| First Semester | First Exam | King Saud University |
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| (without calculators) | Time: 8 to 9:30 am | College of Science |
| Wednesday 29-2-1443 | 240 Math | Math. Department |

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Q1: If $A = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$, then find the following:

(a) tr($BB^T + A^2 + 3I_2$) (6 marks)

$$BB^T = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 5 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 1 & 3 \end{bmatrix}$$

$$3I_2 = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$tr(BB^T + A^2 + 3I_2) = tr\left(\begin{bmatrix} 14 & 5 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} 23 & 7 \\ 6 & 11 \end{bmatrix}\right) = 23 + 11 = 34$$

(b) adj(F) <u>in details</u> (4 marks)

$$adj(F) = adj\left(\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Q2: Put the following matrix in the reduced row echelon form (R.R.E.F.):

(4 marks)

$$A = \begin{bmatrix} 2 & 4 & 2 & 6 \\ 3 & 1 & -2 & 4 \\ 4 & 3 & -1 & 11 \end{bmatrix}$$

$$A=\begin{bmatrix}2&4&2&6\\3&1&-2&4\\4&3&-1&11\end{bmatrix}\xrightarrow{\frac{1}{2}R_1}\begin{bmatrix}1&2&1&3\\3&1&-2&4\\4&3&-1&11\end{bmatrix}\xrightarrow[-4R_{13}]{-3R_{12}}\begin{bmatrix}1&2&1&3\\0&-5&-5&-5\\0&-5&-5&-1\end{bmatrix}$$

$$\xrightarrow{-\frac{1}{5}R_2}\begin{bmatrix}1&2&1&3\\0&1&1&1\\0&-5&-5&-1\end{bmatrix}\xrightarrow{5R_{23}}\begin{bmatrix}1&2&1&3\\0&1&1&1\\0&0&0&4\end{bmatrix}\xrightarrow{\frac{1}{4}R_3}\begin{bmatrix}1&2&1&3\\0&1&1&1\\0&0&0&1\end{bmatrix}$$

$$\xrightarrow[(-1)R_{32}]{(-1)R_{31}}\begin{bmatrix}1&2&1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}\xrightarrow{-2R_{21}}\begin{bmatrix}1&0&-1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}$$

Q3: Find the inverse of the following matrix by using elementary row operations and then find the cofactor $C_{32}$:

(5 marks)

$$A=\begin{bmatrix}1&2&3&4\\1&3&3&4\\1&2&4&4\\1&2&3&5\end{bmatrix}$$

$$[A\,|\,I]=\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\1&3&3&4&0&1&0&0\\1&2&4&4&0&0&1&0\\1&2&3&5&0&0&0&1\end{array}\right]\xrightarrow[(-1)R_{14}]{(-1)R_{12},\ (-1)R_{13}}\left[\begin{array}{cccc|cccc}1&2&3&4&1&0&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]$$

So $$\xrightarrow{(-2)R_{21}}\left[\begin{array}{cccc|cccc}1&0&3&4&3&-2&0&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]\xrightarrow{(-3)R_{31}}\left[\begin{array}{cccc|cccc}1&0&0&4&6&-2&-3&0\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]$$

$$\xrightarrow{(-4)R_{41}}\left[\begin{array}{cccc|cccc}1&0&3&0&10&-2&-3&-4\\0&1&0&0&-1&1&0&0\\0&0&1&0&-1&0&1&0\\0&0&0&1&-1&0&0&1\end{array}\right]=[I\,|\,A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 10 & -2 & -3 & -4 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Now,

$$C_{32} = -\begin{vmatrix} 1 & 3 & 4 \\ 1 & 3 & 4 \\ 1 & 3 & 5 \end{vmatrix} = 0$$

Q4: Solve the following linear system By Gaussian Elimination:

(4 marks)

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 2 \\ x_1 + 3x_2 + 5x_3 &= 4 \end{aligned}$$

We will solve the system by reducing the augmented matrix of the system in the row echelon form (R.E.F.) and then solving the corresponding system of equations:

$$[A\,|\,b] = \left[\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 1 & 3 & 3 & 2 \\ 1 & 3 & 5 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 3 & 3 & 2 \\ 1 & 3 & 5 & 4 \end{array}\right]$$

$$\xrightarrow[(-1)R_{13}]{(-1)R_{12}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 6 & 2 \end{array}\right] \xrightarrow{(-1)R_{23}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

So

$$x_3 = 1$$
$$x_2 + 4x_3 = 0$$
$$\Rightarrow x_2 = -4x_3 = -4(1) = -4,$$
$$x_1 + 2x_2 - x_3 = 2$$
$$\Rightarrow x_1 = -2x_2 + x_3 + 2$$
$$-2(-4) + 1 + 3 = 11,$$

Q5: (a) Prove that if a square matrix A has a row of zeros, then |A|=0.

(1 mark)

Suppose $A$ is of order $n$ and the row of zeros is the row number $i$. Computing the determinant using the cofactor expansion, we get that:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} 0 C_{ij} = 0$$

(b) Prove that if A is a symmetric matrix, then $A^2$ is symmetric.

(1 mark)

$(A^2)^T = (AA)^T = A^T A^T = AA = A^2$

or

Since A is symmetric and commutes with itself, then AA= $A^2$ is symmetric.