First Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Monday 5-6-1445	**240 Math**	**Math. Department**

Q1: If A is a matrix such that $A\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$, then find:

(i) the size of A. (1 mark)

(ii) an eigenvalue of A. (1 mark)

Q2: If A,B∈M_{22}, det(B)=2 and det(A)=3, then find det($2A^TB^{-1}$). (2 marks)

Q3: Let V be the subspace of $\mathbb{R}^4$ **spanned** by the set S={v_1=(1,1,1,0), v_2=(−2,0,0,2), v_3=(−1,3,3,4), v_4=(−5,−1,−1,5)}.

(i) Find a **subset** of S that forms a basis of V. (3 marks)

(ii) **Find** dim(V). (1 mark)

(iii) show that (−6,0,0,7)∈V. (3 marks)

Q4: Let W={(a,0)∈ $\mathbb{R}^2$: a∈ $\mathbb{R}$}. Show that W is a **subspace** of $\mathbb{R}^2$. (3 marks)

Q5: Let B={(1,0),(1,1)} and B'={u,v} be two bases of $\mathbb{R}^2$. If the transition matrix from B' to B is $\begin{bmatrix}1&2\\2&1\end{bmatrix}$, then find u. (2 marks).

Q6: Let $A=\begin{bmatrix}1&a&b\\0&-1&c\\0&0&0\end{bmatrix}$, where a, b and c are real numbers.

(i) Show that A is diagonalizable. (3 marks)

(ii) If P is the matrix that diagonalizes A, then find the product $P^{-1}AP$. (1 mark)

(iii) If x=$[0\ 1\ 0]^T$ is an eigenvector of A, then find the value of a . (2 marks)

Q7: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis {u_1=(0,1,−1),u_2=(0,4,2),u_3=(1,0,0)} into an **orthonormal basis**. (5 marks)

Q8: Let M_{22} be the vector space of square matrices of order 2, and let T: M_{22}→ $\mathbb{R}$ be the function defined by $T\begin{pmatrix}a&b\\c&d\end{pmatrix}=a$ for all $a,b,c,d\in\mathbb{R}$. Show that:

(i) T is a linear transformation. (2 marks)

(ii) Find a basis for ker(T). (3 marks)

(iii) Find $[T]_{B',B}$ where B and B' are the standard bases of M_{22} and $\mathbb{R}$, respectively. (2 marks)

(iv) Find rank(T). (1 mark)

Q9: (i) If B={u,v,w} is a basis of a vector space V, then find the coordinate vector $(v)_B$. (1 mark)

(ii) If *u* and *v* are orthogonal vectors in an inner product space such that ‖u‖=8 and ‖v‖=6, then find ‖u+v‖. (1 mark)

(iii) If B is a 5×7 matrix with nullity(B)=3, then find rank(B^T). (1 mark)

(iv) Show that if A is a diagonalizable matrix of order n such that AP=PA, where P is the matrix that diagonalizes A, then A is a diagonal matrix. (1 mark)

(v) Suppose that A and B are invertible matrices of the same size. Show that if B is similar to A, then B^{-1} is similar to A^{-1}. (1 mark)

Q1: If A is a matrix such that $A\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$, then find:

(i) the size of A. (1 mark)

(ii) an eigenvalue of A. (1 mark)

Answer: (i) Suppose A is of size m×n. Since $\begin{bmatrix}1\\2\end{bmatrix}$ is of size 2×1 and the product $A\begin{bmatrix}1\\2\end{bmatrix}$ is defined, so n=2. But the product $A\begin{bmatrix}1\\2\end{bmatrix}$ is of size m×1 and is equal to $\begin{bmatrix}2\\4\end{bmatrix}$ which is of size 2×1. So, m=2. Hence, A is of size 2×2.

(ii) Since $A\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}=2\begin{bmatrix}1\\2\end{bmatrix}$, so 2 is an eigenvalue of A.

Q2: If A,B∈M_{22}, det(B)=2 and det(A)=3, then find det($2A^TB^{-1}$). (2 marks)

Answer: det($2A^TB^{-1}$)= 2^2det(A^T)det(B^{-1})=4det(A)(det(B))$^{-1}$=4(3)(1/2)=6.

Q3: Let V be the subspace of $\mathbb{R}^4$ **spanned** by the set S={v_1=(1,1,1,0), v_2=(−2,0,0,2), v_3=(−1,3,3,4), v_4=(−5,−1,−1,5)}.

(i) Find a **subset** of S that forms a basis of V. (3 marks)

(ii) **Find** dim(V). (1 mark)

(iii) show that u=(−6,0,0,7)∈V. (3 marks)

Answer: (i) Putting the vectors as columns in the following matrix:

$$\begin{bmatrix}1&-2&-1&-5\\1&0&3&-1\\1&0&3&-1\\0&2&4&5\end{bmatrix}\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\begin{bmatrix}1&-2&-1&-5\\0&2&4&4\\0&2&4&4\\0&2&4&5\end{bmatrix}\xrightarrow[(-1)R_{24}]{(-1)R_{23}}\begin{bmatrix}1&-2&-1&-5\\0&2&4&4\\0&0&0&0\\0&0&0&1\end{bmatrix}$$

$$\xrightarrow[R_{34}]{\frac{1}{2}R_2}\begin{bmatrix}1&-2&-1&-5\\0&1&2&2\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

So, S={v_1, v_2,v_4} is a basis of V.

(ii) dim(V)=|S|=3.

(iii) Suppose u=av_1+bv_2+cv_4, where a, b and c are scalars. So,

$$\begin{bmatrix}1&-2&-5&-6\\1&0&-1&0\\1&0&-1&0\\0&2&5&7\end{bmatrix}\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\begin{bmatrix}1&-2&-5&-6\\0&2&4&6\\0&2&4&6\\0&2&5&7\end{bmatrix}\xrightarrow[(-1)R_{24}]{\substack{1R_{21}\\(-1)R_{23}}}\begin{bmatrix}1&0&-1&0\\0&2&4&6\\0&0&0&0\\0&0&1&1\end{bmatrix}$$

$$\xrightarrow[R_{34}]{\frac{1}{2}R_2}\begin{bmatrix}1&0&-1&0\\0&1&2&3\\0&0&1&1\\0&0&0&0\end{bmatrix}\xrightarrow[(-3)R_{31}]{(-2)R_{32}}\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

$\Rightarrow a=b=c=1$

So, u∈V.

Q4: Let W={(a,0)∈ $\mathbb{R}^2$: a∈ $\mathbb{R}$}. Show that W is a **subspace** of $\mathbb{R}^2$. (3 marks)

Answer: 1- If a=0, then (0,0)∈W. So W≠∅.

2- Take u=(a_1,0),v=(a_2,0)∈W. Now, u+v=(a_1+a_2,0). So u+v∈W.

3- Take u=(a,0)∈W & k∈$\mathbb{R}$. Now, ku=(ka,k0)=(ka,0). So ku∈W.

1,2 and 3 imply that W is a subspace of $\mathbb{R}^2$.

Q5: Let B={(1,0),(1,1)} and B'={u,v} be two bases of $\mathbb{R}^2$. If the transition matrix from B' to B is $\begin{bmatrix}1&2\\2&1\end{bmatrix}$, then find u. (2 marks).

Answer:

$$P_{B'\to B}=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}=\left[[u]_B \quad | \quad [v]_B\right]$$

$$\Rightarrow [u]_B=\begin{bmatrix}1\\ 2\end{bmatrix}$$

$$\Rightarrow u=(1)(1,0)+(2)(1,1)=(1,0)+(2,2)=(3,2)$$

Q6: Let $A=\begin{bmatrix}1 & a & b\\ 0 & -1 & c\\ 0 & 0 & 0\end{bmatrix}$, where a, b and c are real numbers.

(i) Show that A is diagonalizable. (3 marks)

(ii) If P is the matrix that diagonalizes A, then find the product $P^{-1}AP$. (1 mark)

(iii) If $x=[0\ 1\ 0]^T$ is an eigenvector of A, then find the value of a . (2 marks)

Answer: (i) A is an upper triangular. So, the eigenvalues are 1, -1 and 0. Since they are distinct, A is diagonalizable.

(ii) $P^{-1}AP=D=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{bmatrix}$ (the arrangement of the entries in the main diagonal is not important).

(iii) λx=Ax. So,

$$\begin{bmatrix}0\\ \lambda\\ 0\end{bmatrix}=\lambda\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}1 & a & b\\ 0 & -1 & c\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}a\\ -1\\ 0\end{bmatrix}$$

So, $a=0$.

Q7: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis {u_1=(0,1,−1),u_2=(0,4,2),u_3=(1,0,0)} into an **orthonormal basis**. (5 marks)

Answer:

$$u_1=(0,1,-1), u_2=(0,4,2), u_3=(1,0,0),$$

$$v_1=u_1=(0,1,-1)$$

$$v_2=u_2-\frac{<u_2,v_1>}{\|v_1\|^2}v_1$$

$$=(0,4,2)-\frac{<(0,4,2),(0,1,-1)>}{\|(0,1,-1)\|^2}(0,1,-1)=(0,4,2)-\tfrac{2}{2}(0,1,-1)$$

$$=(0,4,2)-(0,1,-1)=(0,3,3)$$

$$v_3=u_3-\frac{<u_3,v_2>}{\|v_2\|^2}v_2-\frac{<u_3,v_1>}{\|v_1\|^2}v_1$$

$$=(1,0,0)-\frac{<(1,0,0),(0,1,-1)>}{\|(0,1,-1)\|^2}(0,1,-1)-\frac{<(1,0,0),(0,3,3)>}{\|(0,3,3)\|^2}(0,3,3)$$

$$=(1,0,0)-\tfrac{0}{2}(0,1,-1)-\tfrac{0}{18}(0,3,3)=(1,0,0)$$

$$w_1=\frac{v_1}{\|v_1\|}=\frac{1}{\sqrt{2}}(0,1,-1)$$

$$w_2=\frac{v_2}{\|v_2\|}=\frac{1}{\sqrt{18}}(0,3,3)=\frac{1}{3\sqrt{2}}(0,3,3)=\frac{1}{\sqrt{2}}(0,1,1)$$

$$w_3=\frac{v_3}{\|v_3\|}=(1,0,0)$$

Q8: Let M_{22} be the vector space of square matrices of order 2, and let T: $M_{22} \to \mathbb{R}$ be the function defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a$ for all $a, b, c, d \in \mathbb{R}$. Show that:

(i) T is a linear transformation. (2 marks)

(ii) Find a basis for ker(T). (3 marks)

(iii) Find $[T]_{B',B}$ where B and B' are the standard bases of M_{22} and $\mathbb{R}$, respectively. (2 marks)

(iv) Find rank(T). (1 mark)

Answer: For all $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in M_{22}, k \in \mathbb{R}$:

(i) 1- T(A+B)= $T\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} = a + a'$ =T(A)+T(B)

2- T(*k*A)= $T\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = ka$ =*k*T(A)

So T is linear.

(ii) ker(T)={A∈ M_{22}|T(A)=0}={$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} | a=0$}={$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \in M_{22} | b,c,d \in \mathbb{R}$ }

$$= \left\{ b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} | b, c, d \in \mathbb{R} \right\}$$

So, the set $S = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ spans ker(T). Observe that

$$b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

implies that

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and hence b=c=d=0. So, S is linearly independent also. Thus, S is a basis of ker(T).

(iii) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1, T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0, T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0$ and $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$.

Now,

$\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_B = 1, \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_B = 0, \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_B = 0, \left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_B = 0.$ Therefore, $[T]_{B',B} = [1 \quad 0 \quad 0 \quad 0]$.

(iv) Since dim(ker(T))=3, so nullity(T)=3 and hence rank(T)=dim(M_{22})-nullity(T)=4-3=1.

Q9: (i) If B={u,v,w} is a basis of a vector space V, then find the coordinate vector $(v)_B$. (1 mark)

Answer: As v=0u+1v+0w and writing a vector as a linear combination of the vectors in B is unique, so $(v)_B$=(0,1,0)

(ii) If *u* and *v* are orthogonal vectors in an inner product space such that ‖u‖=8 and ‖v‖=6, then find ‖u+v‖. (1 mark)

Answer: As *u* and *v* are orthogonal, so $\|u+v\|^2=\|u\|^2+\|v\|^2=64+36=100$. So ‖u+v‖=10.

(iii) If *B* is a 5×7 matrix with nullity(B)=3, then find rank(B^T). (1 mark)

Answer: rank(B^T)=rank(B)=7− nullity(B)= 7−3=4

(iv) Show that if A is a diagonalizable matrix of order n such that AP=PA, where P is the matrix that diagonalizes A, then A is a diagonal matrix. (1 mark)

Answer: Since A is diagonalizable, we have that $P^{-1}AP$ is a diagonal matrix. But we have that AP=PA. So, $P^{-1}AP=P^{-1}(AP)=P^{-1}(PA)= (P^{-1}P)A$ =IA=A. So, A is diagonal.

(v) Suppose that A and B are invertible matrices of the same size. Show that if B is similar to A, then B^{-1} is similar to A^{-1}. (1 mark)

Answer: Since B is similar to A, so B= $P^{-1}AP$. Taking the inverse of the both sides, we have

$$B^{-1}= P^{-1}A^{-1}(P^{-1})^{-1}= P^{-1}A^{-1}P$$

So, B^{-1} is similar to A^{-1}.