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MID TERM EXAMINATION, SEMESTER I, 1444
DEPT. OF MATH., COLLEGE OF SCIENCE, KSU
MATH: 107 FULL MARK: 30 TIME: 2 HOURS

Q1.[1+4+3=8]

Consider the following system of linear equations

$$x+y+z=5$$
$$-4x+y+z=0$$
$$x+y-4z=10.$$

(a) Write the system in the form $AX=B$.

(b) Find the inverse of the matrix A by elementary matrix method.

(c) Use A^{-1} to solve the above system.

Q2.[3+3+3=9]

(a) Evaluate the determinant of B by reducing the matrix to row-echelon form.

$$B=\begin{bmatrix} 1 & 3 & 5 \\ 4 & 14 & 12 \\ -2 & -3 & -20 \end{bmatrix}$$

(b) Let

$$C=\begin{bmatrix} 2 & 0 & -3 \\ 0 & -3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

Find the adjoint of the matrix C.

(c) By using Cramer's Rule solve the system of equations:

$$4x+5y=2$$
$$x+y+2z=3$$
$$x+5y+2z=1.$$

Q3. [3+3=6] (a) Let $A(1,1,0)$, $B(2,1,1)$, $C(1,2,1)$, and $D(0,-1,2)$.
Evaluate (i) the angle between the vector $\overrightarrow{AB}$ and $\overrightarrow{CD}$;
(ii) the component of $\overrightarrow{AB}$ along $\overrightarrow{AC}$.

(b) Find the work done by a constant force $\vec{F}=2\vec{i}+\vec{j}-3\vec{k}$ if its point of application moves from $P(1,3,6)$ to $Q(3,4,5)$.

Q4.[3+4=7]

(a) Find the volume V of a box having adjacent sides AB, AC, and AD where $A(1,1,1)$, $B(1,0,-1)$, $C(2,1,2)$, $D(3,4,5)$.

(b) Find an equation of the plane P determine by the points $A(1,1,1)$, $B(1,0,-1)$, and $C(2,1,2)$.

Model Answer of MID-term Exam S1/1444

Q1 [1+4+3=8]

(a) The system in the form $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix}$$

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(b) $[A|I] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -4 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -4 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{\substack{4R_1+R_2 \\ -R_1+R_3}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 4 & 1 & 0 \\ 0 & 0 & -5 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4/5 & 1/5 & 0 \\ 0 & 0 & -5 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -1/5 & 0 \\ 0 & 1 & 1 & 4/5 & 1/5 & 0 \\ 0 & 0 & -5 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow{-\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -1/5 & 0 \\ 0 & 1 & 1 & 4/5 & 1/5 & 0 \\ 0 & 0 & 1 & 1/5 & 0 & -1/5 \end{array}\right]$

$\xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -1/5 & 0 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & 0 & -1/5 \end{array}\right] = [I|A^{-1}]$

$\therefore$ the inverse of the matrix A is

$$A^{-1} = \begin{bmatrix} 1/5 & -1/5 & 0 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & 0 & -1/5 \end{bmatrix}$$

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(c) The solution of the system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/5 & -1/5 & 0 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & 0 & -1/5 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$$

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$\therefore \; x = 1, \; y = 5$ and $z = -1$

Q2 [3+3+3=9]

(a)

$$B = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 14 & 12 \\ -2 & -3 & -20 \end{bmatrix}$$

$$\det(B) = \begin{vmatrix} 1 & 3 & 5 \\ 4 & 14 & 12 \\ -2 & -3 & -20 \end{vmatrix}, \quad -4R_1 + R_2, \; 2R_1 + R_3$$

$$= \begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & -8 \\ 0 & 3 & -10 \end{vmatrix}$$

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$$= 2\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & -4 \\ 0 & 3 & -10 \end{vmatrix}, \quad -3R_2 + R_3$$

$$\therefore \det(B) = 2\begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{vmatrix} = 2(1)(1)(2) = 4$$

(b)

$C_{11} = M_{11} = \begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix} = -6, \; C_{12} = -M_{12} = -\begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} = -2, \; C_{13} = M_{13} = \begin{vmatrix} 0 & -3 \\ -1 & 0 \end{vmatrix} = -3$

$C_{21} = -M_{21} = \begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix} = 0, \; C_{22} = M_{22} = \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} = 1, \; C_{23} = -M_{23} = \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} = 0$

$C_{31} = M_{31} = \begin{vmatrix} 0 & -3 \\ -3 & 2 \end{vmatrix} = -9, \; C_{32} = -M_{32} = \begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} = -4, \; C_{33} = M_{33} = \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = -6$

Matrix of Cofactors is $\begin{bmatrix} -6 & -2 & -3 \\ 0 & 1 & 0 \\ -9 & -4 & -6 \end{bmatrix}$

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$\therefore$ The adjoint of the matrix C is:

$$adj(C) = \begin{bmatrix} -6 & 0 & -9 \\ -2 & 1 & -4 \\ -3 & 0 & -6 \end{bmatrix}$$

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(c)

$$\det(A) = \begin{vmatrix} 4 & 5 & 0 \\ 1 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = 4(2-10) - 5(2-2) + 0$$

$$= -32$$

$$\det(A_1) = \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = 2(2-10) - 5(6-2) + 0$$

$$= -16 - 20 = -36$$

$$\det(A_2) = \begin{vmatrix} 4 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 4(6-2) - 2(0) + 0 = 16$$

$$\det(A_3) = \begin{vmatrix} 4 & 5 & 2 \\ 1 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = 4(1-15) - 5(1-3) + 2(5-1)$$

$$= -56 + 10 + 8 = -38$$

$\rightarrow$ By using Cramer's rule, we have:

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-36}{-32} = \frac{9}{8}, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{16}{-32} = \frac{-1}{2}$$

and $= \dfrac{\det(A_3)}{\det(A)} = \dfrac{-38}{-32} = \dfrac{19}{16}$

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$\therefore$ The solution of the given system is $x = \frac{9}{8}$, $y = \frac{-1}{2}$ and $z = \frac{19}{16}$.

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Q3 [3+3=6]

(a)

(i) $\overrightarrow{AB} = \langle 1,0,1\rangle$, $\overrightarrow{CD} = \langle -1,-3,1\rangle$

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{\|\overrightarrow{AB}\|\,\|\overrightarrow{CD}\|} = \frac{-1+0+1}{\sqrt{2}\sqrt{1+9+1}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

1.5

(ii)

$$\text{Comp}_{\overrightarrow{AC}}\overrightarrow{AB} = \overrightarrow{AB}\cdot\frac{\overrightarrow{AC}}{\|\overrightarrow{AC}\|}, \quad \overrightarrow{AC} = \langle 0,1,1\rangle$$

$$= \langle 1,0,1\rangle\cdot\frac{\langle 0,1,1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

1.5

(b) $W = \vec{F}\cdot\overrightarrow{PQ}$

$= \langle 2,1,-3\rangle\cdot\langle 2,1,-1\rangle$

$= 4+1+3 = 8$

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Q4 [3+4=7]

(a)

$\vec{a} = \overrightarrow{AB} = \langle 0,-1,-2\rangle$

$\vec{b} = \overrightarrow{AC} = \langle 1,0,1\rangle$

$\vec{c} = \overrightarrow{AD} = \langle 2,3,4\rangle$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} 0 & -1 & -2 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{vmatrix} = -4$$

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$\therefore$ the volume is $V = |(\vec{a}\times\vec{b})\cdot\vec{c}| = |-4| = 4$.

(b) $\vec{a} = \overrightarrow{AB} = \langle 0,-1,-2\rangle$, $\vec{b} = \overrightarrow{AC} = \langle 1,0,1\rangle$

$\vec{n} = \vec{a}\times\vec{b}$ is the normal to the plane

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & -2 \\ 1 & 0 & 1 \end{vmatrix} = -\vec{i} - 2\vec{j} + \vec{k}$$

i.e $\vec{n} = \langle -1,-2,1\rangle$

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$\therefore$ The Eq. of the plane is

$-1(x-1) - 2(y-1) + 1(z-1) = 0$

$-x - 2y + z + \cancel{1} + 2 - \cancel{1} = 0$

$\Rightarrow -x - 2y + z + 2 = 0$

$\boxed{x + 2y - z = 2}$

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