The Model Answer of MATH 107 First MDT Exam S2 1445

Q. 1 [5]

Find the value of δ for which the following linear system of equations

$$x + y + z + t = 4$$

$$x + \delta y + z + t = 4$$

$$x + y + \delta z + (3 - \delta)t = 6$$

$$2\,x + 2y + 2z + (\delta - 5)t = 6$$

has (i) no solution (ii) infinitely many solutions

Answer

The Augmented matrix of the given system is given by

$$\mathbf{A}=\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & \delta & 1 & 1 & 4 \\ 1 & 1 & \delta & 3-\delta & 6 \\ 2 & 2 & 2 & \delta-5 & 6 \end{bmatrix}$$

$$\mathbf{A}\sim\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & \delta-1 & 0 & 0 & 0 \\ 0 & 0 & \delta-1 & 2-\delta & 2 \\ 0 & 0 & 0 & \delta-7 & -2 \end{bmatrix}$$

$$\mathbf{A}\sim\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & \delta-1 & 0 & 0 & 0 \\ 0 & 0 & \delta-1 & -5 & 0 \\ 0 & 0 & 0 & \delta-7 & -2 \end{bmatrix}$$

Hence, (i) the system has no solution if $\delta \in \{1,7\}$ (ii) the system has infinitely many solutions if $\delta \in \{\,\}$.

Q. 2 [4+3=7]

(a) If $\mathbf{A} = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$, Find $\mathbf{A} + \mathbf{A}^{\mathrm{T}} + \mathbf{A}^{-1}$

Answer

$$\mathbf{A} + \mathbf{A}^{\mathrm{T}} + \mathbf{A}^{-1} = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} + \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 2 \\ 7 & 13 \end{bmatrix}$$

(b) If the inverse of $2\mathbf{A}$ is $\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$, Find the matrix $\mathbf{A}$.

Answer

$$(2\mathbf{A})^{-1} = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$\frac{1}{2}(\mathbf{A})^{-1} = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 4 & -4 \\ -6 & 10 \end{bmatrix}$$

$$\therefore \mathbf{A} = \frac{1}{16}\begin{bmatrix} 10 & 4 \\ 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5/8 & 1/4 \\ 3/8 & 1/4 \end{bmatrix}$$

Q. 3 [5]

For the following linear system of equations: $x - z = 6,\ x + y + z = -3,\ -x + y = 12$

Find the inverse of the coefficient matrix by using elementary row operations,

then find the solution of the given system.

Answer

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$\xRightarrow{\substack{-R_1+R_2, \\ R_1+R_3}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\xRightarrow{-R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -3 & 2 & -1 & 1 \end{array}\right)$$

$$\xRightarrow{R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & -3 & 2 & -1 & 1 \end{array}\right)$$

$$\xRightarrow{-\frac{1}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2/3 & 1/3 & -1/3 \end{array}\right)$$

$$\xRightarrow{\substack{R_3+R_1, \\ R_3+R_2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 1/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & 1/3 & 2/3 \\ 0 & 0 & 1 & -2/3 & 1/3 & -1/3 \end{array}\right)$$

$$\therefore \mathbf{A}^{-1} = \begin{pmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & 2/3 \\ -2/3 & 1/3 & -1/3 \end{pmatrix}$$

$$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$=\begin{pmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 1/3 & 2/3 \\ -2/3 & 1/3 & -1/3 \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 12 \end{pmatrix}=\begin{pmatrix} -3 \\ 9 \\ -9 \end{pmatrix}$$

The solution is $x=-3, y=9\,, z=-9$.

Q. 4 [4]

Evaluate the determinant of a matrix $\mathbf{A}$ by using row operations

$$\text{where } \mathbf{A}=\begin{bmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Answer

$$\det(A)=(-1)(2)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\therefore \det(A)=(-1)(2)(1)=-2$$

Q. 5 [4]

Use Cramer's rule to solve the following linear system of equations:

$$x+y=1$$

$$x+2y+z=-1$$

$$x+3y-z=2$$

Answer

$$\det(A)=\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & -1 \end{vmatrix}=-3\quad,\ \det(A_1)=\begin{vmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -1 \end{vmatrix}=-4$$

$$\det(A_2)=\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}=1,\ \det(A_3)=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 2 \end{vmatrix}=5$$

$\therefore$ The solution is given by

$$x=\frac{\det(A_1)}{\det(A)}=\frac{-4}{-3}=\frac{4}{3}, y=\frac{\det(A_2)}{\det(A)}=\frac{1}{-3}=-\frac{1}{3}, z=\frac{\det(A_3)}{\det(A)}=\frac{5}{-3}=-\frac{5}{3}$$