First Semester	**First Exam**	**King Saud University**
(without calculators)	**Time: 90 minutes**	**College of Science**
Monday 10-3-1445	**240 Math**	**Math. Department**

Q1: If $A=\begin{bmatrix}2 & 3\\ 1 & -1\end{bmatrix}$, $B=\begin{bmatrix}2 & -5 & 3\\ 2 & 3 & 0\end{bmatrix}$ and $C=\begin{bmatrix}2 & 9\\ 3 & 4\\ 0 & -1\end{bmatrix}$, then find the following:

(a) $B+3C^T$ (2 marks)

(b) $BC+10I_2$ (2 marks)

(c) $tr(A^2)$ (2 marks)

(d) adj(A) **in details** (2 marks)

Q2: Put the following matrix in the reduced row echelon form (R.R.E.F.):

(4 marks)

$$A=\begin{bmatrix}2 & 4 & 2 & 6\\ 3 & 1 & -2 & 4\\ 4 & 3 & 1 & 11\end{bmatrix}$$

Q3: Find the inverse of the following matrix by using elementary row operations and then find its determinant: (5 marks)

$$\begin{bmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 3 & 4\\ 1 & 2 & 4 & 4\\ 1 & 2 & 3 & 5\end{bmatrix}$$

Q4: Solve the following linear system By Gauss-Jordan Elimination: (4 marks)

$$\begin{aligned}2x_1+4x_2-2x_3&=4\\ x_1+3x_2+3x_3&=2\\ x_1+3x_2+5x_3&=4\end{aligned}$$

Q5: (a) Prove that if a square matrix A has a row of zeros, then $|A|=0$.

(1 mark)

(b) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

(1 mark)

(c) If A is an invertible matrix of size $n\times n$, then find:

(i) det(B), where B is the reduced row echelon form (R.R.E.F.) of A.

(ii) the solution of the linear system Ax=0.

(2 marks)

Solutions of the first mid-term exam

240 Math

Q1(a):

$$B+3C^T=\begin{bmatrix}2&-5&3\\2&3&0\end{bmatrix}+3\begin{bmatrix}2&3&0\\9&4&-1\end{bmatrix}$$
$$=\begin{bmatrix}2&-5&3\\2&3&0\end{bmatrix}+\begin{bmatrix}6&9&0\\27&12&-3\end{bmatrix}=\begin{bmatrix}8&4&3\\29&15&-3\end{bmatrix}$$

Q1(b):

$$BC+10I_2=\begin{bmatrix}2&-5&3\\2&3&0\end{bmatrix}\begin{bmatrix}2&9\\3&4\\0&-1\end{bmatrix}+10\begin{bmatrix}1&0\\0&1\end{bmatrix}$$
$$=\begin{bmatrix}-11&-5\\13&30\end{bmatrix}+\begin{bmatrix}10&0\\0&10\end{bmatrix}=\begin{bmatrix}-1&-5\\13&40\end{bmatrix}$$

Q1(c):

$$tr(A^2)=tr(AA)=tr\left(\begin{bmatrix}2&3\\1&-1\end{bmatrix}\begin{bmatrix}2&3\\1&-1\end{bmatrix}\right)$$
$$=tr\left(\begin{bmatrix}7&3\\1&4\end{bmatrix}\right)=7+4=11$$

Q1(d):

Adjoint A is equal to the transpose of the matrix of cofactors C from A, where

$$C=\begin{bmatrix}C_{11}&C_{12}\\C_{21}&C_{22}\end{bmatrix}$$

Such that

$$C_{11} = (-1)^{1+1}\det[-1] = (-1)^2(-1) = -1$$
$$C_{12} = (-1)^{1+2}\det[1] = (-1)^3(1) = -1$$
$$C_{21} = (-1)^{2+1}\det[3] = (-1)^3(3) = -3$$
$$C_{22} = (-1)^{2+2}\det[2] = (-1)^4(2) = 2$$

So

$$C = \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$$

and then

$$adj(A) = C^T = \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix}$$

Q2:

$$A = \begin{bmatrix} 2 & 4 & 2 & 6 \\ 3 & 1 & -2 & 4 \\ 4 & 3 & 1 & 11 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 1 & -2 & 4 \\ 4 & 3 & 1 & 11 \end{bmatrix} \xrightarrow[-4R_{13}]{-3R_{12}} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -5 & -5 & -5 \\ 0 & -5 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -5 & -3 & -1 \end{bmatrix} \xrightarrow{5R_{23}} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[(-1)R_{32}]{(-1)R_{31}} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{-2R_{21}} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The last matrix is the reduced row echelon form of A.

Q3:

$$[A\,|\,I]=\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\1&3&3&4&0&1&0&0\\1&2&4&4&0&0&1&0\\1&2&3&5&0&0&0&1\end{array}\right]\xrightarrow[(-1)R_{14}]{\substack{(-1)R_{12}\\(-1)R_{13}}}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]$$

So $\xrightarrow{(-2)R_{21}}\left[\begin{array}{cccc|cccc}1&0&3&4&3&-2&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]\xrightarrow{(-3)R_{31}}\left[\begin{array}{cccc|cccc}1&0&0&4&6&-2&-3&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]$

$$\xrightarrow{(-4)R_{41}}\left[\begin{array}{cccc|cccc}1&0&3&0&10&-2&-3&-4\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]=[I\,|\,A^{-1}]$$

$$A^{-1}=\begin{bmatrix}10&-2&-3&-4\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}$$

Now,

$$\det(A)=\begin{vmatrix}1&2&3&4\\1&3&3&4\\1&2&4&4\\1&2&3&5\end{vmatrix}\underset{(-1)R_{14}}{\overset{\substack{(-1)R_{12}\\(-1)R_{13}}}{=}}\begin{vmatrix}1&2&3&4\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1(1)(1)(1)=1$$

Q4:

We will solve the system by reducing the augmented matrix of the system in the reduced row echelon form (R.R.E.F.) and then solving the corresponding system of equations:

$$[A\,|\,b]=\left[\begin{array}{ccc|c}2&4&-2&4\\1&3&3&2\\1&3&5&4\end{array}\right]\xrightarrow{\frac{1}{2}R_1}\left[\begin{array}{ccc|c}1&2&-1&2\\1&3&3&2\\1&3&5&4\end{array}\right]$$

$$\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&4&0\\0&1&6&2\end{array}\right]\xrightarrow{(-1)R_{23}}\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&4&0\\0&0&2&2\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3}\left[\begin{array}{ccc|c}1&2&-1&2\\0&1&4&0\\0&0&1&1\end{array}\right]\xrightarrow[(1)R_{31}]{(-4)R_{21}}\left[\begin{array}{ccc|c}1&2&0&3\\0&1&0&-4\\0&0&1&1\end{array}\right]$$

$$\xrightarrow{(-2)R_{21}}\left[\begin{array}{ccc|c}1&0&0&11\\0&1&0&-4\\0&0&1&1\end{array}\right]$$

$$\Rightarrow x_1=11, x_2=-4, x_3=1$$

Q5(a):

Suppose A is of order n and the row of zeros is the row number i. Computing the determinant using the cofactor expansion, we get that:

$$\det(A)=\sum_{j=1}^{n}a_{ij}C_{ij}=\sum_{j=1}^{n}0C_{ij}=0$$

Q5(b):

From a theorem, we have that

$$\left(A^{-1}\right)^{T}=\left(A^{T}\right)^{-1}$$

But A is symmetric, so

$$\left(A^{T}\right)^{-1}=(A)^{-1}$$

Hence

$$\left(A^{-1}\right)^{T}=(A)^{-1}$$

Q5(c): (i)

Since A is invertible, we have from Equivalence Theorem that $B = I_n$.
Hence, $\det(B)=1$.

Q5(c): (ii)

Since A is invertible, we have from Equivalence Theorem that $x=0$.

OR

Since A is invertible, we have that $x = A^{-1}0 = 0$