King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2 1443
M 380 - Stochastic Processes
Time: 3 hours - **Marks: 40**

Answer the following questions:

Q1: [5+4]

(a) Let X and Y have the joint normal distribution with Var(X) = σ_X^2 , Var(Y) = σ_Y^2 and $\rho(X,Y)$ is the correlation coefficient.

(i) What value of β that minimizes the variance of Z, where Z= β X + (1- β)Y?

(ii) What would be the minimum value of β if $\sigma_X = \frac{1}{4}$, $\sigma_Y = \frac{1}{3}$ and $\rho = 0.5$

(iii) Simplify your result when X and Y are independent.

(b) Let $S_0 = 0$, and for $n \geq 1$, let $S_n = \zeta_1 + \zeta_2 + \cdots + \zeta_n$ be the sum of n independent random variables, each exponentially distributed with mean $E(\zeta_k) = 1$. Show that:

$X_n = 2^n e^{-S_n}, n \geq 0,$ defines a martingale.

Q2: [3+3+2]

Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(i) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(ii) Starting in state 1, determine the mean time to absorption.

(iii) Sketch, the Markov chain diagram, and determine whether it's an absorbing chain or not.

Q3: [4+4]

(a) Suppose that the summands $\xi_1, \xi_2, \ldots$ are continuous random variables having a probability density function $f(z) = \begin{cases} \lambda e^{-\lambda z} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$ and $P_N(n) = \beta(1-\beta)^{n-1}$ for $n = 1, 2, \ldots$

Find the probability density function for $X = \xi_1 + \xi_2 + \ldots + \xi_N$

(b) Demands on a first aid facility in a certain location occur according to a nonhomogeneous Poisson process having the rate function

$$\lambda(t)=\begin{cases} t-2 & \text{for } 0\le t<1 \\ 5 & \text{for } 1\le t<2 \\ 0.5t & \text{for } 2\le t\le 4 \end{cases}$$

where t is measured in hours from the opening time of the facility. What is the probability that two demands occur in the first 2h of operation and two in the second 2h?

Q4: [7]

Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are upper class in the long run?

Q5: [4+4]

(a) Using the differential equations

$$\frac{dp_0(t)}{dt}=-\lambda p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt}=\lambda p_{n-1}(t)-\lambda p_n(t),\; n=1,2,3,\ldots \qquad (2)$$

where all birth parameters are the same constant λ with initial condition $X(0)=0$,

Show that $p_n(t)=\dfrac{(\lambda t)^n e^{-\lambda t}}{n!},\; n=0,1,2,\ldots$

(b) Messages arrive at a telegraph office as a Poisson Process with mean rate of 3 messages per hour.

(i) What is the probability that no messages arrive during the morning hours 8:00 A.M. to noon?

(ii) What is the distribution of the time at which the first afternoon message arrives?

Model Answer

Q1: [5+4]

(a)

(i) We know that for Z= β X + (1- β)Y,

Var(Z)= $\beta^2\sigma_X^2$ +2$\beta(1-\beta)\rho\,\sigma_X\,\sigma_Y$+$(1-\beta)^2\sigma_Y^2$

= $\beta^2\sigma_X^2$ +$(2\beta-2\beta^2)\rho\sigma_X\,\sigma_Y$+(1-2$\beta+\beta^2)\,\sigma_Y^2$

To minimize V= Var(Z), we differentiate V w.r.t β and equate with zero,

i.e. $\frac{\partial V}{\partial \beta}=0$:

2 $\beta\,\sigma_X^2$ + $(2-4\beta)\rho\,\sigma_X\,\sigma_Y$ +(-2+2β) σ_Y^2 = 0

2 β $(\sigma_X^2-2\,\rho\,\sigma_X\,\sigma_Y+\sigma_Y^2)$ = 2$(\sigma_Y^2-\rho\,\sigma_X\,\sigma_Y)$

So, $\beta=\frac{\sigma_Y^2-\rho\,\sigma_X\sigma_Y}{\sigma_X^2-2\,\rho\,\sigma_X\sigma_Y+\sigma_Y^2}$ is the minimum value.

(ii) By substituting, we obtain

$$\beta=\frac{\sigma_Y^2-\rho\,\sigma_X\,\sigma_Y}{\sigma_X^2-2\,\rho\,\sigma_X\,\sigma_Y+\sigma_Y^2}=\frac{(\frac{1}{3})^2-(0.5)(\frac{1}{4})(\frac{1}{3})}{(\frac{1}{4})^2-2(0.5)(\frac{1}{4})(\frac{1}{3})+(\frac{1}{3})^2}=\frac{10}{13}\simeq 0.7692$$

(iii) When X and Y independent, $\rho=0$, so $\beta=\frac{\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}$.

(b)

(1) $E[|X_n|]=E[X_n]=E[2^n\,e^{-S_n}]$

$=2^n\,E[e^{-\zeta_1}\ldots e^{-\zeta_n}]$

$=2^n\,E[e^{-\zeta_1}]\ldots E[e^{-\zeta_n}]$, as $\zeta_{i's}$ are independent

$=2^n\,\frac{1}{2}\ldots\ldots\frac{1}{2}=\frac{2^n}{2^n}=1$, as

$E\left[e^{-\zeta_n}\right]=\int_0^\infty e^{-x}\;e^{-x}\,dx$

$=\int_0^\infty e^{-2x}\;dx=\frac{1}{2}$

So, $E[|X_n|]=1<\infty$.

(2) $E[X_{n+1}|X_0,\ldots\ldots,X_n] = E[2^{n+1}\, e^{-S_{n+1}}|X_0,\ldots\ldots,X_n],\ S_{n+1} = S_n + \zeta_{n+1}$

$$= E\left[2^n\, e^{-S_n}\ 2\, e^{-\zeta_{n+1}}\middle|X_0,\ldots\ldots,X_n\right]$$

$$= 2^n\, e^{-S_n}\, E\left[2\, e^{-\zeta_{n+1}}\middle|X_0,\ldots\ldots,X_n\right]$$

$$= 2^n\, e^{-S_n}\, 2\, E[e^{-\zeta_{n+1}}],$$

as ζ_{n+1} is independent of $X_{i's}$,

$$E[X_{n+1}|X_0,\ldots\ldots,X_n] = 2^n\, e^{-S_n}\, 2 \,.\, \tfrac{1}{2}$$

$$= 2^n\, e^{-S_n}$$

$$= X_n.$$

We have proved from (1) and (2) that X_n is a martingale.

Q2: [3+3+2]

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{array}\right\| \end{array}$$

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

and v_i =E[$T | X_0 = i$] for i=1,2.

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.2 + 0.4u_1 + 0.3u_2$$
$$u_2 = 0.1 + 0.5u_1 + 0.3u_2$$

$\Rightarrow$

$$6u_1 - 3u_2 = 2 \qquad (1)$$
$$5u_1 - 7u_2 = -1 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{17}{27} \text{ and } u_2 = \tfrac{16}{27}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \tfrac{16}{27} = 0.5926$$

(ii) Also, the mean time to absorption can be found as follows.

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.4v_1 + 0.3v_2$$
$$v_2 = 1 + 0.5v_1 + 0.3v_2$$

$\Rightarrow$

$$6v_1 - 3v_2 = 10 \qquad (1)$$
$$5v_1 - 7v_2 = -10 \qquad (2)$$

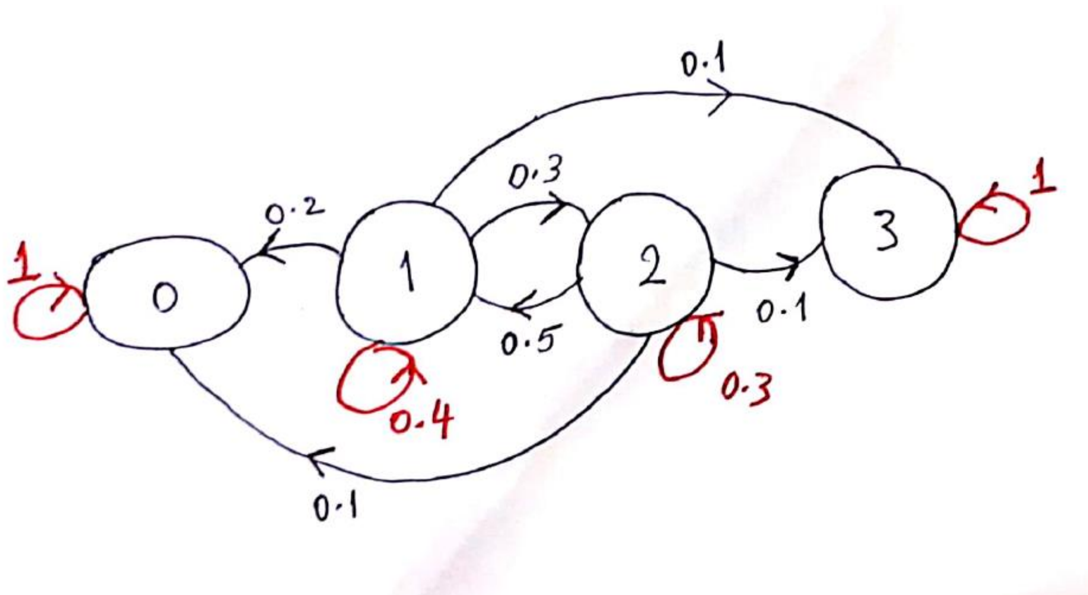
Solving (1) and (2), we get

$$\therefore v_1 = \tfrac{100}{27},\ v_2 = \tfrac{110}{27}$$

Starting in state 1, the mean time to absorption is

$$v_1 = v_{10} = \tfrac{100}{27}$$
$$\simeq 3.7037$$

(iii) It's an absorbing Markov Chain.



Markov Chain Diagram

Q3: [4+4]

(a)

We have $f_X(z) = \sum_{n=1}^{\infty} f^n(z) P_N(n)$

$\because$ The $n-fold$ convolution of $f(z)$ is the Gamma density function, $n \geq 1$

$$\therefore \quad f^n(z) = \begin{cases} \dfrac{\lambda^n}{\Gamma(n)} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

$\Rightarrow$

$$f^n(z) = \begin{cases} \dfrac{\lambda^n}{(n-1)!} z^{n-1} e^{-\lambda z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$

and $\because$ $P_N(n) = \beta(1-\beta)^{n-1}$ for $n = 1, 2, \ ...$

$$\therefore \ f_X(z) = \lambda\beta e^{-\lambda z} \sum_{n=1}^{\infty} \frac{[\lambda(1-\beta)z]^{n-1}}{(n-1)!}$$

$$= \lambda\beta e^{-\lambda z} . e^{\lambda(1-\beta)z}$$

$$= \lambda\beta e^{-\lambda\beta z}, \quad z \geq 0$$

$\therefore$ X has an exponential distribution with parameter $\lambda\beta$.

(b)

(i)

$$\mu_1 = \int_0^2 \lambda(u)du$$
$$= \int_0^1 (t-2)dt + \int_1^2 5dt$$
$$= \left[\frac{t^2}{2} - 2t\right]_0^1 + 5[t]_1^2$$
$$\therefore \mu_1 = 3.5$$

The prob. that two demands occur in the first 2h of operation is

$$\Pr\{X(2)=2\} = \Pr\{X(2)-X(0)=2\}$$
$$= \frac{e^{-\mu_1}\mu_1^k}{k!}$$
$$= \frac{e^{-3.5}\times 3.5^2}{2!}$$
$$\simeq 0.1850$$

(ii)

$$\mu_2 = \int_2^4 \lambda(u)du$$
$$= \int_2^4 (0.5t)dt$$
$$= \tfrac{1}{2}\left[\tfrac{t^2}{2}\right]_2^4$$
$$\therefore \mu_2 = 3$$

The prob. that two demands occur in the second 2h of operation is

$\Pr\{X(4)-X(2)=2\}$

$$=\frac{e^{-\mu_2}\mu_2^k}{k!}$$

$$=\frac{e^{-3}\times 3^2}{2!}$$

$$\simeq 0.2240$$

Q4: [7]

Let $\pi=(\pi_0,\pi_1,\pi_2)$ be the limiting distribution

$\Rightarrow$

$$\pi_0=0.7\pi_0+0.2\pi_1+0.1\pi_2$$

$$\pi_1=0.2\pi_0+0.6\pi_1+0.4\pi_2$$

$$\pi_2=0.1\pi_0+0.2\pi_1+0.5\pi_2$$

$$\pi_0+\pi_1+\pi_2=1$$

Solving the following equations

$$3\pi_0-2\pi_1-\pi_2=0 \qquad (1)$$
$$\pi_0+2\pi_1-5\pi_2=0 \qquad (2)$$
$$\pi_0+\pi_1+\pi_2=1 \qquad (3)$$

By solving equations using Cramer's rule, we get

$$\Delta=\begin{vmatrix}3 & -2 & -1\\ 1 & 2 & -5\\ 1 & 1 & 1\end{vmatrix}=34,\ \Delta_0=\begin{vmatrix}0 & -2 & -1\\ 0 & 2 & -5\\ 1 & 1 & 1\end{vmatrix}=12$$

$$\Delta_1=\begin{vmatrix}3 & 0 & -1\\ 1 & 0 & -5\\ 1 & 1 & 1\end{vmatrix}=14,\ \Delta_2=\begin{vmatrix}3 & -2 & 0\\ 1 & 2 & 0\\ 1 & 1 & 1\end{vmatrix}=8$$

$$\therefore\ \pi_0=\tfrac{\Delta_0}{\Delta}=\tfrac{6}{17},\ \pi_1=\tfrac{\Delta_1}{\Delta}=\tfrac{7}{17},\ \pi_2=\tfrac{\Delta_2}{\Delta}=\tfrac{4}{17}$$

$\therefore$ The limitting distribution is $\pi=(\pi_0,\pi_1,\pi_2)=(6/17,7/17,4/17)$

$\therefore$ In the long run, approximately 23.53% of families are upper class.

Q5: [4+4]

(a)

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t), \;\; n = 1, 2, 3, \ldots \qquad (2)$$

Let $X(t)$ represents the size of the population, and the initial condition is

$$X(0) = 0 \quad \Rightarrow \; p_0(0) = 1$$

$$\Rightarrow \; p_n(0) = \begin{cases} 1 & , \; n = 0 \\ 0 & , \; \text{otherwise} \end{cases}$$

$$(1) \Rightarrow \frac{dp_0(t)}{dt} = -\lambda p_0(t)$$

$$\therefore \; \frac{dp_0(t)}{p_0(t)} = -\lambda dt$$

$$\int_0^t \frac{dp_0(u)}{p_0(u)} = -\lambda \int_0^t du$$

$$\left[\ln p_0(u)\right]_0^t = -\lambda t$$

$$\ln p_0(t) - \ln p_0(0) = -\lambda t$$

$$\ln p_0(t) - \ln 1 = -\lambda t \;, \;\text{ where } \; p_0(0) = 1$$

$$\therefore \; \ln p_0(t) = -\lambda t \;\Rightarrow p_0(t) = e^{-\lambda t} \qquad (3)$$

$$(2) \Rightarrow \frac{dp_n(t)}{dt} = p_{n-1}(t) - \lambda p_n(t), \;\; n = 1, 2, 3, \ldots$$

$$\therefore \; \frac{dp_n(t)}{dt} + \lambda p_n(t) = \lambda p_{n-1}(t)$$

Multiply both sides by $\mathrm{e}^{\lambda t}$

$$\mathrm{e}^{\lambda t}\left[\frac{dp_n(t)}{dt} + \lambda p_n(t)\right] = \lambda p_{n-1}(t)\mathrm{e}^{\lambda t}$$

$$\therefore \; \frac{d}{dt}\left[e^{\lambda t} p_n(t)\right] = \lambda p_{n-1}(t)\mathrm{e}^{\lambda t}$$

$\therefore$ By separation of variables and Integration from 0 to t, we get

$$\int_0^t d\left[e^{\lambda x} p_n(x)\right] = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx$$

$$\left[e^{\lambda x} p_n(x)\right]_0^t = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx$$

$$e^{\lambda t} p_n(t) - p_n(0) = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx, \; n = 1, 2, 3, \ldots$$

$$p_n(t) = \lambda e^{-\lambda t} \int_0^t p_{n-1}(x) e^{\lambda x} dx, \; n = 1, 2, 3, \ldots \qquad (4)$$

which is a recurrence relation

$at \; n = 1$

$$(4) \Rightarrow p_1(t) = \lambda e^{-\lambda t} \int_0^t p_0(x) e^{\lambda x} dx$$

$\because p_0(x) = e^{-\lambda x}$ from eq. (3)

$$\therefore p_1(t) = \lambda e^{-\lambda t} \int_0^t e^{-\lambda x} e^{\lambda x} dx$$

$$= \lambda e^{-\lambda t} \int_0^t dx$$

$$\therefore p_1(t) = \lambda t e^{-\lambda t} \qquad (5)$$

$at \; n = 2$

$$(4) \Rightarrow p_2(t) = \lambda e^{-\lambda t} \int_0^t p_1(x) e^{\lambda x} dx$$

$\because p_1(x) = \lambda x e^{-\lambda x}$ from eq. (5)

$$\therefore p_2(t) = \lambda e^{-\lambda t} \int_0^t \lambda x e^{-\lambda x} e^{\lambda x} dx$$

$$= \lambda^2 e^{-\lambda t} \int_0^t x dx$$

$$\therefore p_2(t) = \lambda^2 e^{-\lambda t} \left[\frac{x^2}{2}\right]_0^t$$

$$\therefore p_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!} \qquad (6)$$

From (3), (5) and (6), we can deduce that $p_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}, \; n = 0, 1, 2, \ldots$

(b)

(i) For Poisson process $\left\{X(t);\ t \geq 0\right\}$, where $X(t)$ is the random variable that represents the number of messages arrive at the telegraph office at any time t.

$$\Pr\left\{X(s+t) - X(s) = k\right\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!},\ k = 0,1,2,...$$

$$\therefore \Pr\left\{X(12) - X(8) = 0\right\} = \frac{(3\times 4)^0 e^{-3(4)}}{0!} = e^{-12},$$

where $\lambda = 3,\ t = 12 - 8 = 4$ and $k = 0$

$$\therefore \Pr\left\{X(12) - X(8) = 0\right\} \simeq 6.1442\times 10^{-6}$$

(ii) Consider T is the random variable that represents the time at which the first afternoon message arrives. Afternoon is the period between 12:00 P.M. and 12:00 A.M. i.e. $t \in (12, 24)$
So, we can write

$$\begin{aligned}\Pr(T > t) &= \Pr\{\text{The first afternoon message arrives after t units of time}\}\\ &= \Pr\left\{X(t) - X(12) = 0\right\}\\ &= \frac{\left[3(t-12)\right]^0 e^{-3(t-12)}}{0!}\end{aligned}$$

$$\therefore \Pr(T > t) = e^{-3(t-12)},$$

which is the survival/reliability function.
Also,

$$\begin{aligned}\Pr(T \leq t) &= 1 - \Pr(T > t)\\ &= 1 - e^{-3(t-12)}\end{aligned}$$

$\therefore\ \Pr(T \leq t) = 1 - e^{-3x}$, where $x = t - 12$
which is the cumulative distribution function.
$\therefore\ T \sim\ \exp(3)$
i.e. $T \sim$ exponential distribution with parameter equals 3.

Another solution for (ii)

Pr{The first afternoon message arrives after t units of time}

$$= \Pr\left\{X(t+12) - X(12) = 0\right\}$$

$$= \frac{\left[3(t)\right]^0 e^{-3t}}{0!}$$

$$\therefore \Pr(T > t) = e^{-3t},$$

which is the survival/reliability function.
Also,

$$\Pr(T \le t) = 1 - \Pr(T > t)$$

$$= 1 - e^{-3t}$$

$$\therefore \; \Pr(T \le t) = 1 - e^{-3t},$$

which is the cumulative distribution function.

$\therefore \; T \sim \exp(3)$

i.e. $T \sim$ exponential distribution with parameter equals 3.