جامعة الملك سعود
King Saud University

FINAL EXAMINATION, SEMESTER III, 2023
DEPT. MATH., COLLEGE OF SCIENCE, KSU
MATH: 107 FULL MARK: 40 TIME: 3 HOURS

Q1. [Marks: 4+4+4=12]

(a) Determine the values of a for which the following system has no solutions, exactly one solution, or infinitely many solutions:

$$\begin{aligned} x+2y+z &= -1 \\ -2x+(a-4)y-3z &= a+1 \\ -x+(a-2)y+(a-3)z &= 2a. \end{aligned}$$

(b) Let M be a matrix:

$$M = \begin{bmatrix} a & 3 & 3 \\ 0 & b & 1 \\ 4 & 3 & 1 \end{bmatrix}$$

Find conditions on a and b such that the matrix M is invertible.

(c) Let A be a matrix where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

Use method of the cofactors to find the inverse of the matrix A.

Q2. [Marks: 3+3+3=9]

(a) Show that the two planes $3x+12-6z=-2$ and $5x+20y-10z=7$ are parallel, and find the distance between them.

(b) Identify the surface $y^2-9x^2-z^2-9=0$. Find its traces on the coordinate planes and sketch the surface.

(c) Let C be the curve determined by the vector-valued function $\mathbf{r}(t)=\langle 4\cos t, 4\sin t, 3t\rangle$. Verify that the unit tangent vector and unit normal vector are orthogonal.

Q3. [Mark: 4+3+3=10]

(a) The position vector of a moving point at time t is $\mathbf{r}(t)=\langle \cos 4t, \sin 4t, 3t\rangle$. Find the tangential and normal components of acceleration at time t.

(b) If $w=f(x^2+y^2)$, show that $y\frac{\partial w}{\partial x}-x\frac{\partial w}{\partial y}=0$

(c) Find the directional derivative of the function $f(x,y)=x^2+y^2-4z$ at $P(2,-1,1)$ in the direction of $\mathbf{a}=\mathbf{i}+3\mathbf{j}-2\mathbf{k}$.

Q4. [Marks: 3+3+3=9]

(a) Find the equations of the tangent plane and normal line to the graph of $x^3-2xy+z^3+7y+6=0$ at the point $P(1,4,-3)$.

(b) Find $\frac{\partial z}{\partial x}$ if $F(x,y,z)=2xz^3-3yz^2+x^2y^2+4z=0$.

(c) If $f(x,y)=x^2+y^2+xy^2+9$, find the local extrema and saddle point of f.

Solution to M 107 Final Examination
S-III / 2023

Q_1 (a)
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ -2 & a-4 & -3 & a+1 \\ -1 & a-2 & a-3 & 2a \end{array}\right) \xrightarrow[r_1+r_3]{2r_1+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & a & -1 & a-1 \\ 0 & a & a-2 & 2a-1 \end{array}\right)$$

(2)
$$\xrightarrow{-r_1+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & a & -1 & a-1 \\ 0 & 0 & a-1 & a \end{array}\right)$$

(2)
case I : If $a=1$, no solution
case II : If $a\neq 1$, unique solution
case III : If $a=0$, $-z=0 \Rightarrow z=0$
and the system becomes,

$$\begin{cases} x+2y=-1 \\ y=t \\ z=0 \end{cases} \Rightarrow \begin{cases} x=-1-2t \\ y=t \\ z=0 \end{cases}$$

infinitely many soln.

(b) M is invertible iff $\operatorname{Det} M \neq 0$ iff

$$\begin{vmatrix} a & 3 & 3 \\ 0 & b & 1 \\ 4 & 3 & 1 \end{vmatrix} \neq 0 \text{ iff } ab-3a-12b+12\neq 0$$

(4)

(c) Matrix of the cofactors

$$C=\begin{pmatrix} 0 & 1 & 1 \\ -4 & 3 & 1 \\ -2 & 1 & 1 \end{pmatrix}, \quad A^{-1}=\frac{1}{2}\begin{pmatrix} 0 & -4 & -2 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(2) (2)

P_2

Q2 (a) Here $\underline{a} = \langle 3, 12, -6 \rangle$ and $\underline{b} = \langle 5, 20, -10 \rangle$.
$\underline{a}$ is parallel to $\underline{b}$ since $\underline{b} = \frac{5}{3}\underline{a}$, hence planes are parallel. Since $(0, 0, \frac{1}{3})$ is on the first plane, its distance from the second plane is

$$d = \frac{|5(0)+20(0)-10(\frac{1}{3})-7|}{\sqrt{5^2+20^2+(-10)^2}} = \frac{31}{15\sqrt{21}}$$

3

(b) $y^2 - 9x^2 - z^2 - 9 = 0 \Leftrightarrow \frac{y^2}{9} - x^2 - \frac{z^2}{9} = 1$

a hyperboloid of two sheets with axis on the y-axis

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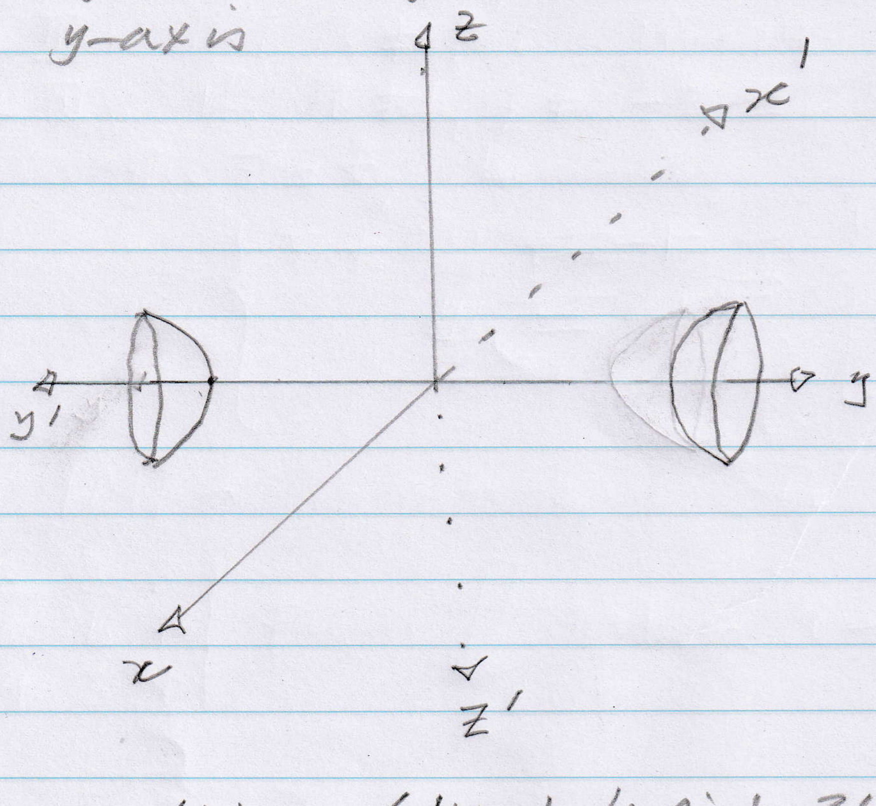

1

(c) $\underline{r}(t) = \langle 4\cos t, 4\sin t, 3t \rangle$

$\underline{r}'(t) = \langle -4\sin t, 4\cos t, 3 \rangle$

1

$\underline{T}(t) = \frac{\langle -4\sin t, 4\cos t, 3 \rangle}{5}$

$$\underline{N}(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle}{\sqrt{\frac{16}{25} + 0}}$$

$$\Rightarrow \underline{N}(t) = \langle -\cos t, -\sin t, 0\rangle \quad 1$$

Now

$$\underline{T}(t)\cdot\underline{N}(t) = \langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, 0\rangle \cdot \langle -\cos t, -\sin t, 0\rangle$$

$$= \frac{4}{5}\sin t\cos t - \frac{4}{5}\sin t\cos t + 0$$

$$= 0 \quad 1$$

Q_3 (a) $\underline{r}(t) = \langle \cos 4t, \sin 4t, 3t\rangle$

$$\underline{r}'(t) = \langle -4\sin 4t, 4\cos 4t, 3\rangle$$

$$\underline{r}''(t) = \langle -16\cos 4t, -16\sin 4t, 0\rangle$$

$$a_T = \frac{\underline{r}'(t)\cdot\underline{r}''(t)}{\|\underline{r}'(t)\|} = \frac{64\sin 4t\cos 4t + 64}{} \quad 2$$

$$= \frac{64\sin 4t\cos 4t - 64\cos 4t\sin 4t}{\sqrt{25}}$$

$$a_T = 0$$

$$\underline{r}'(t)\times\underline{r}''(t) = \begin{vmatrix} i & j & k \\ -4\sin 4t & 4\cos 4t & 3 \\ -16\cos 4t & -16\sin 4t & 0 \end{vmatrix}$$

$$= (0 + 48\sin 4t)\underline{i} - (0 + 48\cos 4t)\underline{j} + 64\underline{k}$$

$$= 48\sin 4t\,\underline{i} - 48\cos 4t\,\underline{j} + 64\underline{k}$$

$$a_N = \frac{\sqrt{48^2\sin^2 4t + 48^2\cos^2 4t + 64^2}}{\sqrt{25}}$$ 2

$$= \frac{\sqrt{48^2 + 64^2}}{5} = \frac{\sqrt{2304 + 4096}}{5} = \frac{80}{5} = 16$$

(b) Let $u = x^2 + y^2$ 3

$$\begin{tikzcd} & & x \\ w \arrow[r, "dw/du"] & u \arrow[ur, "\partial u/\partial x"] \arrow[dr, "\partial u/\partial y"'] & \\ & & y \end{tikzcd}$$

$$y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y} = y\left(\frac{dw}{du}\,2x\right) - x\left(\frac{dw}{du}\cdot 2y\right)$$

$$= 2xy\frac{dw}{du} - 2xy\frac{dw}{du}$$

$$= 0$$

$$\Rightarrow\ y\frac{\partial w}{\partial x} - x\frac{\partial w}{\partial y} = 0$$

(c) $\nabla f = \langle 2x, 2y, -4\rangle$

$\therefore\ \nabla f(2,-1,1) = \langle 4, -2, -4\rangle$

P5

Q3 (c) $u = \frac{a}{\|a\|} = \frac{\langle 1,3,-2\rangle}{\sqrt{1+9+4}} = \left\langle \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}} \right\rangle$ 1'

So, the directional derivative is given by

2

$$D_u f(2,-1,1) = \langle 4,-2,-4\rangle \cdot \left\langle \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}} \right\rangle = \frac{6}{\sqrt{14}}$$

Q4 (a) $\nabla f = \langle 3x^2-2y, -2x+7, 3z^2\rangle$

$\Rightarrow \nabla f(1,4,-3) = \langle -5, 5, 27\rangle$

So, the eq of the tangent plane is

2 + 1

$$-5(x-1) + 5(y-4) + 27(z+3) = 0$$

$$\Rightarrow -5x + 5y + 27z = 66 \qquad \begin{cases} x = 1-5t \\ y = 4+5t \\ z = -3+27t \end{cases}$$

(b) $F(x,y,z) = 2xz^3 - 3yz^2 + x^2y^2 + 4z = 0$

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$$\frac{\partial F}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{2z^3+2xy^2}{6xz^2-6yz+4}$$

(c) $f_x = 2x + y^2 = 0$, $f_y = 2y + 2xy = 0$

$\Rightarrow y^3 - 2y = 0 \Rightarrow y = 0, \pm\sqrt{2}$

and $x = 0, -1$

Critical points are $(0,0), (-1,\sqrt{2}), (-1,-\sqrt{2})$

1 + 1 + 1

Q4 (c) $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

$= \begin{vmatrix} 2 & 2y \\ 2y & 2+2x \end{vmatrix}$

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- $\therefore D(0,0) = 4 > 0, \quad f_{xx} = 2 > 0$

$\Rightarrow f(0,0) = 9$ is a local minimum

$$\left.\begin{array}{l} - \; D(-1, \sqrt{2}) = -8 < 0 \\ - \; D(-1, -\sqrt{2}) = -8 < 0 \end{array}\right\} \text{saddle points}$$