First Semester	**Final Exam**	**King Saud University**
(without calculators)	**Time allowed: 3 hours**	**College of Science**
Sunday 22-5-1443	**240 Math**	**Math. Department**

Q1: Find the values of m (if possible) such that the following system:

$$x_1 + x_2 - x_3 = 1$$
$$2x_1 + 3x_2 - 3x_3 = 1$$
$$x_1 + x_2 - mx_3 = m$$

has: (i)unique solution. (ii) infinitely many solutions. (iii) no solutions. (4 marks)

Answer:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & -3 & 1 \\ 1 & 1 & -m & m \end{array}\right] \xrightarrow[-1R13]{-2R_{12}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -m+1 & m-1 \end{array}\right]$$

(i) m∈ℝ−{1} (ii) m=1. (iii) no values.

Q2: Let V be the subspace of $\mathbb{R}^3$ **spanned** by the set S={v_1=(1, 1,3), v_2=(2, 2,6), v_3=(4, 5, 6)}. Find a **subset** of S that forms a basis of V. (4 marks)

Answer:

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 5 \\ 3 & 6 & 6 \end{bmatrix} \xrightarrow[-3R_{13}]{-1R_{12}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{bmatrix} \xrightarrow{6R_{23}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So {v_1, v_3} is a basis of V.

Q3: Show that $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is diagonalizable and find the matrix P that diagonalizes A. (6 marks)

Answer:

Since A is upper triangular, then the Eigenvalues are λ=1,-1,0. Since they are distinct, A is diagonalizable. To find P, take the equation (λI-A)x=0 and substitute λ=1,-1,0, as follows:

$$\lambda I - A = \begin{bmatrix} \lambda-1 & -1 & -1 \\ 0 & \lambda+1 & -1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\lambda = 1 \Rightarrow (1)I - A = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[1R_{32}]{1R_{31}} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow x = t, y = z = 0 \,\&\, t = 1 \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = -1 \Rightarrow (-1)I - A = \begin{bmatrix} -2 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[-1R_{32}]{-1R_{31}} \begin{bmatrix} -2 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow y = -2x = -2t, z = 0 \,\&\, t = 1 \Rightarrow x = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$\lambda = 0 \Rightarrow (0)I - A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{1R_{21}} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = -2z = -2t, y = z = t \,\&\, t = 1 \Rightarrow x = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

So P=$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Q4: Let $\mathbb{R}^3$ be the Euclidean inner product space. Apply the Gram-Schmidt process to transform the following basis vectors (1, 2,1), (2,2,0), (3,1,1) into an **orthonormal basis**. (7 marks)

Answer:

Let u_1=(1,2,1), u_2=(2,2,0), u_3=(3,1,1). To transform to orthonormal basis w_1,w_2,w_3, we will do as follows:

$$v_1 = u_1 = (1,2,1)$$

$$v_2 = u_2 - \frac{<u_2, v_1>}{\|v_1\|^2} v_1 = (2,2,0) - \frac{<(2,2,0),(1,2,1)>}{\|(1,2,1)\|^2}(1,2,1)$$

$$= (2,2,0) - \frac{6}{6}(1,2,1) = (2,2,0) - (1,2,1) = (1,0,-1)$$

$$v_3 = u_3 - \frac{<u_3, v_1>}{\|v_1\|^2} v_1 - \frac{<u_3, v_2>}{\|v_2\|^2} v_2$$

$$= (3,1,1) - \frac{<(3,1,1),(1,2,1)>}{\|(1,2,1)\|^2}(1,2,1) - \frac{<(3,1,1),(1,0,-1)>}{\|(1,0,-1)\|^2}(1,0,-1)$$

$$= (3,1,1) - \frac{6}{6}(1,2,1) - \frac{2}{2}(1,0,-1) = (3,1,1) - (1,2,1) - (1,0,-1) = (1,-1,1)$$

$$Now, w_1 = \frac{1}{\sqrt{6}}(1,2,1), w_{12} = \frac{1}{\sqrt{2}}(1,0,-1), w_1 = \frac{1}{\sqrt{3}}(1,-1,1),$$

Q5: Let M_{nn} be the vector space of square matrices of order n, let P∈ M_{nn} be an invertible matrix, and let T: M_{nn}→M_{nn} be the map defined by T(A)=P^{-1}AP for all matrices A in M_{nn}. Show that:

(a) T is a linear transformation. (4 marks)

(b) Find ker(T) and **deduce** that T is one-to-one. (2 marks)

(c) Show that T is onto. (2 marks)

Answer:

(a) For all A,B∈M_{nn} and k∈$\mathbb{R}$, we have:

(i) T(A+B)=P^{-1}(A+B)P=P^{-1}AP+P^{-1}BP=T(A)+T(B)

(ii) T(kA)=P^{-1}(kA)P=k(P^{-1}AP)=kT(A)

(b) A∈ker(T)⇒0=T(A)=P^{-1}AP⇒A=0. So ker(T)=0 and T is 1-1.

(c) For all B∈M_{nn}, we have T(PBP^{-1})=P^{-1}(PBP^{-1})P=B and T is onto.

Or

Since T is a 1-1 linear operator and M_{nn} is finite dimensional, then T is onto (by a theorem).

Or

Since ker(T)={0}, nullity(T)=dim(ker(T))=0 and hence:

rank(T)=dim(M_{nn})-nullity(T)=n^2-0=n^2.

But rank(T)=dim(R(T)). So dim(R(T))=dim(M_{nn}) and hence R(T)= M_{nn} and T is onto.

Q6: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by:
$T(x_1, x_2) = (x_1 - x_2, -2x_1, x_2)$.

(a) Find $[T]_{S,B}$ where S is the standard basis of $\mathbb{R}^3$ and B={v_1=(1,1),v_2=(1,0)} is a basis of $\mathbb{R}^2$. (3 marks)

(b) Find a basis of R(T). (2 marks)

Answer:

(a) T(1,1)=(0,-2,1), T(1,0)=(1,-2,0)⇒$[T(1,1)]_S$=$[(0,-2,1)]_S$=(0,−2,1),

$[T(1,0)]_S$=,$[(1,-2,0)]_S$=(1,−2,0). So $[T]_{S,B}=\begin{bmatrix} 0 & 1 \\ -2 & -2 \\ 1 & 0 \end{bmatrix}$

(b)As R(T) is generated by the images of the basis vectors and T(1,1)=(0,-2,1), T(1,0)=(1,-2,0) and neither vector is a scalar multiple of the other, then (0,-2,1) and (1,-2,0) are the basis vectors of R(T).

Q7: (a) If $T : V \to W$ is a linear transformation, then prove that the image of T is a subspace of W. (3 marks)

Answer:

Firstly, R(T) is not empty since T(0)=0 and hence 0∈R(T). Now, take w_1 and w_2 from R(T) and k∈$\mathbb{R}$. So w_1=T(v_1) and w_2=T(v_2) for some v_1 and v_2 belongs to V. Observe that w_1+ w_2= T(v_1)+T(v_2)= T(v_1+v_2) and hence w_1+ w_2∈R(T).
Also, kw_1=kT(v_1)= T(kv_1) and hence kw_1 ∈R(T). So R(T) is a subspace of W.

(b) If u and v are orthogonal vectors in an inner product space, then prove that:
$\|u+v\|^2 = \|u\|^2 + \|v\|^2$. (2 marks)

Answer:

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$
$$= \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle$$
$$= \langle u,u \rangle + \langle v,v \rangle = \|u\|^2 + \|v\|^2$$

(c) If we have the following polynomial f(x)=x^5+x^4-3x^3-x^2+2x-7, where x∈$\mathbb{R}$, then show that $\det(A) = \begin{vmatrix} f(13) & f(13) & f(13) & f(13) \\ f(11) & f(12) & f(14) & f(15) \\ f(16) & f(17) & f(18) & f(19) \\ f(13)+1 & f(13)+1 & f(13)+1 & f(13)+1 \end{vmatrix} = 0$. (1 mark)

Answer:

Since the first row and the fourth row are proportional, so the determinant equals to zero. To see this, multiply the first row by (f(13)+1)/f(13) and you will get the fourth row (Clearly f(13)≠0).

Other way, you can do the following:

$$\begin{vmatrix} f(13) & f(13) & f(13) & f(13) \\ f(11) & f(12) & f(14) & f(15) \\ f(16) & f(17) & f(18) & f(19) \\ f(13)+1 & f(13)+1 & f(13)+1 & f(13)+1 \end{vmatrix}$$

$$= f(13)(f(13)+1)\begin{vmatrix} 1 & 1 & 1 & 1 \\ f(11) & f(12) & f(14) & f(15) \\ f(16) & f(17) & f(18) & f(19) \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= f(13)(f(13)+1)(0) = 0$$

<u>Or</u>

$$\begin{vmatrix} f(13) & f(13) & f(13) & f(13) \\ f(11) & f(12) & f(14) & f(15) \\ f(16) & f(17) & f(18) & f(19) \\ f(13)+1 & f(13)+1 & f(13)+1 & f(13)+1 \end{vmatrix} \overset{-1R_{14}}{=} \begin{vmatrix} f(13) & f(13) & f(13) & f(13) \\ f(11) & f(12) & f(14) & f(15) \\ f(16) & f(17) & f(18) & f(19) \\ f(13)+1 & f(13)+1 & f(13)+1 & f(13)+1 \end{vmatrix}$$

$$= \begin{vmatrix} f(13) & f(13) & f(13) & f(13) \\ f(11) & f(12) & f(14) & f(15) \\ f(16) & f(17) & f(18) & f(19) \\ 1 & 1 & 1 & 1 \end{vmatrix} \overset{(-f(13))R_{41}}{=} \begin{vmatrix} 0 & 0 & 0 & 0 \\ f(11) & f(12) & f(14) & f(15) \\ f(16) & f(17) & f(18) & f(19) \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$