Second Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Sunday 6-11-1443	**240 Math**	**Math. Department**

Q1:(a) If A is a square matrix of degree 2 such that det(A)=3, then find det(A+A). (2 marks)

Answer:
det(A+A)= det(2A)=2^2det(A)=4(3)=12

(b) If A and B are square matrices of degree 2 such that BA=I, where $B=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$, then find the matrix A. (2 marks)

Answer:
$A=B^{-1}=\frac{1}{\det(B)}\operatorname{adj}(B)=\frac{1}{1}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}$.

(c) Suppose (1,2) is a solution of the following linear system:

$$x+2y=b_1$$
$$2x+3y=b_2$$

Find the **values** of b_1,b_2. (2 marks)

Answer:
$b_1=1+2(2)=5,$
$b_2=2+3(2)=8.$

Q2: Let V be the subspace of $\mathbb{R}^4$ **spanned** by the set S={v_1=(1,5,3,1), v_2=(2,3,6,2), v_3=(3,8,9,3), v_4=(4,6,6,6)}. Find a **subset** of S that forms a basis of V. (4 marks)

Answer:

$$\begin{bmatrix}1&2&3&4\\5&3&8&6\\3&6&9&6\\1&2&3&6\end{bmatrix}\xrightarrow[\substack{-3R_{13}\\-1R_{14}}]{-5R_{12}}\begin{bmatrix}1&2&3&4\\0&-7&-7&-14\\0&0&0&-6\\0&0&0&2\end{bmatrix}$$

$$\xrightarrow[\frac{-1}{6}R_3]{\frac{-1}{7}R_2}\begin{bmatrix}1&2&3&4\\0&1&1&2\\0&0&0&1\\0&0&0&2\end{bmatrix}\xrightarrow{-2R_{34}}\begin{bmatrix}1&2&3&4\\0&1&1&2\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Using the leading ones, {v_1,v_2,v_4} is a basis of V.

Q3: Show that $A=\begin{bmatrix}1&1&1\\0&-1&1\\0&0&0\end{bmatrix}$ is diagonalizable and find the matrix P that diagonalizes A.

(6 marks)

Answer:

Since A is upper triangular, then the Eigenvalues are λ=1,-1,0. Since they are distinct, A is diagonalizable. To find P, take the equation (λI-A)x=0 and substitute λ=1,-1,0, as follows:

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda + 1 & -1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\lambda = 1 \Rightarrow (1)I - A = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[1R_{32}]{1R_{31}} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow x = t, y = z = 0 \,\&\, t = 1 \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = -1 \Rightarrow (-1)I - A = \begin{bmatrix} -2 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[-1R_{32}]{-1R_{31}} \begin{bmatrix} -2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow y = -2x = -2t, z = 0 \,\&\, t = 1 \Rightarrow x = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\lambda = 0 \Rightarrow (0)I - A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{1R_{21}} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x = -2z = -2t, y = z = t \,\&\, t = 1 \Rightarrow x = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

So P= $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Q4: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis vectors (1, 2,1), (2,2,0), (3,1,1) into an **orthonormal basis**.
(6 marks)

Answer:

Let u_1=(1,2,1), u_2=(2,2,0), u_3=(3,1,1). To transform to orthonormal basis w_1,w_2,w_3, we will do as follows:

$$v_1 = u_1 = (1,2,1)$$

$$v_2 = u_2 - \frac{<u_2,v_1>}{\|v_1\|^2}v_1 = (2,2,0) - \frac{<(2,2,0),(1,2,1)>}{\|(1,2,1)\|^2}(1,2,1)$$

$$= (2,2,0) - \frac{6}{6}(1,2,1) = (2,2,0) - (1,2,1) = (1,0,-1)$$

$$v_3 = u_3 - \frac{<u_3,v_1>}{\|v_1\|^2}v_1 - \frac{<u_3,v_2>}{\|v_2\|^2}v_2$$

$$= (3,1,1) - \frac{<(3,1,1),(1,2,1)>}{\|(1,2,1)\|^2}(1,2,1) - \frac{<(3,1,1),(1,0,-1)>}{\|(1,0,-1)\|^2}(1,0,-1)$$

$$= (3,1,1) - \frac{6}{6}(1,2,1) - \frac{2}{2}(1,0,-1) = (3,1,1) - (1,2,1) - (1,0,-1) = (1,-1,1)$$

Now ,

$$w_1 = \frac{1}{\sqrt{6}}(1,2,1), w_{12} = \frac{1}{\sqrt{2}}(1,0,-1), w_1 = \frac{1}{\sqrt{3}}(1,-1,1).$$

Q5: Let M_{nn} be the vector space of square matrices of order n and T: $M_{nn} \rightarrow M_{nn}$ the map defined by T(A)=kA for all matrices A in M_{nn}, where k is a non-zero real number.

(a) Show that T is a linear transformation. (2 marks)

(b) Find ker(T). (2 marks)

(c) Find rank(T). (2 marks)

Answer:

(a) For all A,B∈M_{nn} and m∈ℝ, we have:

(i) T(A+B)=k(A+B) =kA+kB=T(A)+T(B)

(ii) T(mA)=k(mA) =(km)A=(mk)A=m(kA)=mT(A)

(b) A∈ker(T)⇒0=T(A)=kA⇒A=0. So ker(T)={0}.

(c) For all B∈M_{nn}, we have $T(k^{-1}B)=k(k^{-1}B)=(kk^{-1})B=B$ and T is onto. Hence, R(T)=M_{nn} and rank(T)=dim(R(T))=n^2.

Or

Since ker(T)={0}, nullity(T)=dim(ker(T))=0 and hence:
rank(T)=dim(M_{nn})-nullity(T)=n^2-0=n^2.

Q6: Let T: $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by $T(x_1,x_2)=(x_1-x_2,-2x_1,x_2)$.

(a) Find $[T]_{S,B}$ where S is the standard basis of $\mathbb{R}^3$ and B={v_1=(1,1),v_2=(1,0)} is a basis of $\mathbb{R}^2$. (3 marks)

(b) Find a basis of R(T) (the range of T). (3 marks)

Answer:

(a) T(1,1)=(0,-2,1), T(1,0)=(1,-2,0)⇒$[T(1,1)]_S=[(0,-2,1)]_S=[0 \;\; -2 \;\; 1]^T$,

$[T(1,0)]_S$=,$[(1,-2,0)]_S=[1 \;\; -2 \;\; 0]^T$. So $[T]_{S,B}=\begin{bmatrix} 0 & 1 \\ -2 & -2 \\ 1 & 0 \end{bmatrix}$.

(b)As R(T) is generated by the images of the basis vectors and T(1,1)=(0,-2,1), T(1,0)=(1,-2,0) and neither vector is a scalar multiple of the other, then (0,-2,1) and (1,-2,0) are the basis vectors of R(T).

Or

$$\begin{bmatrix} 0 & 1 \\ -2 & -2 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_{13}} \begin{bmatrix} 1 & 0 \\ -2 & -2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_{23}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{bmatrix} \xrightarrow[2R_{23}]{2R_{13}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Using the leading ones, {(0,-2,1), (1,-2,0)} is a basis of R(T).

Q7: (a) If $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ is the transition matrix of $\mathbb{R}^2$ from a basis S={u,v} to a basis B={(1,1),(2,3)}, then find the vector u. (2 marks)

Answer:

$(u)_B$=(1,2). So u=1(1,1)+2(2,3)= (1,1)+ (4,6)=(5,7).

(b) Show that if 1 and -1 are the eigenvalues of a square matrix A of order 2, then we have that $A^{100} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (2 marks)

Answer:

Since 1 and -1 are distinct eigenvalues of A, then A is diagonalizable and A is similar to $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ such that $A=PDP^{-1}$. So $A^{100} = PD^{100}P^{-1} = PIP^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) If A and B are square matrices of order 2 such that $A^2+3A=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ and $BA+2B=\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$, then find the matrices A and B. (2 marks)

Answer:

Observe that $B(A+2I)=\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$ and hence $|B||A+2I|=\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix}=2\neq 0$. So $|A+2I|\neq 0$ and then A+2I is invertible. Now, $A^2+3A=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ implies that $A^2+3A=-2I$ and then $A^2+3A+2I=0$. So (A+I)(A+2I)=0. But A+2I is invertible, thus A+I=0 and A=-I. Now, $\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$=B(A+2I)= B(I)=B.