



Answer the following questions:

Q1: [4+5]

a) For the Markov process $\{X_t\}$, $t=0,1,2,\dots,n$ with states $i_0, i_1, i_2, \dots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$

b) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.1$, $\Pr\{\xi_n = 1\} = 0.5$, $\Pr\{\xi_n = 2\} = 0.4$ and suppose $s=0$ and $S=2$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n .

Q2: [3+6]

a) Let X_n denote the weather of the n th day with $X_n = 1$ meaning “rainy” and $X_n = 2$ meaning “dry”. Suppose that $\{X_n\}$, $n = 0, 1, 2, \dots$ evolves as a Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{vmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{vmatrix} \end{matrix}$$

Given that, the probability of dry weather on 1st June equals $\frac{5}{8}$. What’s the probability that the weather will be rainy on 3rd June.

b) Determine whether the transition matrix

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{vmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{vmatrix} \end{matrix}$$

represents an absorbing Markov chain or not, sketch Markov chain diagram and then find each of the following:

- i) Starting in state 1, determine the probability that the Markov chain ends in state 0.
- ii) Determine the mean time to absorption.

Q3: [3+4]

a) Let $X = \begin{cases} 0 & \text{if } N=0 \\ \xi_1 + \xi_2 + \dots + \xi_N & \text{if } N>0 \end{cases}$ be a random sum and assume that $E(\xi_k) = \mu$, $E(N) = \nu$

Prove that $E(X) = \mu\nu$

b) The number of accidents occurring in a factory in a week is a Poisson random variable with mean 2. The number of individuals injured in different accidents is independently distributed, each with mean 3 and variance 4. Determine the mean and variance of the number of individuals injured in a week.



The Model Answer

Q1: [4+5]

a)

$$\begin{aligned} & \because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} \\ &= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \cdot \Pr\{X_n = i_n \mid X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \\ &= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \cdot P_{i_{n-1}i_n} \quad \text{Definition of Markov} \end{aligned}$$

By repeating this argument $n - 1$ times

$$\begin{aligned} & \therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\} \\ &= p_{i_0} P_{i_0i_1} P_{i_1i_2} \dots P_{i_{n-2}i_{n-1}} P_{i_{n-1}i_n} \quad \text{where } p_{i_0} = \Pr\{X_0 = i_0\} \text{ is obtained from the initial distribution of the process.} \end{aligned}$$

b)

$$\begin{array}{c} -1 \quad 0 \quad 1 \quad 2 \\ -1 \left\| \begin{array}{cccc} 0 & 0.4 & 0.5 & 0.1 \\ 0 & 0.4 & 0.5 & 0.1 \\ 1 & 0.4 & 0.5 & 0.1 \\ 2 & 0 & 0.4 & 0.5 \end{array} \right\| \end{array}$$

Where

$$P_{ij} = \begin{cases} pr = (\xi_n = 2 - j), i \leq 0 & \text{replenishment} \\ pr = (\xi_n = i - j), 0 < i \leq 2 & \text{without replenishment} \end{cases}$$

Q2: [3+6]

a) X_n , $n = 0, 1, 2, \dots$ denotes the weather of the n th day with $X_n = 1$ meaning “rainy” and $X_n = 2$ meaning “dry”

The probability of dry weather on 1st June equals $\frac{5}{8}$

$$\therefore P^0 = \left[\frac{3}{8} \quad \frac{5}{8} \right] \text{ is the initial Prob. distribution}$$

$$\mathbf{P}^2 = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.48 & 0.52 \\ 0.39 & 0.61 \end{bmatrix}$$

$$\therefore \Pr(X_2 = 1) = P_1^2$$

$$= \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix} \begin{bmatrix} 0.48 \\ 0.39 \end{bmatrix}$$

$$= 0.4238$$

b)

i)

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$u = \Pr\{X_T = 0 | X_0 = 1\}$$

$$u_1 = p_{10} + p_{11}u_1$$

$$u_1 = 0.1 + 0.6u_1$$

$\therefore u_1 = u_{10} = \frac{1}{4}$ is the prob. that Markov chains ends in state 0

ii)

The mean time to absorption can be found as follows

$$v = E\{T | X_0 = 1\}$$

$$v_1 = 1 + p_{11}v_1$$

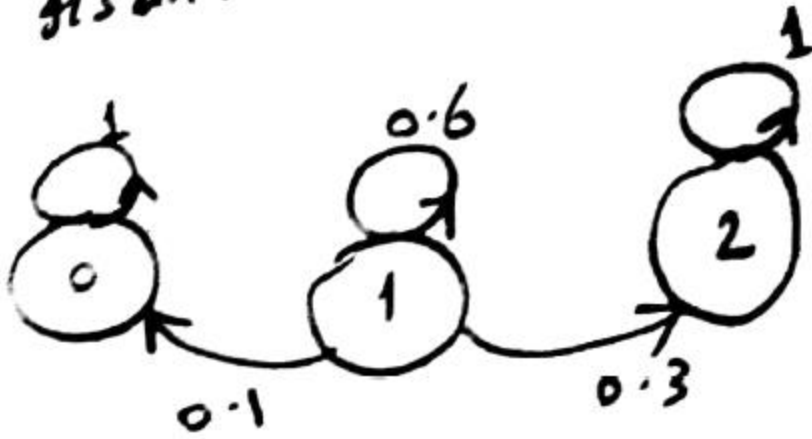
\Rightarrow

$$v_1 = 1 + 0.6v_1$$

$$v_1 = \frac{5}{2}$$

$$\therefore v_1 = v_{10} = \frac{5}{2}$$

It's an absorbing Markov chain



Q3: [3+4]

a)

The random sum is

$$X = \begin{cases} 0 & \text{if } N=0 \\ \xi_1 + \xi_2 + \dots + \xi_N & \text{if } N>0 \end{cases}$$

$$\begin{aligned}
\therefore E(X) &= \sum_{n=0}^{\infty} E[X|N=n]P_N(n) \\
&= \sum_{n=1}^{\infty} E[\xi_1 + \xi_2 + \dots + \xi_n|N=n]P_N(n) \\
&= \sum_{n=1}^{\infty} E[\xi_1 + \xi_2 + \dots + \xi_n|N=n]P_N(n) \dots \dots \dots \text{Prop. of cond. expectation} \\
&= \sum_{n=1}^{\infty} E[\xi_1 + \xi_2 + \dots + \xi_n]P_N(n), \text{ where } N \text{ is independent of } \xi_1, \xi_2, \dots
\end{aligned}$$

and $\therefore E(\xi_k) = \mu, k=1,2,\dots,n$

$$\begin{aligned}
\therefore E(X) &= \sum_{n=1}^{\infty} n\mu P_N(n) \\
&= \mu \sum_{n=1}^{\infty} nP_N(n) \\
&= \mu E(N), E(N) = v
\end{aligned}$$

$$\therefore E(X) = \mu v$$

b)

$N \sim \text{Poisson}(2)$

N is the # of accidents in a week

ξ_k is the # of individuals injured for k th accident

$$E(\xi_k) = 3, \text{ var}(\xi_k) = 4$$

$$E(N) = 2, \text{ var}(N) = 2$$

$$\therefore E(X) = \mu v = 3(2) = 6$$

$$\text{var}(X) = v\sigma^2 + \mu^2\tau^2$$

$$\therefore \text{var}(X) = 2(4) + 9(2) = 26$$