FINAL EXAMINATION, SEMESTER I, 1445
DEPT. MATH., COLLEGE OF SCIENCE, KSU
MATH: 107 FULL MARK: 40 TIME: 3 HOURS

Q1. [Marks: 3+4=7]

(a) Find conditions on a, b such that the following system is consistent:

$$x + 2y + z + 3t = a$$
$$2x + y + 3z + 2t = b$$
$$-x + 7y - 4z + 9t = 1.$$

(b) Evaluate the determinant by reducing the matrix to row echelon form:

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & -4 & 7 & 1 \end{bmatrix}$$

Q2. [Marks: 2+3+3+3=11]

(a) Let g be a function of two variables defined on $\Re^2$ by: $w = g(x, y) = \sin(x - y)$. Show that the function satisfies $\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 0$.

(b) Find symmetric equations for the line passing through point $P(4, 3, 2)$ and parallel to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

(c) Identify the surface $9(x^2 + z^2) + 4y^2 = 36$. Give its traces, and sketch.

(d) Let C be the curve with parametric equations $x = t$, $y = t^2$, $z = t^3$, $t \geq 0$. Find the parametric equations for the tangent line to C at the point corresponding to $t = 2$.

Q3. [Mark: 3+4+3=10]

(a) A moving particle in the space has position vector $\mathbf{r}(t) = \langle 2\sin t, \sin^2 t, t - \sin t \cos t \rangle$. Show that the speed of the particle is constant during the motion.

(b) A moving particle in the plane has position vector $\mathbf{r}(t) = \langle \ln(1 + t^2), 2\tan^{-1} t - t \rangle$. Find unit tangent vector $\mathbf{T}(t)$ and the principal unit normal vector $\mathbf{N}(t)$.

(c) Find the position vector $\mathbf{r}(t)$ of a moving particle with acceleration $\mathbf{a}(t) = \langle 6t + 2, e^t - 2, -\sin t \rangle$ and initial position vector and velocity by $\mathbf{r}(0) = \langle 1, -1, 0 \rangle$ and $\mathbf{v}(0) = \langle 1, 0, 0 \rangle$, respectively.

Q4. [Marks: 2+3+4+3=12]

(a) Let $z = x^2 y$ where $x = r\cos\theta$ and $y = r\sin\theta$. Use Chain rule to find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

(b) Let $f(x, y, z) = 4x - y^2 e^{3xz}$. Find the directional derivative of f at the point $A(3, -1, 0)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(c) If $g(x, y) = x^2 + y^2 + xy^2 + 16$. Find the local extrema and saddle point, if any.

(d) Let $f(x, y, z) = xyz$ where x,y and z are three positive numbers. Use Lagrange multiplier to maximize f subject to the constraint $x + y + z = 1$.

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P. 1

Model Answer of M 107
Final Exam S1/1445

Q1. Marks: 3+4=7

(a)

$$\begin{bmatrix} 1 & 2 & 1 & 3 & a \\ 2 & 1 & 3 & 2 & b \\ -1 & 7 & -4 & 9 & 1 \end{bmatrix}$$

$-2R_1+R_2,\ R_1+R_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & a \\ 0 & -3 & 1 & -4 & b-2a \\ 0 & 9 & -3 & 12 & a+1 \end{bmatrix}$$

3

$3R_2+R_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 3 & a \\ 0 & -3 & 1 & -4 & b-2a \\ 0 & 0 & 0 & 0 & 3b-5a+1 \end{bmatrix}$$

the system is consistent if $3b-5a+1=0$ i.e. $b=\dfrac{5a-1}{3}$ where $a,b\in\mathbb{R}$.

(b)

$$\det(A)=\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & -4 & 7 & 1 \end{vmatrix}$$

$-5R_1+R_2,\ R_1+R_3,\ -2R_1+R_4$

$$\det(A)=\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 1 & -1 \end{vmatrix}$$

$R_3 \longleftrightarrow R_4$

$$\det(A)=-\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -3 & -1 \end{vmatrix}$$

4

$3R_3+R_4$

$$\det(A)=-\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

$\therefore \det(A)=-(1)(1)(1)(-4)=4$ #

Q2. Marks: 2+3+3+3=11

(a) $w=g(x,y)=\sin(x-y)$

$\dfrac{\partial w}{\partial x}=\cos(x-y),\ \dfrac{\partial^2 w}{\partial x^2}=-\sin(x-y)$

$\dfrac{\partial w}{\partial y}=-\cos(x-y),\ \dfrac{\partial^2 w}{\partial y^2}=-\sin(x-y)$

2

$\therefore \dfrac{\partial^2 w}{\partial x^2}-\dfrac{\partial^2 w}{\partial y^2}=0$

(b) the direction vector is $\vec{a}=\langle 1,2,3\rangle$ and a given point is $P(4,3,2)$

3

So, the symmetric Eqn of the line is

$$\frac{x-4}{1}=\frac{y-3}{2}=\frac{z-2}{3}$$

(c) Ellipsoid

"the traces and sketch in the following page"

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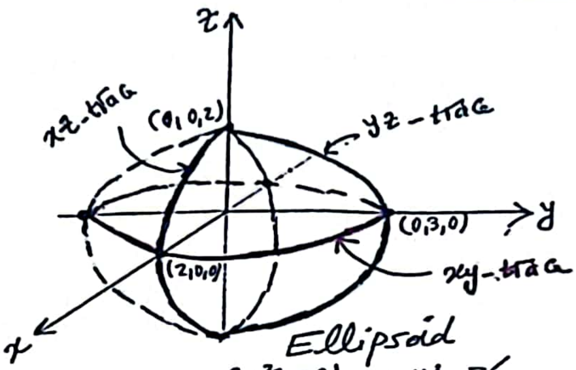
Q2(c)

Trace	Equation of trace	Description
On xy-plane ($z = 0$)	$\frac{x^2}{4}+\frac{y^2}{9}=1$	Ellipse
On yz-plane ($x = 0$)	$\frac{y^2}{9}+\frac{z^2}{4}=1$	Ellipse
On xz-plane ($y = 0$)	$\frac{x^2}{4}+\frac{z^2}{4}=1$ $\Rightarrow x^2+z^2=4$	Circle

3



Ellipsoid

$9(x^2+z^2)+4y^2=36$

$\Rightarrow \frac{x^2}{4}+\frac{z^2}{4}+\frac{y^2}{9}=1$

Q2(d)

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$

3

At $t=2$, $\vec{r}'(2) = \langle 1, 4, 12 \rangle$,

the point at $t=2$ is $P(2,4,8)$

So, the parametric Eqns for the tangent line is

$x = 2+t,\ y = 4+4t,\ z = 8+12t.$ #

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Q3. Marks: 3+4+3=10

(a) $\vec{r}(t) = \langle 2\sin t, \sin^2 t, t - \sin t\cos t \rangle$

$\vec{v} = \frac{d\vec{r}}{dt} = \langle 2\cos t, 2\sin t\cos t, \langle 1 - \cos^2 t + \sin^2 t \rangle$

$\therefore \vec{v} = 2\cos t\,\vec{i} + 2\sin t\cos t\,\vec{j} + 2\sin^2 t\,\vec{k}$

speed $= \|\vec{v}\|$

$= \sqrt{4\cos^2 t + 4\sin^2 t\cos^2 t + 4\sin^4 t}$

$= \sqrt{4\cos^2 t + 4\sin^2 t(1 - \sin^2 t) + 4S^4 t}$

$= \sqrt{4\cos^2 t + 4S^2 t}$ 3

$\therefore$ Speed $= 2\sqrt{\cos^2 t + S^2 t} = 2 =$ Const.

(b) $\vec{r}(t) = \langle \ln(1+t^2), 2\tan^{-1} t - t \rangle$

$\vec{r}'(t) = \langle \frac{2t}{1+t^2}, \frac{2}{1+t^2} - 1 \rangle$

$\vec{r}'(t) = \frac{1}{1+t^2} \langle 2t, 1 - t^2 \rangle$

$\|\vec{r}'(t)\| = \frac{1}{1+t^2}\sqrt{1 + 2t^2 + t^4} = 1$

• The unit tangent vector is 2

$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \langle \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \rangle$

$\vec{T}'(t) = \langle \frac{2-2t^2}{(1+t^2)^2}, \frac{-4t}{(1+t^2)^2} \rangle$

$\|\vec{T}'(t)\| = \sqrt{\frac{4 - 8t^2 + 4t^4 + 16t^2}{(1+t^2)^2}}$

$\|\vec{T}'(t)\| = 2\sqrt{\frac{(1+t^2)^2}{(1+t^2)^2}} = 2$

• The principal unit normal vector is

$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$ 2

$\therefore \vec{N}(t) = \langle \frac{1-t^2}{(1+t^2)^2}, \frac{-2t}{(1+t^2)^2} \rangle$

(c) $\vec{a}(t) = \langle 6t+2, e^t - 2, -\sin t \rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt$

$\vec{v}(t) = (3t^2 + 2t)\vec{i} + (e^t - 2t)\vec{j} + \cos t\,\vec{k} + \vec{C}$

$\because \vec{v}(0) = \langle 1, 0, 0 \rangle$

$\Rightarrow \vec{C} = \langle 1, 0, 0 \rangle - \langle 0, 1, 1 \rangle$

$\therefore \vec{C} = \langle 1, -1, -1 \rangle$

$\therefore \vec{v}(t) = \langle 3t^2 + 2t + 1, e^t - 2t - 1, \cos t - 1 \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt$

$\vec{r}(t) = \langle t^3 + t^2 + t, e^t - t^2 - t, \sin t - t \rangle + \vec{C}$

$\because \vec{r}(0) = \langle 1, -1, 0 \rangle$

$\Rightarrow \vec{C} = \langle 1, -1, 0 \rangle - \langle 0, 1, 0 \rangle$

$\therefore \vec{C} = \langle 1, -2, 0 \rangle$ 3

$\therefore \vec{r}(t) = \langle t^3 + t^2 + t + 1,$
$e^t - t^2 - t - 2,$
$\sin t - t \rangle$

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Q4. Marks: 2+3+4+3=12

(a) $z = x^2 y$, $x = r\cos\theta$, $y = r\sin\theta$

$\because \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$ chain Rule

$= 2xy\cos\theta + x^2\sin\theta$

$= 2r^2\cos^2\theta\sin\theta + r^2\cos^2\theta\sin\theta$

$\therefore \frac{\partial z}{\partial r} = 3r^2\sin\theta\cos^2\theta$ 1

$\because \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$ chain Rule

$= 2xy(-r\sin\theta) + x^2(r\cos\theta)$

$\therefore \frac{\partial z}{\partial \theta} = -2r^3\sin^2\theta\cos\theta + r^3\cos^3\theta$ 1

(b) $f(x,y,z) = 4x - y^2 e^{3xz}$

$\nabla f = \langle 4 - 3zy^2e^{3xz}, -2ye^{3xz}, -3xy^2e^{3xz}\rangle$

$\nabla f(x,y,z)\Big|_{(3,-1,0)} = \langle 4, 2, -9\rangle$ 1

$\vec{u} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{1}{3}\langle 2,2,1\rangle$ 1

$D_u f = \nabla f \cdot \vec{u}$

$D_u f(3,-1,0) = \langle 4,2,-9\rangle \cdot \frac{1}{3}\langle 2,2,1\rangle$

$= \frac{1}{3}(8+4-9) = 1$ 1

(c) $g(x,y) = x^2 + y^2 + xy^2 + 16$

we have $g_x = 2x + y^2 = 0$ 1, $g_y = 2y + 2yx = 0$ 2

By solving 1 and 2

1 $\Rightarrow x = -\frac{1}{2}y^2$, 2 $\Rightarrow 2y - y^3 = 0 \Rightarrow y(2-y^2) = 0$

$\Rightarrow$ Critical points are $(0,0)$, $(-1,\sqrt{2})$ and $(-1,-\sqrt{2})$

The discriminant D is

$D = g_{xx}g_{yy} - (g_{xy})^2$ 4

$\therefore D = 2(2+2x) - 4y^2 = 4 + 4x - 4y^2$,

- at $(0,0)$, $D = 4 > 0$, $g_{xx} = 2 > 0$,
  g has a local minimum, $g(0,0) = 16$
- at $(-1,\sqrt{2})$, $D = -8 < 0 \Rightarrow (-1,\sqrt{2}, g(-1,\sqrt{2}))$ is a saddle point
- at $(-1,-\sqrt{2})$, $D = -8 < 0 \Rightarrow (-1,-\sqrt{2}, g(-1,-\sqrt{2}))$ is a saddle point.

(d) we have

$f(x,y,z) = xyz$ 1

$g(x,y,z) = x + y + z - 1 = 0$ 2
Constraint

$\nabla f = \lambda \nabla g$ 3, λ is the Lagrange multiplier.

$\Rightarrow$

$\langle yz, xz, xy\rangle = \lambda\langle 1,1,1\rangle$

$\Rightarrow yz = \lambda$, $xz = \lambda$, $xy = \lambda$

$\therefore x = y = z$ 3

2 $\Rightarrow 3x - 1 = 0$

$\therefore x = y = z = \frac{1}{3}$

$\therefore$ f has a maximum value at $P(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

i.e. the max. value of f is

$f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{1}{27}$. #