Q1: Let $F=\begin{bmatrix}1 & 1 & 0\\ 2 & 1 & -1\\ 3 & 2 & 0\end{bmatrix}$. Find

(a) the inverse of F. Then **deduce** the inverse of (113F). (4 marks)

(b) the cofactor C_{32}. (2 marks)

Answer: (a) We have:

$$[F\,|\,I]=\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 2 & 1 & -1 & 0 & 1 & 0\\ 3 & 2 & 0 & 0 & 0 & 1\end{array}\right]\xrightarrow[-3R_{13}]{-2R_{12}}\left[\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & -1 & -1 & -2 & 1 & 0\\ 0 & -1 & 0 & -3 & 0 & 1\end{array}\right]$$

$$\xrightarrow[-1R_{23}]{1R_{21}}\left[\begin{array}{ccc|ccc}1 & 0 & -1 & -1 & 1 & 0\\ 0 & -1 & -1 & -2 & 1 & 0\\ 0 & 0 & 1 & -1 & -1 & 1\end{array}\right]\xrightarrow[1R_{32}]{1R_{31}}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -2 & 0 & 1\\ 0 & -1 & 0 & -3 & 0 & 1\\ 0 & 0 & 1 & -1 & -1 & 1\end{array}\right]$$

$$\xrightarrow{-1R_{2}}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -2 & 0 & 1\\ 0 & 1 & 0 & 3 & 0 & -1\\ 0 & 0 & 1 & -1 & -1 & 1\end{array}\right]=[I\,|\,F^{-1}]$$

$$(113F)^{-1}=\tfrac{1}{113}F^{-1}=\frac{1}{113}\begin{bmatrix}-2 & 0 & 1\\ 3 & 0 & -1\\ -1 & -1 & 1\end{bmatrix}$$

(b)

$$C_{32}=(-1)\begin{vmatrix}1 & 0\\ 2 & -1\end{vmatrix}=(-1)(-1)=1$$

Q2: Solve the following linear system By Gauss-Jordan Elimination:

(5 marks)

$$\begin{aligned}2x_1+4x_2-2x_3&=2\\ x_1+3x_2+3x_3&=2\\ x_1+3x_2+5x_3&=4\end{aligned}$$

Answer: We will solve the system by reducing the augmented matrix of the system in the reduced row echelon form (R.R.E.F.) and then solving the corresponding system of equations:

$$[A\,|\,b]=\left[\begin{array}{ccc|c}2&4&-2&2\\1&3&3&2\\1&3&5&4\end{array}\right]\xrightarrow{\frac{1}{2}R_1}\left[\begin{array}{ccc|c}1&2&-1&1\\1&3&3&2\\1&3&5&4\end{array}\right]$$

$$\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\left[\begin{array}{ccc|c}1&2&-1&1\\0&1&4&1\\0&1&6&3\end{array}\right]\xrightarrow[(-1)R_{23}]{(-2)R_{21}}\left[\begin{array}{ccc|c}1&0&-9&-1\\0&1&4&1\\0&0&2&2\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3}\left[\begin{array}{ccc|c}1&0&-9&-1\\0&1&4&1\\0&0&1&1\end{array}\right]\xrightarrow[(-4)R_{32}]{(9)R_{31}}\left[\begin{array}{ccc|c}1&0&0&8\\0&1&0&-3\\0&0&1&1\end{array}\right]$$

$$\Rightarrow (x_1,x_2,x_3)=(8,-3,1)$$

Q3: Let V be any nonempty set which has two operations are defined: addition and scalar multiplication. State the 10 axioms that should be satisfied by all scalars and all objects in V that make V a vector space. (5 marks)

Answer: For all u,v,w∈V and k,m∈ℝ:

1- u+v∈V
2- u+v=v+u
3- u+(v+w)=(u+v)+w
4- there is a zero vector 0 in v such that u+0=u for all u∈V
5- for each vector u in V, there is a negative vector –u such u+(-u)=0
6- ku∈V
7- k(u+v)=ku+kv
8- (k+m)u=ku+mu
9- k(mu)=(km)u
10- 1u=u

Q4: Let V=M_{22} and W={A∈ M_{22}|det(A)=0}. Show that W is a **not** subspace of V. (2 marks)

Answer: Let A=$\begin{bmatrix}1&1\\1&1\end{bmatrix}$ and B=$\begin{bmatrix}1&2\\0&0\end{bmatrix}$. Then |A|=|B|=0 and A,B∈W. Now, A+B=$\begin{bmatrix}2&3\\1&1\end{bmatrix}$ and |A+B|=2−3=−1≠0. So, A+B∉W and W is a not subspace of V.

Q5: Use the Wronskian to show that the vectors: 1, x and cos(x) are linearly independent in the vector space $C^\infty(-\infty,\infty)$. (3 marks)

Answer: As

$$W(x) = \begin{vmatrix} 1 & x & \cos(x) \\ 0 & 1 & -\sin(x) \\ 0 & 0 & -\cos(x) \end{vmatrix} = -\cos(x)$$

$$W(0) = -\cos(0) = -1 \neq 0$$

So the vectors 1, x and cos(x) are linearly independent.

Q6: (a) Prove that if A has an inverse, then it is unique. (2 marks)

Answer: Suppose A has two inverses B and C. So

$$B = BI = B(AC) = (BA)C = IC = C$$

So the inverse is unique.

(b) Suppose A has an inverse. Show that $\det(A^{-1})=(\det(A))^{-1}$. (2 marks)

Answer: Since $AA^{-1}=I$ and $\det(A)\neq 0$, So

$$|A||A^{-1}| = |AA^{-1}| = |I| = 1$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|} = |A|^{-1}$$

(c) Suppose S is a subset of the vector space P_5 and suppose S has five different vectors. Is S a basis of P_5? Why? (1 mark)

Answer: No, since $\dim(P_5)=6>5$.

(d) If A is an invertible matrix of size 2×2 and $|A|=3$, then find $|3((A^T)^2)^{-1}|$.

(2 marks)

Answer:

$$\left|3\left(\left(A^T\right)^2\right)^{-1}\right| = 3^2\left|\left(\left(A^T\right)^2\right)^{-1}\right| = 9\times\frac{1}{\left|\left(A^T\right)^2\right|}$$

$$= \frac{9}{|A^T|^2} = \frac{9}{|A|^2} = \frac{9}{3^2} = \frac{9}{9} = 1$$

(e) If the general solution of a nonhomogeneous linear system is {(2r−s+1,r,s,5)}, then find the general solution of the corresponding homogeneous linear system. (2 marks)

Answer: S.S={(2r−s,r,s,0)}