Q1: (a) Find the inverse of $F=\begin{bmatrix}1&1&0\\2&1&-1\\3&2&0\end{bmatrix}$. Then find the cofactor C_{32}. (5 marks)

Answer: We have:

$$[F\mid I]=\left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\2&1&-1&0&1&0\\3&2&0&0&0&1\end{array}\right]\xrightarrow[-3R_{13}]{-2R_{12}}\left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&-1&-1&-2&1&0\\0&-1&0&-3&0&1\end{array}\right]$$

$$\xrightarrow[-1R_{23}]{1R_{21}}\left[\begin{array}{ccc|ccc}1&0&-1&-1&1&0\\0&-1&-1&-2&1&0\\0&0&1&-1&-1&1\end{array}\right]\xrightarrow[1R_{32}]{1R_{31}}\left[\begin{array}{ccc|ccc}1&0&0&-2&0&1\\0&-1&0&-3&0&1\\0&0&1&-1&-1&1\end{array}\right]$$

$$\xrightarrow{-1R_{2}}\left[\begin{array}{ccc|ccc}1&0&0&-2&0&1\\0&1&0&3&0&-1\\0&0&1&-1&-1&1\end{array}\right]=[I\mid F^{-1}]$$

Now:

$$C_{32}=(-1)\begin{vmatrix}1&0\\2&-1\end{vmatrix}=(-1)(-1)=1$$

(b) Find tr(F) and $F^2+(2F)^T$. (4 marks)

Answer: tr(F)=1+1+0=2 and $F^2+(2F)^T=$

$$F^2+(2F)^T=\begin{bmatrix}1&1&0\\2&1&-1\\3&2&0\end{bmatrix}\begin{bmatrix}1&1&0\\2&1&-1\\3&2&0\end{bmatrix}+2\left(\begin{bmatrix}1&1&0\\2&1&-1\\3&2&0\end{bmatrix}\right)^T$$

$$=\begin{bmatrix}3&2&-1\\1&1&-1\\7&5&-2\end{bmatrix}+2\begin{bmatrix}1&2&3\\1&1&2\\0&-1&0\end{bmatrix}$$

$$=\begin{bmatrix}3&2&-1\\1&1&-1\\7&5&-2\end{bmatrix}+\begin{bmatrix}2&4&6\\2&2&4\\0&-2&0\end{bmatrix}=\begin{bmatrix}5&6&5\\3&3&3\\7&3&-2\end{bmatrix}$$

Q2: Solve the following linear system By Gauss-Jordan Elimination:

(5 marks)

$$2x_1 + 4x_2 - 2x_3 = 4$$
$$x_1 + 3x_2 + 3x_3 = 2$$
$$x_1 + 3x_2 + 5x_3 = 4$$

Answer: We will solve the system by reducing the augmented matrix of the system in the reduced row echelon form (R.R.E.F.) and then solving the corresponding system of equations:

$$[A\,|\,b] = \left[\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 1 & 3 & 3 & 2 \\ 1 & 3 & 5 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 3 & 3 & 2 \\ 1 & 3 & 5 & 4 \end{array}\right]$$

$$\xrightarrow[(-1)R_{13}]{(-1)R_{12}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 6 & 2 \end{array}\right] \xrightarrow[(-1)R_{23}]{(-2)R_{21}} \left[\begin{array}{ccc|c} 1 & 0 & -9 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -9 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[(-4)R_{32}]{(9)R_{31}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\Rightarrow (x_1, x_2, x_3) = (11, -4, 1)$$

Q3: Let V be any nonempty set which has two operations are defined: addition and scalar multiplication. State the 10 axioms that should be satisfied by all scalars and all objects in V that make V a vector space. (5 marks)

Answer: For all u,v,w∈V and k,m∈ℝ:

1- u+v∈ℝ
2- u+v=v+u
3- u+(v+w)=(u+v)+w
4- there is a zero vector 0 in v such that u+0=u for all u∈V
5- for each vector u in V, there is a negative vector –u such u+(-u)=0
6- ku∈V
7- k(u+v)=ku+kv
8- (k+m)u=ku+mu
9- K(mu)=(km)u
10- 1u=u

Q4: Let V=M_{22} and W={A∈ M_{22}|tr(A)=0}. Prove that W is a subspace of V. (3 marks)

Answer: For all A=$\begin{bmatrix} a & a' \\ a'' & a''' \end{bmatrix}$,B=$\begin{bmatrix} b & b' \\ b'' & b''' \end{bmatrix}$∈W and k∈ℝ:

1- W is not empty since tr(0)=0. Hence 0∈W

2- tr(A+B) =tr$\left(\begin{bmatrix} a+b & a'+b' \\ a''+b'' & a'''+b''' \end{bmatrix}\right)$=$a$+$b$+$a'''$+$b'''$ = a +a''' +b+b''' =tr(A)+tr(B)=0+0=0. So A+B∈W.

3- tr(kA) = tr$\left(\begin{bmatrix} ka & ka' \\ ka'' & ka''' \end{bmatrix}\right)$=$ka$+$ka'''$ = $k(a$+$a''')$ =ktr(A)=k0=0. So kA∈W

1, 2 and 3 implies that W is a subspace of V=M_{nn}.

Q5: Use the Wronskian to show that the vectors: 1, x and cos(x) are linearly independent in the vector space $C^\infty(-\infty,\infty)$. (3 marks)

Answer: As

$$W(x)=\begin{vmatrix} 1 & x & \cos(x) \\ 0 & 1 & -\sin(x) \\ 0 & 0 & -\cos(x) \end{vmatrix}=-\cos(x)$$
$$W(0)=-\cos(0)=-1\neq 0$$

So the vectors 1, x and cos(x) are linearly independent.

Q6: (a) Prove that if A has an inverse, then it is unique. (1 mark)

Answer: Suppose A has two inverses B and C. So

$$B=BI=B(AC)=(BA)C=IC=C$$

So the inverse is unique.

(b) Suppose A has an inverse. Show that $\det(A^{-1})=(\det(A))^{-1}$. (1 mark)

Answer: Since AA^{-1}=I and det(A)≠0, So

$$|A||A^{-1}|=|AA^{-1}|=|I|=1$$
$$\Rightarrow |A^{-1}|=\frac{1}{|A|}=|A|^{-1}$$

(c) Suppose S is a subset of the vector space $\mathbb{R}^5$ and suppose S has seven different vectors. Is S linearly independent? Why? (1 mark)

Answer: No, since 7>5.

(d) If A is an invertible matrix of size 2×2 and |A|=3, then find $|3((A^T)^2)^{-1}|$.

(2 marks)

Answer:

$$\left|3\left(\left(A^T\right)^2\right)^{-1}\right| = 3^2\left|\left(\left(A^T\right)^2\right)^{-1}\right| = 9\times\frac{1}{\left|\left(A^T\right)^2\right|}$$

$$= \frac{9}{\left|A^T\right|^2} = \frac{9}{\left|A\right|^2} = \frac{9}{3^2} = \frac{9}{9} = 1$$