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| **Third Semester** | **Final Exam** | **King Saud University** |
| **(without calculators)** | **Time allowed: 3 hours** | **College of Science** |
| **Sunday 29-11-1444** | **240 Math** | **Math. Department** |

**<u>Q1</u>**: If A=$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, then find $|2A^T|A^{-1}+A$. (4 marks)

**<u>Q2</u>**: Solve the following system:

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 2x_1 + 2x_2 - 2x_3 &= 2 \\ 3x_3 &= -3 \end{aligned}$$

(2 marks)

**<u>Q3</u>**: Let V be the subspace of $\mathbb{R}^4$ **<u>spanned</u>** by the set S={$v_1$=(1,1,1,1), $v_2$=(−2,−2,−2,−2), $v_3$=(3,3,3,3), $v_4$=(−5,−5,−5,−4)}. Find a **<u>subset</u>** of S that forms a basis of V. (3 marks)

**<u>Q4</u>**: Let W={(a,0)∈$\mathbb{R}^2$: a∈$\mathbb{R}$}. Show that W is a **<u>subspace</u>** of $\mathbb{R}^2$. (3 marks)

**<u>Q5</u>**: Let B={(3,3,3),(2,2,0),(1,0,0)} be a basis of $\mathbb{R}^3$. If $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is the transition matrix from S to B, where S={$u_1$,$u_2$,$u_3$} is another basis of $\mathbb{R}^3$. Then find $u_1$. (3 marks)

**<u>Q6</u>**: Show that the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is diagonalizable and find the matrix P that diagonalizes A. (6 marks)

**<u>Q7</u>**: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis {$u_1$=(1,1,1),$u_2$=(0,1,−1),$u_3$=(0,4,2)} into an **<u>orthonormal basis</u>**. (6 marks)

**<u>Q8</u>**: Let $M_{22}$ be the vector space of square matrices of order 2, C=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and let T: $M_{22}$→$M_{22}$ be the function defined by T(A)=AC for all matrices A in $M_{22}$. Show that:

(i) T is a linear operator. (2 marks)

(ii) Find a basis for ker(T). (2 marks)

(iii) Find $[T]_S$ where S is the standard basis of $M_{22}$. (2 marks)

(iv) Find rank(T). (2 marks)

**Q9**: (i) If B=$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, D=$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and A−C=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then find BAD−BCD. (1 mark)

(ii) If g is a function from $\mathbb{R}^3\times\mathbb{R}^3$ to $\mathbb{R}$ defined by: g(u,v)=0 for all u and v in $\mathbb{R}^3$, then show that g is **<u>not</u>** an inner product function. (1 mark)

(iii) If T:$P_5$→$M_{22}$ is a linear transformation with nullity(T)=3, then find rank(T). (1 mark)

(iv) If 2 and 3 are all the eigenvalues of a matrix A, then find the eigenvalues of $A^{-1}$. (1 mark)

(v) If B={$v_1$,$v_2$,$v_3$} is a basis of a vector space V, then $c_1v_1+c_2v_2+c_3v_3=0$, where $c_1$, $c_2$ and $c_3$ are scalars, implies that (choose the correct answer): (1 mark)

(a) $v_1=v_2=v_3=0$ (b) $c_1=c_2=c_3=0$ (c) $c_1=v_1,c_2=v_2,c_3=v_3$ (d) B is a subspace of V

**Q1**: If A=$\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$, then find $|2A^T|A^{-1}$+A. (4 marks)

Answer: $A^{-1}=\frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ and $|2A^T|=4|A|=4(2)=8$. So, $|2A^T|A^{-1}+A=8(\frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix})+\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$

$$=\begin{bmatrix}4 & -4\\ 4 & 4\end{bmatrix}+\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}=\begin{bmatrix}5 & -3\\ 3 & 5\end{bmatrix}$$

**Q2**: Solve the following system:

$$\begin{aligned} x_1+x_2-x_3&=1\\ 2x_1+2x_2-2x_3&=2\\ 3x_3&=-3 \end{aligned}$$

(2 marks)

Answer: Using the augmented matrix of the system

$$\left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\ 2 & 2 & -2 & 2\\ 0 & 0 & 3 & -3\end{array}\right]\xrightarrow[(-2)R_{12}]{\frac{1}{3}R_3}\left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1\end{array}\right]\xrightarrow{R_{23}}\left[\begin{array}{ccc|c}1 & 1 & -1 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{array}\right]\xrightarrow{(1)R_{21}}\left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\Rightarrow x_3=-1, x_1=-x_2=-t, x_2=t\in\mathbb{R}$$

**Q3**: Let V be the subspace of $\mathbb{R}^4$ **spanned** by the set S={$v_1$=(1,1,1,1), $v_2$=(−2,−2,−2,−2), $v_3$=(3,3,3,3), $v_4$=(−5,−5,−5,−4)}. Find a **subset** of S that forms a basis of V. (3 marks)

Answer: Putting the vectors as columns in the following matrix:

$$\begin{bmatrix}1 & -2 & 3 & -5\\ 1 & -2 & 3 & -5\\ 1 & -2 & 3 & -5\\ 1 & -2 & 3 & -4\end{bmatrix}\xrightarrow[(-1)R_{13}]{\substack{(-1)R_{14}\\(-1)R_{12}}}\begin{bmatrix}1 & -2 & 3 & -5\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\xrightarrow{R_{24}}\begin{bmatrix}1 & -2 & 3 & -5\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So {$v_1$,$v_4$} is a basis of V.

**Q4**: Let W={(a,0)∈$\mathbb{R}^2$: a∈$\mathbb{R}$}. Show that W is a **subspace** of $\mathbb{R}^2$. (3 marks)

Answer: 1- If a=0, then (0,0)∈W. So W≠∅.

2- Suppose u=($a_1$,0),v=($a_2$,0). Then u,v∈W. Now, u+v=($a_1$,0)+($a_2$,0)=($a_1$+$a_2$,0)∈W.

3- Take u=(a,0)∈W & k∈$\mathbb{R}$. Now, ku=(ka,k0)=(ka,0)∈W.

So, 1, 2 and 3 imply that W is a subspace of $\mathbb{R}^2$.

**Q5**: Let B={(3,3,3),(2,2,0),(1,0,0)} be a basis of $\mathbb{R}^3$. If $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 1 & 0 & 1\end{bmatrix}$ is the transition matrix from S to B, where S={$u_1$,$u_2$,$u_3$} is another basis of $\mathbb{R}^3$. Then find $u_1$. (3 marks)

Answer:

$(u_1)_B$=(1,0,1). So $u_1$=1(3,3,3)+0(2,2,0)+1(1,0,0)=(4,3,3).

**Q6**: Show that the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 0\\ 0 & 1 & 0\end{bmatrix}$ is diagonalizable and find the matrix P that diagonalizes A. (6 marks)

Answer: The characteristic equation:

$$0=\det(\lambda I-A)=\det\left(\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}-\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 0\\ 0 & 1 & 0\end{bmatrix}\right)=\begin{vmatrix}\lambda-1 & -1 & -1\\ 0 & \lambda-2 & 0\\ 0 & -1 & \lambda\end{vmatrix}$$

$$=(\lambda-1)\begin{vmatrix}\lambda-2 & 0\\ -1 & \lambda\end{vmatrix}=(\lambda-1)(\lambda-2)\lambda$$

and hence the Eigenvalues are λ=1,2,0. Since the Eigenvalues are distinct, A is diagonalizable. To find P, take the equation (λI−A)x=0 and substitute λ=1,2,0, respectively as follows:

$$\lambda I - A = \begin{bmatrix} \lambda-1 & -1 & -1 \\ 0 & \lambda-2 & 0 \\ 0 & -1 & \lambda \end{bmatrix}$$

$$\lambda = 1 \Rightarrow (1)I - A = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[(-1)R_{13}]{(-1)R_{12}} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[(-2)R_{23}]{(1)R_{21}} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow y = z = 0, x = t \,\&\, t = 1 \Rightarrow C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 2 \Rightarrow (2)I - A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{(-1)R_{31}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow x = 3z = 3t, y = 2z = 2t \,\&\, t = 1 \Rightarrow C_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda = 0 \Rightarrow (0)I - A = -A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow[(-1)R_{31}]{(-2)R_{32}} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow x = -z = -t, y = 0 \,\&\, t = 1 \Rightarrow C_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

So P= $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix}$.

**Q7**: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis {$u_1$=(1,1,1),$u_2$=(0,1,−1),$u_3$=(0,4,2)} into an **orthonormal basis**. (6 marks)

Answer:

$$u_1 = (1,1,1), u_2 = (0,1,-1), u_3 = (0,4,2)$$

$$v_1 = u_1 = (1,1,1)$$

$$v_2 = u_2 - \frac{<u_2, v_1>}{\|v_1\|^2} v_1$$

$$(0,1,-1) - \frac{<(0,1,-1),(1,1,1)>}{\|(1,1,1)\|^2}(1,1,1) = (0,1,-1) - \tfrac{0}{3}(1,1,1)$$

$$= (0,1,-1) - (0,0,0) = (0,1,-1)$$

$$v_3 = u_3 - \frac{<u_3, v_2>}{\|v_2\|^2} v_2 - \frac{<u_3, v_1>}{\|v_1\|^2} v_1$$

$$= (0,4,2) - \frac{<(0,4,2),(0,1,-1)>}{\|(0,1,-1)\|^2}(0,1,-1) - \frac{<(0,4,2),(1,1,1)>}{\|(1,1,1)\|^2}(1,1,1)$$

$$= (0,4,2) - \tfrac{2}{2}(0,1,-1) - \tfrac{6}{3}(1,1,1) = (0,4,2) - (0,1,-1) - (2,2,2) = (-2,1,1)$$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(1,1,1)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}(0,1,-1)$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{6}}(-2,1,1)$$

**Q8**: Let $M_{22}$ be the vector space of square matrices of order 2, C=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and let T: $M_{22} \to M_{22}$ be the function defined by T(A)=AC for all matrices A in $M_{22}$. Show that:

(i) T is a linear operator. (2 marks)

(ii) Find a basis for ker(T). (2 marks)

(iii) Find $[T]_S$ where S is the standard basis of $M_{22}$. (2 marks)

(iv) Find rank(T). (2 marks)

Answer: (i) For all A,B∈ $M_{22}$, k∈ ℝ:

1- T(A+B)=(A+B)C=AC+BC=T(A)+T(B)

2- T(kA)=(kA)C=k(AC)=kT(A)

So T is linear.

(ii) ker(T)={A∈ $M_{22}$|T(A)=0}={A∈ $M_{22}$|AC=0}=$\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} | \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0\}$

=$\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} | a + b = 0 \& c + d = 0\}$=$\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} | b = -a \& d = -c\}$

=$\{\begin{bmatrix} a & -a \\ c & -c \end{bmatrix} \in M_{22} | a,c \in \mathbb{R}\}$=$\{a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \in M_{22} | a,c \in \mathbb{R}\}$. So, $\left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right\}$ is a basis of ker(T).

(iii) $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$,
$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Now,

$$\left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_S = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_S = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, $[T]_S$=$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(iv) Since nullity(T)=dim(ker(T))=2, so rank(T)=dim($M_{22}$)-nullity(T)=4-2=2.

**Q9**: (i) If B=$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, D=$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and A−C=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then find BAD−BCD. (1 mark)

Answer: BAD−BCD=B(A−C)D=$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$.

(ii) If g is a function from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$ defined by: g(u,v)=0 for all u and v in $\mathbb{R}^3$, then show that g is **not** an inner product function. (1 mark)

Answer: g((1,1,1),(1,1,1))=0 whereas (1,1,1)≠(0,0,0).

(iii) If $T:P_5 \rightarrow M_{22}$ is a linear transformation with nullity(T)=3, then find rank(T). (1 mark)

Answer: rank(T) =dim($P_5$)−nullity(T)=6−3=3

(iv) If 2 and 3 are all the eigenvalues of a matrix A, then find the eigenvalues of $A^{-1}$. (1 mark)

Answer: $Ax=\lambda x$ implies $x=\lambda A^{-1}x$. So, $\lambda^{-1}x=A^{-1}x$. Hence, λ is an eigenvalue of A if and only if $\lambda^{-1}$ is an eigenvalue of $A^{-1}$. So the eigenvalues of $A^{-1}$ are $\frac{1}{2}$ and $\frac{1}{3}$.

(v) If B={$v_1$,$v_2$,$v_3$} is a basis of a vector space V, then $c_1v_1+c_2v_2+c_3v_3=0$, where $c_1$, $c_2$ and $c_3$ are scalars, implies that (choose the correct answer): (1 mark)

(a) $v_1=v_2=v_3=0$ (b) $c_1=c_2=c_3=0$ (c) $c_1=v_1, c_2=v_2, c_3=v_3$ (d) B is a subspace of V

Answer: (b) $c_1=c_2=c_3=0$