First Semester	**Second Exam**	**King Saud University**
(without calculators)	**Time allowed: 1 h and 30 m**	**College of Science**
Wednesday 28-3-1443	**240 Math**	**Math. Department**

Q1: Let V be any nonempty set which has two operations are defined: addition and scalar multiplication. State the 10 axioms that should be satisfied by all scalars and all objects in V that make V a vector space. (5 marks)
A1: For all u,v,w∈V and k,m∈ℝ:

1- u+v∈ℝ
2- u+v=v+u
3- u+(v+w)=(u+v)+w
4- there is a zero vector 0 in v such that u+0=u for all u∈V
5- for each vector u in V, there is a negative vector –u such u+(-u)=0
6- ku∈V
7- k(u+v)=ku+kv
8- (k+m)u=ku+mu
9- K(mu)=(km)u
10- 1u=u

Q2: Let $V=M_{22}$ and $W=\{A\in M_{22} | tr(A)=0\}$. Prove that W is a subspace of V. (3 marks)

A2: For all $A=\begin{bmatrix} a & a' \\ a'' & a''' \end{bmatrix}$,$B=\begin{bmatrix} b & b' \\ b'' & b''' \end{bmatrix}$∈W and k∈ℝ:

1- W is not empty since tr(0)=0. Hence 0∈W
2- $tr(A+B)=tr\left(\begin{bmatrix} a+b & a'+b' \\ a''+b'' & a'''+b''' \end{bmatrix}\right)=a+b+a'''+b'''=a+a'''+b+b'''$ $=tr(A)+tr(B)=0+0=0$. So A+B∈W.
3- $tr(kA)=tr\left(\begin{bmatrix} ka & ka' \\ ka'' & ka''' \end{bmatrix}\right)=ka+ka'''=k(a+a''')=k\,tr(A)=k0=0$. So kA∈W
   1, 2 and 3 implies that W is a subspace of $V=M_{nn}$.

Q3: Use the Wronskian to show that xsin(x) and xcos(x) are linearly independent in the vector space $C^{\infty}(-\infty,\infty)$. (3 marks)

A3:

$$W(x) = \begin{vmatrix} x\sin(x) & x\cos(x) \\ \sin(x)+x\cos(x) & \cos(x)-x\sin(x) \end{vmatrix}$$

$$= x\sin(x)\cos(x) - x^2\sin^2(x) - x\cos(x)\sin(x) - x^2\cos^2(x)$$

$$= -x^2\sin^2(x) - x^2\cos^2(x) = -x^2(\sin^2(x)+\cos^2(x))$$

$$= -x^2(1) = -x^2$$

Since W(1)=−1≠0, so xsin(x) and xcos(x) are linearly independent.

Q4: show that the vectors (1,1,2), (2,1,0), (1,1,0) form a basis for $\mathbb{R}^3$. (3 marks)
A4:

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2(2-1) = 2 \neq 0$$

So the vectors (1,1,2), (2,1,0), (1,1,0) form a basis for $\mathbb{R}^3$

Q5: Let B={(1,2),(2,5)} and B'={(1,1),(2,0)} be two bases of $\mathbb{R}^2$. Find the transition matrix from B' to B. (3 marks).
A5:

$$[B \mid B'] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 2 & 5 & 1 & 0 \end{array}\right] \xrightarrow{(-2)R_{12}} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$\xrightarrow{(-2)R_{21}} \left[\begin{array}{cc|cc} 1 & 0 & 3 & 10 \\ 0 & 1 & -1 & -4 \end{array}\right]$$

$$= [I \mid P_{B'\to B}]$$

$$P_{B'\to B} = \begin{bmatrix} 3 & 10 \\ -1 & -4 \end{bmatrix}$$

Q6: Find a basis for the column space of the matrix:

$$A=\begin{bmatrix}1 & 2 & 6 & -1\\ 2 & 4 & 4 & 6\\ 3 & 6 & 10 & 5\end{bmatrix}$$

and **deduce** nullity(A^T) without solving any linear system. (4 marks)
A6:

$$A=\begin{bmatrix}1 & 2 & 6 & -1\\ 2 & 4 & 4 & 6\\ 3 & 6 & 10 & 5\end{bmatrix}\xrightarrow[(-3)R_{13}]{(-2)R_{12}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & -8 & 8\\ 0 & 0 & -8 & 8\end{bmatrix}$$

$$\xrightarrow{(-1)R_{23}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & -8 & 8\\ 0 & 0 & 0 & 0\end{bmatrix}\xrightarrow{(-1/8)R_{2}}\begin{bmatrix}1 & 2 & 1 & 2\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Using the leading ones, $\{[1\ \ 2\ \ 3]^T, [6\ \ 4\ \ 10]^T\}$ is a basis of col(A).

Now, rank(A)+nullity(A^T)=m

So nullity(A^T)=m- rank(A)=3-2=1

Q7:**(a)** Let S={$\mathbf{v_1}$, $\mathbf{v_2}$,…,$\mathbf{v_n}$} be a basis for a vector space **V**. Suppose **u** is a vector in **V** such that

$$\mathbf{u}=|A_1|\mathbf{v_1}+2|A_2|\mathbf{v_2}+3|A_3|\mathbf{v_3}+\ldots+n|A_n|\mathbf{v_n}$$

where, A_i is a matrix of order 2 for all $i\in\{1,2,\ldots,n\}$. Find $(\mathbf{u})_S$ (1 mark)

A7(a): $(u)_S=(|A_1|,2|A_2|,3|A_3|,\ldots,n|A_n|)$

(b) If S={$\mathbf{v_1}$, $\mathbf{v_2}$,…,$\mathbf{v_n}$} is a basis for a vector space **V**, then prove that every vector **v** in **V** can be expressed in the form $\mathbf{v}=c_1\mathbf{v_1}+c_2\mathbf{v_2}+\ldots+c_n\mathbf{v_n}$ in exactly one way, where $c_1, c_2, \ldots, c_n$ are real numbers. (2 marks)

A5(b): Suppose v∈V has two expressions:

$v = c_1v_1 + c_2v_2 +\cdots+c_nv_n$ and $v = k_1v_1 + k_2v_2 +\cdots+k_nv_n$, so

$0= (c_1-k_1)v_1 + (c_2-k_2)v_2 +\cdots+(c_n-k_n)v_n$

But $S = \{v_1, v_2, \ldots, v_n\}$ is a basis, so it is linearly independent. Thus,

$c_1-k_1= c_2-k_2=\ldots= c_n-k_n=0$ and hence $c_i=k_i$ for all $i\in\{1,2,\ldots,n\}$ and hence v has exactly one expression.

(c) Suppose S is a subset of the vector space P_5 and suppose S has seven different vectors. Is S linearly independent? Why? (1 mark)

No, since $7>6=\dim(P_5)$