First Semester 1443	First Exam	King Saud University
(without calculators)	Time: 90 Minutes	College of Science
Wednesday 29-7-1443	240 Math	Math. Department

Q1: If $A=\begin{bmatrix}1 & 2\\ 1 & -2\end{bmatrix}$, $B=\begin{bmatrix}1 & 0 & 2\\ 2 & 2 & 0\end{bmatrix}$, $C=\begin{bmatrix}1 & 2 & 0\\ 2 & 1 & 1\end{bmatrix}$ and $P(x)=2x^2-x-2$, then find the following:

(a) tr(P(A)). (4 marks)

(b) adj(BC^T). (3 marks)

(c) the inverse of A. (3 marks)

(a) $P(A)=2A^2-A-2I$

$$P(A)=2\begin{bmatrix}1 & 2\\ 1 & -2\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & -2\end{bmatrix}-\begin{bmatrix}1 & 2\\ 1 & -2\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=2\begin{bmatrix}3 & -2\\ -1 & 6\end{bmatrix}-\begin{bmatrix}1 & 2\\ 1 & -2\end{bmatrix}-\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

$$=\begin{bmatrix}6 & -4\\ -2 & 12\end{bmatrix}+\begin{bmatrix}-3 & -2\\ -1 & 0\end{bmatrix}=\begin{bmatrix}3 & -6\\ -3 & 12\end{bmatrix}$$

$$\Rightarrow tr(P(A))=3+12=15$$

(b) adj(BC^T)=

$$adj\left(\begin{bmatrix}1 & 0 & 2\\ 2 & 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 2 & 1\\ 0 & 1\end{bmatrix}\right)=adj\left(\begin{bmatrix}1 & 4\\ 6 & 6\end{bmatrix}\right)=\begin{bmatrix}6 & -4\\ -6 & 1\end{bmatrix}$$

$$C=\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}$$

Such that

$$C_{11}=(-1)^{1+1}\det[6]=(-1)^2(6)=6$$
$$C_{12}=(-1)^{1+2}\det[6]=(-1)^3(6)=-6$$
$$C_{21}=(-1)^{2+1}\det[4]=(-1)^3(4)=-4$$
$$C_{22}=(-1)^{2+2}\det[1]=(-1)^4(1)=1$$

So

$$Cof = \begin{bmatrix} 6 & -6 \\ -4 & 1 \end{bmatrix}$$

and then

$$adj(BC^T) = Cof^T = \begin{bmatrix} 6 & -4 \\ -6 & 1 \end{bmatrix}$$

(c) the inverse of A

$$A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{-4} \begin{bmatrix} -2 & -2 \\ -1 & 1 \end{bmatrix}$$

Q2: Solve the system by Gauss-Jordan elimination:

$$\begin{aligned} x + 2y + z &= 0 \\ 2x + 7y + 5z &= 3 \\ 2x + 5y + 3z &= 1 \end{aligned}$$

(4 marks)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 7 & 5 & 3 \\ 2 & 5 & 3 & 1 \end{array}\right] \xrightarrow[(-2)R_{13}]{(-2)R_{12}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 3 & 3 & 3 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[(-1)R_{23}]{(-2)R_{21}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow z = t \in \mathbb{R}$$

$$x = z - 2 = t - 2$$

$$y = -z + 1 = -t + 1$$

Q3: Find the determinant of the following matrix, then find the cofactor C_{43}:

(5 marks)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 5 & 4 \end{bmatrix}$$

$$\begin{vmatrix}1&2&3&4\\1&3&3&4\\1&2&3&5\\1&2&5&4\end{vmatrix} \overset{(-1)R_{12}}{\underset{(-1)R_{14}}{\overset{(-1)R_{13}}{=}}} \begin{vmatrix}1&2&3&4\\0&1&0&0\\0&0&0&1\\0&0&2&0\end{vmatrix} \overset{R_{34}}{=} -\begin{vmatrix}1&2&3&4\\0&1&0&0\\0&0&2&0\\0&0&0&1\end{vmatrix} = -1(1)(2)(1) = -2$$

$$C_{43} = (-1)^{4+3}\begin{vmatrix}1&2&4\\1&3&4\\1&2&5\end{vmatrix} \overset{(-1)R_{12}}{\underset{(-1)R_{13}}{=}} -\begin{vmatrix}1&2&4\\0&1&0\\0&0&1\end{vmatrix} = (-1)(1) = -1$$

Q4: (a) Prove that if A is an invertible matrix, then AA^T is also invertible. (1 mark)
If A is invertible, then A^T is invertible. Hence, AA^T is invertible, since the product of two invertible matrices is invertible.

(b) Prove that the inverse of any invertible matrix A is unique. (1 mark)

If B and C are two inverses of A, then

$$C = CI = C(AB) = (CA)B = IB = B$$

Hence, the invers is unique.

(c) If A and B are two symmetric matrices of order 2 Such that $AB = \begin{bmatrix}1&2\\2&3\end{bmatrix}$, then find BA. (1 mark)

Since A, B and AB are symmetric so AB=BA and hence $BA = \begin{bmatrix}1&2\\2&3\end{bmatrix}$

(d) If $E = \begin{bmatrix}a&b&a\\e&a&e\\a&a&a\end{bmatrix}$, then find det(E). (1 mark)

Since the first and the third columns are the same, det(E)=0.

(e) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric. (1 mark)

$$(A^{-1})^T=(A^T)^{-1}=A^{-1}$$

So A^{-1} is symmetric.

(f) If A is an invertible matrix of size 3×3 and |A|=2, then find $|2(A^T)^{-1}|$. (1 mark)

$$det\left(2\left(A^T\right)^{-1}\right) = 2^3 det\left(\left(A^T\right)^{-1}\right) = 8det\left(\left(A^{-1}\right)^T\right)$$
$$= 8det\left(A^{-1}\right) = 8(\tfrac{1}{2}) = 4$$