Second Semester	**Second Exam**	**King Saud University**
(without calculators)	**Time allowed: 1 h and 30 m**	**College of Science**
Thursday 11-10-1443	**240 Math**	**Math. Department**

Q1: **(a)** Let $V=M_{nn}$ and W is the set of all symmetric matrices of degree n. Prove that W is a subspace of V. (3 marks)

A1(a): For all A,B∈W and k∈$\mathbb{R}$:

1- W is not empty since $0^T=0$. Hence 0∈W
2- $(A+B)^T=A^T+B^T=A+B$. So A+B∈W.
3- $(kA)^T=kA^T=kA$. So kA∈W

1, 2 and 3 implies that W is a subspace of $V=M_{nn}$.

(b) show that the vectors (1,1,2), (2,1,1), (1,1,0) form a basis for $\mathbb{R}^3$. (3 marks)

A1(b):

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} \underset{(-2)R13}{\overset{(-1)R_{12}}{=}} \begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -3 & -2 \end{vmatrix} = 1(-1)(-2) = 2 \neq 0$$

So the vectors (1,1,2), (2,1,1), (1,1,0) form a basis for $\mathbb{R}^3$.

Q2: **(a)** Use the Wronskian to show that the vectors 1, sin(x), cos(x) are linearly independent in the vector space $C^2(-\infty,\infty)$. (3 marks)

A2(a):

$$W(x) = \begin{vmatrix} 1 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \\ 0 & -\sin(x) & -\cos(x) \end{vmatrix} =$$

$$-(\cos^2(x)+\sin^2(x)) = -1 \neq 0$$

So 1, sin(x), cos(x) are linearly independent.

(b) Let B={(1,2),(2,5)} and B'={(1,1),(2,0)} be two bases of $\mathbb{R}^2$. Find the transition matrix from B' to B. (3 marks).

A2(b):

$$[B \mid B'] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 2 & 5 & 1 & 0 \end{array}\right] \xrightarrow{(-2)R_{12}} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$\xrightarrow{(-2)R_{21}} \left[\begin{array}{cc|cc} 1 & 0 & 3 & 10 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$= [I \mid P_{B' \to B}]$$

$$P_{B' \to B} = \begin{bmatrix} 3 & 10 \\ -1 & -4 \end{bmatrix}$$

Q3: Find a basis for the column space of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 6 & -1 \\ 2 & 4 & 4 & 6 \\ 3 & 6 & 10 & 5 \end{bmatrix}$$

and **deduce** nullity(A^T) without solving any linear system. (4 marks)

A3:

$$A = \begin{bmatrix} 1 & 2 & 6 & -1 \\ 2 & 4 & 4 & 6 \\ 3 & 6 & 10 & 5 \end{bmatrix} \xrightarrow[(-3)R_{13}]{(-2)R_{12}} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -8 & 8 \\ 0 & 0 & -8 & 8 \end{bmatrix}$$

$$\xrightarrow{(-1)R_{23}} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -8 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(-\frac{1}{8})R_2} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the leading ones, $\{[1\ \ 2\ \ 3]^T, [6\ \ 4\ \ 10]^T\}$ is a basis of col(A).

Now, rank(A)+nullity(A^T)=m

So nullity(A^T)=m- rank(A)=3-2=1

Q4: Assume that the vector space $\mathbb{R}^3$ has the Euclidean inner product. Apply the Gram-Schmidt process to transform the following basis vectors (1,-2,0), (2,1,-1), (0,1,1) into an **orthonormal basis**. (5 marks)

A4: Let v_1=(1,-2,0) , v_2=(2,1,-1) , v_3=(0,1,1) .

Now define u_1, u_2 and u_3 as follows:

$$u_1 = v_1 = (1,-2,0)$$

$$u_2 = v_2 - \frac{<v_2, u_1>}{\|u_1\|^2} u_1 = (2,1,-1) - 0 = (2,1,-1)$$

$$u_3 = v_3 - \frac{<v_3, u_2>}{\|u_2\|^2} u_2 - \frac{<v_3, u_1>}{\|u_1\|^2} u_1$$

$$= (0,1,1) - 0 - \tfrac{-2}{5}(1,-2,0) = (\tfrac{2}{5}, \tfrac{1}{5}, 1)$$

$$w_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{5}}(1,-2,0)$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}}(2,1,-1)$$

$$w_3 = \frac{u_3}{\|u_3\|} = \frac{\sqrt{5}}{\sqrt{6}}(\tfrac{2}{5}, \tfrac{1}{5}, 1)$$

So $\{w_1, w_2, w_3\}$ is the wanted orthonormal basis.

Q5:**(a)** If *u* and *v* are orthogonal vectors in an inner product space, then: $\|u+v\|^2 = \|u\|^2 + \|v\|^2$. (1 mark)

A5(a): As *u* and *v* are orthogonal, so <u,v>=0 and hence:

$$\|u+v\|^2 = <u+v, u+v>$$

$$= <u,u> + <u,v> + <v,u> + <v,v>$$

$$= <u,u> + <v,v> = \|u\|^2 + \|v\|^2$$

(b) If S={$\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$} is a basis for a vector space **V**, then prove that every vector **v** in **V** can be expressed in the form $\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_n\mathbf{v_n}$ in exactly one way, where $c_1, c_2, \ldots, c_n$ are real numbers. (1 mark)

A5(b): Suppose v∈V has two expressions:

$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ and $v = k_1v_1 + k_2v_2 + \cdots + k_nv_n$, so

$0 = (c_1-k_1)v_1 + (c_2-k_2)v_2 + \cdots + (c_n-k_n)v_n$

But $S = \{v_1, v_2, \ldots, v_n\}$ is a basis, so it is linearly independent. Thus,

c_1-k_1= c_2-k_2=…= c_n-k_n=0 and hence c_i=k_i for all i∈{1,2,…,n} and hence v has exactly one expression.

(c) Assume that S={$\mathbf{v_1}$,$\mathbf{v_2}$,$\mathbf{v_3}$,$\mathbf{v_4}$,$\mathbf{v_5}$} is an orthonormal set of P_4. Is it a basis of P_4? Why? (1 mark)

A5(c): Yes, because any orthonormal set is linearly independent and dim(P_4)=5 which is equal to the number of vectors of S.

(d) Show that the function $<,>$ defined by: $<(x,y),(z,w)>=xz$ for all (x,y),(z,w) in $\mathbb{R}^2$ is **not** an inner product on $\mathbb{R}^2$. (1 mark).

A5(d): $<(0,1),(0,1)>=0$, but $(0,1)\neq(0,0)$.