Mid Term Exam, S2 1443
M 380 – Stochastic Processes
Time: 90 minutes - Marks: 25

Answer the following questions.

Q1: [2+6]

(a) Define a martingale.

(b) Suppose $X_1, X_2, X_3, \ldots$ are identically independent distributed random variables where

$\Pr\{X_k = 1\} = \Pr\{X_k = -1\} = \frac{1}{2}$ and $S_n = \sum_{k=1}^{n} X_k$. Show that S_n is a martingale.

Q2: [4+4]

(a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \mathrm{pr}\{X_0 = i_0\}$

(b) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = \mathrm{pr}\{X_0 = 0\} = 0.3$, $p_1 = \mathrm{pr}\{X_0 = 1\} = 0.5$ and $p_2 = \mathrm{pr}\{X_0 = 2\} = 0.2$.

Determine the probabilities $\mathrm{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\}$ and $\mathrm{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\}$.

Q3: [5+4]

(a) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.5$, $\Pr\{\xi_n = 1\} = 0.4$, $\Pr\{\xi_n = 2\} = 0.1$, and suppose $s = 0$ and $S = 3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

(b) Let X_n denote the quality of the nth item that produced in a certain factory with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". Suppose that $\{X_n\}$ be a Markov chain whose transition matrix is

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 0.99 & 0.01 \\ 0.12 & 0.88 \end{array} \right\| \end{array}$$

i) What is the probability that the fourth item is good given that the first item is defective?

ii) In the long run, what is the probability that an item produced by this system is good?

The Model Answer

Q1: [2+6]

(a)

A stochastic process $\{X_n; n = 0,1,2,...\}$ is a martingale if

(i) $E[|X_n|] < \infty$,

(ii) $E[X_{n+1} | X_0, ..., X_n] = X_n$.

(b)

(1) To show that $E[|S_n|] < \infty$,

$$|S_n| = |X_1 + ... + X_n| \le |X_1| + ... + |X_n|$$
$$\le 1 + ... + 1 = n$$

$$E[|S_n|] \le E[n] = n < \infty.$$

(2) To show that $E[S_{n+1} | X_1, ..., X_n] = S_n$,

$$E[S_{n+1} | X_1, ..., X_n] = E[S_n + X_{n+1} | X_1, ..., X_n]$$
$$= E[S_n | X_1, ..., X_n] + E[X_{n+1} | X_1, ..., X_n]$$
$$= S_n + E[X_{n+1}],$$

where S_n is determined by $X_1, ..., X_n$ and X_{n+1} is independent of $X_{i's}$,

and $\because E[X_{n+1}] = (1).\Pr\{X_{n+1} = 1\} + (-1).\Pr\{X_{n+1} = -1\}$
$$= (1)(1/2) + (-1)(1/2) = 0$$

$$\therefore E[S_{n+1} | X_1, ..., X_n] = S_n$$

That is from (1) and (2), we have proved that S_n is a martingale.

Q2: [4+4]

(a)

$$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$$
$$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}$$
$$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n} \quad \text{Definition of Markov}$$

By repeating this argument $n-1$ times

$$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$$
$$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

(b)

i) $\mathrm{pr}\{X_0 = 1, X_1 = 1, X_2 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \mathrm{pr}\{X_0 = 1\}$
$$= 0.5(0.2)(0.4)$$
$$= 0.04$$

ii) $\mathrm{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = p_1 P_{11} P_{10}$, $p_1 = \mathrm{pr}\{X_1 = 1\}$

$$\mathrm{pr}\{X_1 = 1\} = \Pr(X_1 = 1 | X_0 = 0)\Pr(X_0 = 0) + \Pr(X_1 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2)\Pr(X_0 = 2)$$
$$= P_{01}p_0 + P_{11}p_1 + P_{21}p_2$$
$$= 0.3(0.3) + 0.2(0.5) + 0.3(0.2) = 0.25$$

$$\therefore \mathrm{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = 0.25(0.2)(0.4) = 0.02$$

Q3: [5+4]

(a)

The transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} -1 & 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{ccccc} 0 & 0 & 0.1 & 0.4 & 0.5 \\ 0 & 0 & 0.1 & 0.4 & 0.5 \\ 0.1 & 0.4 & 0.5 & 0 & 0 \\ 0 & 0.1 & 0.4 & 0.5 & 0 \\ 0 & 0 & 0.1 & 0.4 & 0.5 \end{array} \right\| \end{array}$$

where,

$$P_{ij} = \Pr\{X_{n+1} = j | X_n = i\}$$
$$= \begin{cases} \Pr(\xi_{n+1} = 3 - j), \ i \le 0 & \text{replenishment} \\ \Pr(\xi_{n+1} = i - j), \ 0 < i \le 3 & \text{without replenishment} \end{cases}$$

(b)

i)

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{bmatrix} \end{array}$$

$$\begin{aligned} P^2 &= P.P \\ &= \begin{bmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{bmatrix}\begin{bmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{bmatrix} \\ &= \begin{bmatrix} 0.9813 & 0.0187 \\ 0.2244 & 0.7756 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} P^3 &= P.P^2 \\ &= \begin{bmatrix} 0.99 & 0.01 \\ 0.12 & 0.88 \end{bmatrix}\begin{bmatrix} 0.9813 & 0.0187 \\ 0.2244 & 0.7756 \end{bmatrix} = \begin{bmatrix} 0.9737 & 0.0263 \\ 0.3152 & 0.6848 \end{bmatrix} \end{aligned}$$

$$\therefore pr\left\{X_3 = 0 \middle| X_0 = 1\right\} = p_{10}^3 = 0.3152,$$

or $pr\left\{X_4 = 0 \middle| X_1 = 1\right\} = p_{10}^3 = 0.3152.$

ii)

In the long run, the probability that an item produced by this system is good is given by

$$\begin{aligned} b/(a+b) &= \frac{0.12}{0.01+0.12} \\ &= \frac{12}{13} = 92.13\ \%\ , \end{aligned}$$

where $\lim\limits_{n\to\infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}.$