King Saud University
College of Sciences
Department of Mathematics

First Mid Term, S2-1441H
M 380 - Stochastic Processes
Time: 90 minutes

Answer the following questions:

Q1: [4+4]

a) If $T \sim \exp(\lambda)$ prove that: $pr(T > t + s | T > s) = pr(T > t) \quad \forall\ t,\ s \geq 0$

b) Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 5000 miles. If a person desires to take a 1000-mile trip, what is the probability that he will be able to complete his trip without having to replace the car battery?

Q2: [4+4]

a) Suppose $X \sim Bin\,(p,\ N)$ and $N \sim$ Poisson$\,(\lambda)$. Find the marginal probability mass function for X and then determine the mean and variance of X.

b) Determine the mean and median of an exponentially distributed random variable with parameter λ

Q3: [5+4]

a) Given the following joint distribution. Calculate E(X), E(Y), Var(X), Var(Y), Cov(X,Y), ρ(X,Y) and verify E(X) using the law of total Expectation.

Y \ X	0	1
0	0.1	0.3
1	0.4	0.2

b) The following experiment is performed: An observation is made of a Poisson random variable N with parameter λ. Then N independent Bernoulli trials are performed, each with probability p of success. Let Z be the total number of successes observed in the N trials.

i) Formulate Z as a random sum and thereby determine its mean and variance.

ii) What is the distribution of Z?

The Model Answer

Q1: [4+4]

a) If $T \sim \exp(\lambda)$ prove that: $pr(T > t+s | T > s) = pr(T > t) \quad \forall\ t,\ s \geq 0$

Proof:

$$pr(T > t+s | T > s) = \frac{pr(T > t+s, T > s)}{pr(T > s)}$$

$$= \frac{pr(T > t+s)}{pr(T > s)}$$

$\because\ T \sim\ \exp(\lambda)$

$$\therefore\ pr(T > t+s | T > s) = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = R(t)$$

$$= pr(T > t)$$

b)

$\because\ X \sim\ \exp(\lambda)$

$\therefore\ X \sim\ \exp(\frac{1}{5000})$

$\therefore\ \Pr(X > 1000)$

$= e^{\frac{-1000}{5000}}$

$= e^{-0.2} \simeq 0.8187$

Q2: [4+4]

a) $\because\ X \sim\ \text{Bin}\ (p,\ N),\ \ N \sim\ \text{Poisson}\ (\lambda)$

$$\Rightarrow\ P_{X|N}(x|n) = \binom{n}{x} p^x (1-p)^{n-x},\ \ x = 0,1,2,\ldots,n$$

and

$$P_N(n) = \frac{e^{-\lambda}\lambda^n}{n!},\ \ n = 0,1,2,\ldots$$

$\Rightarrow$

$$\Pr(X=x)=\sum_{n=0}^{\infty}\mathrm{P}_{X|N}(x|n)\mathrm{P}_{N}(n)$$

$$=\sum_{n=0}^{\infty}\binom{n}{x}p^x(1-p)^{n-x}.\frac{e^{-\lambda}\lambda^n}{n!}$$

$$=p^x e^{-\lambda}\sum_{n=0}^{\infty}\frac{n!}{x!(n-x)!}(1-p)^{n-x}.\frac{\lambda^n}{n!}$$

$$=\frac{\lambda^x e^{-\lambda}p^x}{x!}\sum_{n=x}^{\infty}\frac{\lambda^{n-x}(1-p)^{n-x}}{(n-x)!}$$

$$=\frac{(\lambda p)^x e^{-\lambda}}{x!}\sum_{r=0}^{\infty}\frac{[\lambda(1-p)]^r}{r!},\quad r=n-x$$

$$=\frac{(\lambda p)^x e^{-\lambda}}{x!}e^{\lambda(1-p)}$$

$$\therefore\ \Pr(X=x)=\frac{(\lambda p)^x e^{-\lambda p}}{x!},\quad x=0,1,2,\ldots$$

$\therefore$ $X\sim$ Poisson (λp) with mean and variance λp.

b)

$\because$ $X\sim$ $\exp(\lambda)$, $\lambda>0$

$\therefore$ p.d.f $\quad f(x)=\lambda e^{-\lambda x},\quad x\geq 0$

$$\Rightarrow\ E(X)=\lambda\int_0^{\infty}xe^{-\lambda x}dx$$

Let $u=\lambda x\ \Rightarrow x=\frac{u}{\lambda},\ dx=\frac{1}{\lambda}du$

$$\Rightarrow\ E(X)=\tfrac{1}{\lambda}\int_0^{\infty}ue^{-u}du$$

$$=\tfrac{1}{\lambda}\Gamma(2)$$

$$\therefore\ E(X)=\tfrac{1}{\lambda}$$

For $\Pr(X \le a) \ge \frac{1}{2}$

$\Rightarrow$ $1-e^{-\lambda a} \ge \frac{1}{2}$

$e^{-\lambda a} \le \frac{1}{2}$

$\therefore$ $a \ge \frac{\ln 2}{\lambda}$ (1)

For $\Pr(X \ge a) \ge \frac{1}{2}$

$e^{-\lambda a} \ge \frac{1}{2}$

$\therefore$ $a \le \frac{\ln 2}{\lambda}$ (2)

(1) and (2) $\Rightarrow$ $a = \frac{\ln 2}{\lambda}$

$\therefore$ Median= $\frac{\ln 2}{\lambda}$

Clearly, Median <Mean for the r.v $X \sim \exp(\lambda)$

Q3: [5+4]

a)

Y \ X	0	1	$P_Y(y)$
0	0.1	0.3	0.4
1	0.4	0.2	0.6
$P_X(x)$	0.5	0.5	Sum=1

$E(X)=0.5$, $E(X^2)=0.5$, $Var(X)=0.25$

$E(Y)=0.6$, $E(Y^2)=0.6$, $Var(Y)=0.24$

$E(XY)=0.2$, $Cov(X,Y)=-0.10$, $\rho(X,Y)=-0.4$

$$P(X|Y=y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$P(X=0|Y=0) = \frac{0.1}{0.4} = \frac{1}{4}, \quad P(X=1|Y=0) = \frac{0.3}{0.4} = \frac{3}{4}$$

$$P(X=0|Y=1) = \frac{0.4}{0.6} = \frac{2}{3}, \quad P(X=1|Y=1) = \frac{0.2}{0.6} = \frac{1}{3}$$

X|Y	0	1	E[X|Y]
y=0	1/4	3/4	3/4
y=1	2/3	1/3	1/3

$$\mathrm{E(X)}=\sum_{y}\mathrm{E(X}|\mathrm{Y=y)P_Y(y)}$$

$$\mathrm{E(X)}=\frac{3}{4}\mathrm{P_Y(0)}+\frac{1}{3}\mathrm{P_Y(1)}$$

$$\mathrm{E(X)}=\frac{3}{4}(0.4)+\frac{1}{3}(0.6)=0.5$$

b)

i) $Z=\xi_1+\xi_2+...+\xi_N,\ N>0$

$$E(\xi_k)=\mu=p,\ \mathrm{Var}(\xi_k)=\sigma^2=p(1-p)$$

$$E(N)=v=\lambda,\ \ \mathrm{Var}(N)=\tau^2=\lambda$$

$$\because E(Z)=\mu v$$

$$\therefore E(Z)=\lambda p$$

$$\because\ \mathrm{Var(Z)}=v\sigma^2+\mu^2\tau^2$$

$$\therefore\ \mathrm{Var(Z)}=\lambda p(1-p)+p^2\lambda$$

$$=\lambda p$$

ii) $Z\sim$ Poisson(λp)