Second Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Sunday 6-8-1444	**240 Math**	**Math. Department**

<u>Q1</u>: Solve the following system:

$$x_1 + x_2 - x_3 = 1$$
$$x_2 - 3x_3 = 1 \qquad \text{(2 marks)}$$
$$2x_3 = -4$$

<u>Q2</u>: If A,B∈M_{22}, det(B)=2 and det(A)=3, then find det($2A^TB^{-1}$). (2 marks)

<u>Q3</u>: Let V be the subspace of $\mathbb{R}^4$ **<u>spanned</u>** by the set S={v_1=(1,1,1,0), v_2=(−2,0,0,2), v_3=(−1,3,3,4), v_4=(−5,−1,−1,5)}.

(i) Find a **<u>subset</u>** of S that forms a basis of V. (3 marks)

(ii) **<u>Find</u>** dim(V). (1 mark)

(iii) **<u>Express</u>** each vector that is not in the basis as a linear combination of the basis vectors. (2 marks)

<u>Q4</u>: Let W={(2a+1,0)∈ $\mathbb{R}^2$: a∈ $\mathbb{R}$}. Show that W is a **<u>subspace</u>** of $\mathbb{R}^2$. (3 marks)

<u>Q5</u>: Let B={(1,0),(1,1)} and B'={(1,3),(2,0)} be two bases of $\mathbb{R}^2$. Find the transition matrix from B' to B. (2 marks).

<u>Q6</u>: (i) Show that the Eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ is 0, 1 and 2. (3 marks)

(ii) Show that A is diagonalizable and find the matrix P that diagonalizes A. (3 marks)

(iii) Find A^{1444} . (2 marks)

<u>Q7</u>: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis {u_1=(1,0,0),u_2=(0,1,−1),u_3=(0,4,2)} into an **<u>orthonormal basis</u>**. (5 marks)

<u>Q8</u>: Let M_{22} be the vector space of square matrices of order 2, and let T: M_{22}→M_{22} be the map defined by T(A)=A^T for all matrices A in M_{22}. Show that:

(i) T is a linear operator. (2 marks)

(ii) Find ker(T). (2 marks)

(iii) Find $[T]_B$ where B is the standard basis of M_{22}. (2 marks)

(iv) Find rank(T). (2 marks)

Q9: (i) If B={u,v,w} is a basis of a vector space V, then find the coordinate vector $(u)_B$. (1 mark)

(ii) If *u* and *v* are orthogonal vectors in an inner product space such that ‖u‖=4 and ‖v‖=3, then find ‖u+v‖. (1 mark)

(iii) If B is a 5×9 matrix with nullity(B)=4, then find rank(B^T). (1 mark)

(iv) Show that if u and v are orthogonal in an inner product space V, then au and bv are orthogonal for every a and b in $\mathbb{R}$. (1 mark)

Q1: Solve the following system:

$$x_1 + x_2 - x_3 = 1$$
$$x_2 - 3x_3 = 1 \qquad \text{(2 marks)}$$
$$2x_3 = -4$$

Answer: Using the augmented matrix of the system

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 2 & -4 \end{array}\right] \xrightarrow[(-1)R_{21}]{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[(-2)R_{31}]{3R_{32}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\Rightarrow (x_1, x_2, x_3) = (4, -5, -2)$$

Q2: If A,B∈M_{22}, det(B)=2 and det(A)=3, then find det($2A^TB^{-1}$). (2 marks)

Answer: det($2A^TB^{-1}$)= 2^2det(A^T)det(B^{-1})=4det(A)(det(B))$^{-1}$=4(3)(1/2)=6.

Q3: Let V be the subspace of $\mathbb{R}^4$ **spanned** by the set S={v_1=(1,1,1,0), v_2=(−2,0,0,2), v_3=(−1,3,3,4), v_4=(−5,−1,−1,5)}.

(i) Find a **subset** of S that forms a basis of V. (3 marks)

(ii) **Find** dim(V). (1 mark)

(iii) **Express** each vector that is not in the basis as a linear combination of the basis vectors. (2 marks)

Answer: (i) Putting the vectors as columns in the following matrix:

$$\begin{bmatrix} 1 & -2 & -1 & -5 \\ 1 & 0 & 3 & -1 \\ 1 & 0 & 3 & -1 \\ 0 & 2 & 4 & 5 \end{bmatrix} \xrightarrow[(-1)R_{13}]{(-1)R_{12}} \begin{bmatrix} 1 & -2 & -1 & -5 \\ 0 & 2 & 4 & 4 \\ 0 & 2 & 4 & 4 \\ 0 & 2 & 4 & 5 \end{bmatrix} \xrightarrow[(-1)R_{24}]{\substack{1R_{21} \\ (-1)R_{23}}} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_{34}]{\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So {v_1, v_2,v_4} is a basis of V.

(ii) dim(V)=3.

(iii) From the last matrix in (i), we have that $v_3=3v_1+2v_2$.

Q4: Let W={(2a+1,0)∈ $\mathbb{R}^2$: a∈ $\mathbb{R}$}. Show that W is a **subspace** of $\mathbb{R}^2$. (3 marks)

Answer: 1- If a=1, then (3,0)∈W. So W≠∅.

2- Suppose u=($2a_1$+1,0),v=($2a_2$+1,0)∈W. Now,

u+v=(($2a_1$+1)+ ($2a_2$+1),0)=(($2a_1$+$2a_2$+1)+ 1,0)= (2(a_1+a_2+0.5)+ 1,0)=(2a+1,0). So u+v∈W.

3- Suppose u=($2a_1$+1,0)∈W & k∈$\mathbb{R}$. Now, ku=(k($2a_1$+1),k0)=($2ka_1$+k,0)=(2(ka_1+(k−1)/2)+1,0) =(2a+1,0). So ku∈W. 1,2 and 3 imply that W is a subspace of $\mathbb{R}^2$.

Q5: Let B={(1,0),(1,1)} and B'={(1,3),(2,0)} be two bases of $\mathbb{R}^2$. Find the transition matrix from B' to B. (2 marks).

Answer:

$$[B \mid B'] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{(-1)R_{21}} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & 0 \end{array}\right] = [I \mid P_{B'\to B}] \Rightarrow P_{B'\to B} = \begin{bmatrix} -2 & 2 \\ 3 & 0 \end{bmatrix}$$

Q6: (i) Show that the Eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ is 0, 1 and 2. (3 marks)

(ii) Show that A is diagonalizable and find the matrix P that diagonalizes A. (3 marks)

(iii) Find A^{1444} . (2 marks)

Answer: (i) The characteristic equation:

$$0=\det(\lambda I-A)=\det\left(\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}-\begin{bmatrix}1&1&1\\1&1&-1\\1&1&1\end{bmatrix}\right)=\begin{vmatrix}\lambda-1&-1&-1\\-1&\lambda-1&1\\-1&-1&\lambda-1\end{vmatrix}$$
$$=(\lambda-1)[(\lambda-1)^2+1]-(\lambda-1)+1-(1+(\lambda-1))=(\lambda-1)^3-(\lambda-1)$$
$$=(\lambda-1)((\lambda-1)^2-1)=(\lambda-1)(\lambda^2-2\lambda)=(\lambda-1)\lambda(\lambda-2)$$

and hence the Eigenvalues are λ=1,0,2.

(ii) Since the Eigenvalues are distinct, A is diagonalizable. To find P, take the equation (λI−A)x=0 and substitute λ=1,0,2, respectively as follows:

$$\lambda I-A=\begin{bmatrix}\lambda-1&-1&-1\\-1&\lambda-1&1\\-1&-1&\lambda-1\end{bmatrix}$$

$$\lambda=1\Rightarrow(1)I-A=\begin{bmatrix}0&-1&-1\\-1&0&1\\-1&-1&0\end{bmatrix}\xrightarrow[(-1)R_1]{(-1)R_{23}}\begin{bmatrix}0&1&1\\-1&0&1\\0&-1&-1\end{bmatrix}\xrightarrow[(-1)R_2]{(1)R_{13}}\begin{bmatrix}0&1&1\\1&0&-1\\0&0&0\end{bmatrix}$$

$$\Rightarrow y=-z=-t, x=z=t \ \& t=1\Rightarrow C_1=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\lambda=0\Rightarrow(0)I-A=-A=\begin{bmatrix}-1&-1&-1\\-1&-1&1\\-1&-1&-1\end{bmatrix}\xrightarrow[(-1)R_{13}]{(-1)R_{12}}\begin{bmatrix}-1&-1&-1\\0&0&2\\0&0&0\end{bmatrix}$$

$$\xrightarrow[(\frac{1}{2})R_2]{(-1)R_1}\begin{bmatrix}1&1&1\\0&0&1\\0&0&0\end{bmatrix}\xrightarrow{(-1)R_{21}}\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$\Rightarrow x=-y=-t, z=0 \ \& t=1\Rightarrow C_2=\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$\lambda=2\Rightarrow(2)I-A=\begin{bmatrix}1&-1&-1\\-1&1&1\\-1&-1&1\end{bmatrix}\xrightarrow[(1)R_{13}]{(1)R_{12}}\begin{bmatrix}1&-1&-1\\0&0&0\\0&-2&0\end{bmatrix}\xrightarrow{(-\frac{1}{2})R_{31}}\begin{bmatrix}1&0&-1\\0&0&0\\0&-2&0\end{bmatrix}$$

$$\Rightarrow x=z=t, y=0 \ \& t=1\Rightarrow C_3=\begin{bmatrix}1\\0\\1\end{bmatrix}$$

So P= $\begin{bmatrix}1&-1&1\\-1&1&0\\1&0&1\end{bmatrix}$.

(iii) $A^{1444}=PD^{1444}P^{-1}$. Firstly, we need to find P^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[(-1)R13]{1R_{12}} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_{23}} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{1R_{21}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{(-1)R_{31}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\Rightarrow P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

So

$$A^{1444} = PD^{1444}P^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2^{1444} \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2^{1444} \\ -1 & 0 & 0 \\ 1 & 0 & 2^{1444} \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2^{1444}-1 & 2^{1444}-1 & 1 \\ 1 & 1 & -1 \\ 2^{1444}-1 & 2^{1444}-1 & 1 \end{bmatrix}$$

Q7: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis $\{u_1=(1,0,0), u_2=(0,1,-1), u_3=(0,4,2)\}$ into an **orthonormal basis**. (5 marks)

Answer:

$$u_1 = (1,0,0), u_2 = (0,1,-1), u_3 = (0,4,2)$$

$$v_1 = u_1 = (1,0,0)$$

$$v_2 = u_2 - \frac{<u_2, v_1>}{\|v_1\|^2} v_1$$

$$(0,1,-1) - \frac{<(0,1,-1),(1,0,0)>}{\|(1,0,0)\|^2}(1,0,0) = (0,1,-1) - \tfrac{0}{1}(1,0,0)$$

$$= (0,1,-1) - (0,0,0) = (0,1,-1)$$

$$v_3 = u_3 - \frac{<u_3, v_2>}{\|v_2\|^2} v_2 - \frac{<u_3, v_1>}{\|v_1\|^2} v_1$$

$$= (0,4,2) - \frac{<(0,4,2),(0,1,-1)>}{\|(0,1,-1)\|^2}(0,1,-1) - \frac{<(0,4,2),(1,0,0)>}{\|(1,0,0)\|^2}(1,0,0)$$

$$= (0,4,2) - \tfrac{2}{2}(0,1,-1) - (0,0,0) = (0,4,2) - (0,1,-1) = (0,3,3)$$

$$w_1 = \frac{v_1}{\|v_1\|} = (1,0,0)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}(0,1,-1)$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{18}}(0,3,3) = \frac{1}{3\sqrt{2}}(0,3,3) = \frac{1}{\sqrt{2}}(0,1,1)$$

Q8: Let M_{22} be the vector space of square matrices of order 2, and let T: $M_{22} \to M_{22}$ be the map defined by $T(A)=A^T$ for all matrices A in M_{22}. Show that:

(i) T is a linear operator. (2 marks)

(ii) Find ker(T). (2 marks)

(iii) Find $[T]_B$ where B is the standard basis of M_{22}. (2 marks)

(iv) Find rank(T). (2 marks)

Answer: (i) For all A,B∈ M_{22}, k∈ ℝ:

1- $T(A+B)=(A+B)^T=A^T+B^T=T(A)+T(B)$

2- $T(kA)=(kA)^T=kA^T=kT(A)$

So T is linear.

(ii) ker(T)={A∈ M_{22}|T(A)=0}={A∈ M_{22}|A^T=0}={A∈ M_{22}|A=0}. So ker(T)={0}.

(iii) $T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, $T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$, $T\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ and $T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)=\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$.

Now,

$$\left[T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)\right]_B=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\left[T\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)\right]_B=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\left[T\left(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right)\right]_B=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\left[T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\right]_B=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

Therefore, $[T]_B=\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$.

(iv) Since ker(T)={0}, so nullity(T)=0 and hence rank(T)=dim(M_{22})-nullity(T)=4-0=4.

Q9: (i) If B={u,v,w} is a basis of a vector space V, then find the coordinate vector $(u)_B$. (1 mark)

Answer: As u=1u+0v+0w and writing a vector as a linear combination of the vectors in B is unique, so $(u)_B=(1,0,0)$

(ii) If u and v are orthogonal vectors in an inner product space such that $\|u\|=4$ and $\|v\|=3$, then find $\|u+v\|$. (1 mark)

Answer: As u and v are orthogonal, so $\|u+v\|^2=\|u\|^2+\|v\|^2=16+9=25$. So $\|u+v\|=5$.

(iii) If B is a 5×9 matrix with nullity(B)=4, then find rank(B^T)

Answer: rank(B^T)=rank(B)=9− nullity(B)= 9−4=5

(iv) Show that if u and v are orthogonal in an inner product space V, then au and bv are orthogonal for every a and b in ℝ. (1 mark)

Answer: <au,bv>=ab<u,v>=ab(0)=0, since <u,v>=0.